A comprehensive review of implicit constitutive theories and their subclasses for elastic and inelastic solids

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Abstract

Implicit constitutive relations, wherein in general we cannot express the stresses in terms of the strains or vice-versa, have been proposed more than 20 years ago for the modelling of solid media. Since then, many works have appeared in the literature, considering elastic and inelastic deformations, small strains (small gradient of the displacement field) and large strains, and applications of such constitutive theories (and some of their subclasses where the strains are functions of the stresses) for different problems in biomechanics, rubber-mechanics, the study of stress concentration, rock mechanics, and the modelling of bone (among many other potential applications). In this paper a complete review of all of such works is provided, unifying the notation, classifying the different papers accordingly with the applications considered therein, and providing the most important equations and theoretical considerations of each of such works.

1 Introduction

More than 20 years ago Rajagopal [152] proposed the use of implicit constitutive relations for the modelling of solid elastic media, opening a new field of research in solid mechanics[1]. Up to date, when people speak about constitutive theories in solid mechanics, they tacitly work under the assumption that some measures of stresses are functions of some measures of strains [209]. In many works in elasticity through the use of a Helmholtz potential, we can obtain such stresses expressed in terms of the derivative of such scalar potential (Green elastic body). However, in [152, 154, 155, 156, 158] it has been shown that if for an elastic body, we mean a body wherein mechanical work is not dissipated as heat, then we can define different other types of constitutive relations and equations. Some of these new constitutive theories correspond to truly implicit constitutive relations, defined in different ways in terms of the energy potential [80, 152, 154, 155, 156, 158, 198], while other are constitutive equations, where some measures of strains are functions of some measure of stresses [155]. This last type of constitutive equation has been obtained through the use of a Gibbs' potential [156, 203].

The above implicit constitutive relations and equations, originally analyzed for elastic bodies, have inspired different researchers (who work in areas such as nonlinear elasticity, biomechanics, inelastic deformations, electro- and magneto-elasticity, and in rock and concrete mechanics) to extend them to other problems, wherein we have solids showing energy dissipation and problems involving multi-physics.

The purpose of this review article is to provide a complete summary of all the works on implicit constitutive relations and their subclasses that have been proposed for solid bodies, which have been published since the year 2003 up to approximately July of[2] 2024. This review not only summarizes the main results obtained in such communications, but also organize them in a rational manner, such that people interested in working with these relatively new constitutive theories, can access rapidly the important material for their

[1]Such implicit constitutive relations had been considered already by Rajagopal himself in [151], and prior to that by Morgan in [135] for an application of implicit constitutive relations for plasticity (see also [7]). In the field of nonlinear fluid mechanics, such implicit constitutive relations had already been studied and used by different researchers, see, for example, [153, 174] and the references therein.

[2]During the review process and the final publication of this work, some additional references that have been published after July of 2024 have been added.

particular research. Some parts of this work have appeared in [35].

This communication is divided in the following sections: In Section 2 some basic equations of continuum mechanics are shown. In Section 3 non-linear implicit constitutive relations are presented for elastic bodies, and several subclasses, wherein some measure of the strain is assumed to be a function of the stress, are also reviewed. In Section 4 implicit constitutive relations and their subclasses are presented for electro- and magneto-elastic bodies, for thermo-elastic bodies, and for bodies with initial stresses (residually stressed bodies). In Section 5 the case the norm of the gradient of the displacement field is small is considered, for the different constitutive relations and equations presented in the previous sections. In Section 6 constraints on the possible deformations (such as incompressibility and inextensibility) are reviewed for these constitutive theories, in particular for the constitutive equations, wherein the strains are functions of the stresses. In Section 7 different applications of these constitutive theories are presented for soft tissue, rubber-like solids, metallic alloys, concrete, rock and bone (among other applications). In Section 8 different boundary value problems are presented considering the particular case of small strains for the above implicit constitutive relations and their subclasses. In Section 9 other boundary value problems are reviewed for the case of large elastic deformations. In Section 10 different implicit constitutive relations are presented, which have been developed for problems involving inelastic deformations, and visco-elastic solids, including the analysis of some boundary value problems for some of them. Finally, in Section 11 a list of some open problems is presented.

List of symbols

$\mathscr{B}$	Abstract body,
$\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})$	Reference configuration,
$\boldsymbol{\kappa}_t(\mathscr{B})$	Current configuration,
X	Point in an abstract body $\mathscr{B}$,
$\boldsymbol{\chi}$	Motion,
$\mathbf{X}$	Position of X in the reference configuration,
$\mathbf{x}$	Position of X in the current configuration,
t	Time
$\mathbf{F}$	Deformation gradient,
$\mathbf{B}$	Left Cauchy-Green tensor,
$\mathbf{C}$	Right Cauchy-Green tensor,
$\mathbf{u}$	Displacement field,
$\mathbf{E}$	Green Saint-Venant strain tensor,
$\boldsymbol{\varepsilon}$	Linearized strain tensor,
$\mathbf{V}$	Tensor from polar decomposition,
$\boldsymbol{\eta}$	Hencky strain tensor,
λ_i	Principal stretches,
$\mathbf{T}$	Cauchy stress tensor,
$\mathbf{P}$	1st Piola-Kirchhoff stress tensor,
$\mathbf{S}$	2nd Piola-Kirchhoff stress tensor,
$\boldsymbol{\tau}$	Kirchhoff stress tensor,
$\mathbf{T}_{\mathrm{D}}$	Deviatoric stress tensor,
σ_{S}	Spherical stress,
$\mathbf{T}_{\mathrm{R}}$	Residual stress tensor,
σ_i	Principal stresses of $\mathbf{T}$,
$\mathbf{t}^{(i)}$	Principal directions of $\mathbf{T}$,
$\mathbf{v}^{(i)}$	Principal directions of $\mathbf{B}$,
ρ	Mass density in the current configuration,
ρ_{r}	Mass density in the reference configuration,
U	Internal specific energy,
s	Entropy,
θ	Absolute temperature,

$\mathbf{Q}$	Heat flux in the current configuration,
$\mathbf{Q}_{\mathrm{L}}$	Heat flux in the reference configuration,
$\mathbf{b}$	Body forces ,
ψ	Helmohltz free energy,
$\mathcal{G}$	Gibbs potential,
$\mathbf{I}$	Identity tensor,
$\mathfrak{F}$	Implicit constitutive relation,
$\mathfrak{G}$	Implicit constitutive relation,
$\mathfrak{H}$	Implicit constitutive relation,
$\mathfrak{f}$	Constitutive equation,
$\mathfrak{g}$	Constitutive equation,
$\mathfrak{h}$	Constitutive equation,
$\mathfrak{p}$	Constitutive equation,
$\mathbb{E}$	Electric field,
$\mathbb{D}$	Electric displacement,
$\mathbb{P}$	Electric polarization,
$\mathbb{H}$	Magnetic field,
$\mathbb{B}$	Magnetic induction,
$\mathbb{M}$	Magnetization,
$\boldsymbol{\tau}_{\mathrm{m}}$	Maxwell stress tensor,
$\mathbb{E}_{\mathrm{L}}$	Lagrangian electric field,
$\mathbb{D}_{\mathrm{L}}$	Lagrangian electric displacement,
$\mathbb{P}_{\mathrm{L}}$	Lagrangian electric polarization,
ϵ_o	Electric permittivity in vaccum,
μ_o	Magnetic permeability in vaccum,
$\hat{I}$	Imaginary unit.

2 Basic equations

Let a particle X belongs to an abstract body $\mathscr{B}$, and let $\boldsymbol{\kappa}_{\mathrm{r}}$ denote a one to one mapping that maps $\mathscr{B}$ into a three dimensional Euclidean space, and let $\mathbf{X} = \boldsymbol{\kappa}_{\mathrm{r}}(X)$. Let us refer to $\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})$ as the reference configuration of $\mathscr{B}$. Let $\boldsymbol{\kappa}_t$, $t \in \mathcal{R}$ be a one parameter family of placers such that $\mathbf{x} = \boldsymbol{\kappa}_t(X)$. Since $\boldsymbol{\kappa}_{\mathrm{r}}$ is invertible we can define a mapping such that $\mathbf{x} = \boldsymbol{\chi}_{\boldsymbol{\kappa}_{\mathrm{r}}}(\mathbf{X}, t)$. We will assume $\boldsymbol{\chi}_{\boldsymbol{\kappa}_{\mathrm{r}}}$ to be sufficiently smooth and also for ease of notation we shall drop the suffix $\boldsymbol{\kappa}_{\mathrm{r}}$ and refer to the mapping as $\boldsymbol{\chi}$. The deformation gradient, left and right Cauchy-Green tensors, the Lagrange-Saint Venant strain tensor, displacement field, and linearized strain tensor are defined, respectively as:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \quad \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}), \tag{1}$$

$$\mathbf{u} = \mathbf{x} - \mathbf{X}, \quad \boldsymbol{\varepsilon} = \frac{1}{2}(\nabla_{\mathbf{X}}\mathbf{u} + \nabla_{\mathbf{X}}\mathbf{u}^{\mathrm{T}}), \tag{2}$$

where $\nabla_{\mathbf{X}}$ is the gradient operator with respect to the reference configuration and we assume that $J = \det \mathbf{F} > 0$.

The Cauchy stress tensor, the nominal stress tensor, the second Piola-Kirchhoff stress tensor and the Kirchhoff stress tensor are denoted $\mathbf{T}$, $\mathbf{P}$, $\mathbf{S}$ and $\boldsymbol{\tau}$, respectively, and they are related through

$$\mathbf{P} = J\mathbf{F}^{-1}\mathbf{T}, \quad \mathbf{S} = \mathbf{P}\mathbf{F}^{-\mathrm{T}} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-\mathrm{T}}, \quad \boldsymbol{\tau} = J\mathbf{T}. \tag{3}$$

The balance of linear momentum, the balance of mass and the first law of thermodynamics for elastic bodies (if there is no dissipation) read, respectively:

$$\rho\ddot{\mathbf{x}} = \operatorname{div}\mathbf{T} + \rho\mathbf{b}, \quad \dot{\rho} + \rho\operatorname{div}\dot{\mathbf{x}} = 0, \quad \operatorname{tr}(\mathbf{S}\dot{\mathbf{E}}) = \rho_{\mathrm{r}}\dot{U}, \tag{4}$$

where $\mathbf{b}$ represents the body force, div is the divergence operator in the current configuration, ρ is the mass density of the body in the configuration at time t, where the density of the body in the reference configuration is denoted ρ_{r} and $\rho_{\mathrm{r}} = J\rho$, and where U is the internal energy of the body.

For more details about the above definitions and equations see, for example, [207, 208].

3 Nonlinear implicit theories for elastic bodies

In this section we list different types of implicit constitutive relations, which have been proposed for elastic bodies (there is no dissipation of energy and no heat transfer). Some subclassed, wherein some measure of strain is a function of some measure of stress are also reviewed.

3.1 Preliminary discussion

In the classical theory of elasticity, it is customary to assume a measure of stress as a function of some measure of strain (or deformation gradient) as in the case of a Cauchy elastic body (see [61, 62]), where $\mathbf{T} = \mathfrak{f}(\mathbf{F})$. If one assumes that there exists a stored energy function $W = W(\mathbf{F})$, then one can show that $\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}$, and such bodies are referred to as Green elastic bodies (see [89, 90]). In the special case of the body being linearized elastic isotropic, the constitutive expression for the Cauchy stress takes the form $\mathbf{T} = 2\mu\boldsymbol{\varepsilon} + \lambda \operatorname{tr}(\boldsymbol{\varepsilon})\mathbf{I}$, where μ, λ are constants. The expression for the Cauchy stress of the linearized elastic body can be inverted and we obtain $\boldsymbol{\varepsilon} = \frac{(1+\nu)}{E}\mathbf{T} - \frac{\nu}{E}\operatorname{tr}(\mathbf{T})\mathbf{I}$, where $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, $\mu = \frac{E}{2(1+\nu)}$, where E and ν are the Young modulus and Poisson ratio, respectively. In this last expression we have material constants E, ν that have a clear physical meaning, in contrast with the previous expression for the stress in terms of the linearized strain, where the Lame' constants μ, λ are determined in a circuitous fashion by using the values for the shear modulus and the bulk modulus. The functions $\mathfrak{f}(\mathbf{F})$ and $W(\mathbf{F})$ in general cannot be inverted.

3.2 Implicit constitutive relations for elastic solids

A more general relation between stresses and strains, from which we can obtain as special cases the previous constitutive equations, is the implicit relation[3] [152, 155, 156, 167, 177]

$$\mathfrak{F}(\mathbf{S}, \mathbf{E}) = \mathbf{0}. \tag{5}$$

[3] In [187] there is an analysis of an implicit constitutive relation $\mathfrak{F}(\mathbf{S}, \mathbf{C}) = \mathbf{0}$, which is similar to (5). The main question studied in that paper is the analysis on conditions on $\mathfrak{F}$ such that $\mathbf{S}$ can be expressed in terms of $\mathbf{C}$ or vice-versa. To answer that question the implicit function theorem is used, in order to obtain conditions on the local solvability of the above implicit relation. One special case for $\mathfrak{F}$ is considered, namely when $\mathbf{SC} + \mathbf{CS} = \mathfrak{H}(\mathbf{S})$. A solution for the above equation for $\mathbf{C}$ in terms of $\mathbf{S}$ is found, namely $\mathbf{C} = \mathfrak{H}(\mathbf{S})$. Another similar case that is studied in that paper is $(\alpha_1\mathbf{S} + \alpha_2\mathbf{S}^2)\mathbf{C} + \mathbf{C}(\alpha_3\mathbf{S} + \alpha_4\mathbf{S}^2) = \mathfrak{H}(\mathbf{S})$, which is solved for $\mathbf{S}$, where $\alpha_i = \alpha_i(\mathbf{S})$. The above are special subclasses of $\mathfrak{F}(\mathbf{S}, \mathbf{C})$ isotropic, which is of the form (9), changing $\mathbf{T}$, $\mathbf{B}$ by $\mathbf{S}$ and $\mathbf{C}$, respectively, where again it is assumed that $\alpha_i = \alpha_i(\mathbf{S})$.
In [188] there is a continuation of the above analysis, assuming that $\mathfrak{F}(\mathbf{S}, \mathbf{C}) = \mathbf{0}$ cannot be solved for either $\mathbf{C}$ as a function of $\mathbf{S}$ or vice-versa. In [188] it is shown that locally there exists $W = W(\mathbf{F}, \mathbf{p})$ such that for the 1st Piola-Kirchhoff stress tensor (here we use the same notation as the nominal stress) $\mathbf{P} = \frac{\partial W}{\partial \mathbf{C}}$, $\mathbf{0} = \frac{\partial W}{\partial \mathbf{p}}$, where $\mathbf{p}$ is an auxiliary vector

From the first law of thermodynamics, when there is no dissipation of energy $(4)_3$, if we assume that the stored energy of the body is expressed as $U = U(\mathbf{S}, \mathbf{E})$, then for a body to be elastic, we need U and $\mathfrak{F}$ to satisfy

$$S_{\alpha\beta}\dot{E}_{\alpha\beta} = \rho_{\mathrm{r}}\left(\frac{\partial U}{\partial S_{\alpha\beta}}\dot{S}_{\alpha\beta} + \frac{\partial U}{\partial E_{\alpha\beta}}\dot{E}_{\alpha\beta}\right), \quad \frac{\partial \mathfrak{F}_{\gamma\delta}}{\partial S_{\alpha\beta}}\dot{S}_{\alpha\beta} + \frac{\partial \mathfrak{F}_{\gamma\delta}}{\partial E_{\alpha\beta}}\dot{E}_{\alpha\beta} = 0. \quad (6)$$

If we assume that $\mathbf{S} = \mathfrak{p}(\mathbf{E})$ and $U = U(\mathbf{E})$, then from (6) it is possible to show that $\mathfrak{p}_{\alpha\beta}(\mathbf{E}) = \rho_{\mathrm{r}}\frac{\partial U}{\partial E_{\alpha\beta}}$, i.e., the Green elastic body is just a special case of (5).

A more general class of implicit constitutive relations than (5) is that wherein $\mathbf{S}$ and $\mathbf{E}$ must be found from (see Eq. (3.2) in [155])

$$\mathscr{A}_{\alpha\beta\gamma\delta}(\mathbf{S}, \mathbf{E})\dot{S}_{\gamma\delta} + \mathscr{B}_{\alpha\beta\gamma\delta}(\mathbf{S}, \mathbf{E})\dot{E}_{\gamma\delta} = 0, \quad (7)$$

subject to the restriction $(6)_1$.

Another implicit constitutive relation proposed in [154] is:

$$\mathfrak{G}(\mathbf{T}, \mathbf{B}, \rho) = \mathbf{0}, \quad (8)$$

from where we can obtain as a special case the Cauchy elastic bodies $\mathbf{T} = \mathfrak{f}(\rho, \mathbf{B})$, and the subclass of elastic body[4] $\mathbf{B} = \mathfrak{g}(\rho, \mathbf{T})$. In the case $\mathfrak{G}$ is an isotropic relation, and on the use of the classical theory of invariants by Spencer Eq. (8) becomes (see [163, 202]):

$$\alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{B} + \alpha_3\mathbf{T}^2 + \alpha_4\mathbf{B}^2 + \alpha_5(\mathbf{TB} + \mathbf{BT}) + \alpha_6(\mathbf{T}^2\mathbf{B} + \mathbf{BT}^2) + \alpha_7(\mathbf{B}^2\mathbf{T} + \mathbf{TB}^2) + \alpha_8(\mathbf{T}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{T}^2) = \mathbf{0}, \quad (9)$$

field. In the case $W = W(\mathbf{C}, \mathbf{p})$ we have

$$W = W(\mathbf{C}, \mathbf{p}) = W(\mathrm{tr}\,\mathbf{C}, \mathrm{tr}\,(\mathbf{C}^2), \mathrm{tr}\,(\mathbf{C}^3), \mathbf{p}\cdot\mathbf{p}, \mathbf{p}\cdot(\mathbf{Cp}), \mathbf{p}\cdot(\mathbf{C}^2\mathbf{p})).$$

This expression is analyzed in [188], and some conditions are studied for W such that $\mathbf{S}$ can be expressed (locally) in terms of $\mathbf{C}$, $\mathbf{p}$.

[4]This last subclass of elastic bodies $\mathbf{B} = \mathfrak{g}(\rho, \mathbf{T})$ can satisfy (6) if one assumes that there exists an implicit constitutive relation $\mathfrak{H}(\mathbf{S}, \mathbf{C}, \rho) = \mathbf{0}$ and also $U = U(\mathbf{S}, \mathbf{C}, \rho)$ that satisfies (6), and that for the special case in which such functions can be written as $\mathfrak{H}(\mathbf{S}, \mathbf{C}, \rho) = \tilde{\mathfrak{H}}(J^{-1}\mathbf{SC}) - \mathbf{C} = \mathbf{0}$ and $U(\mathbf{S}, \mathbf{C}, \rho) = U(J^{-1}\mathbf{SC})$, defining $\mathbf{G} = J^{-1}\mathbf{SC}$, the condition (6) becomes $S_{\alpha\beta}\frac{\partial \tilde{\mathfrak{H}}}{\partial G_{\gamma\delta}} - \rho_{\mathrm{r}}\frac{\partial U}{\partial G_{\gamma\delta}} = 0$. If we assume there exists a function $\hat{W}(\mathbf{G})$ such that $\tilde{\mathfrak{H}}(\mathbf{G}) = \frac{\partial \hat{W}}{\partial \mathbf{G}}$, then if $\tilde{\mathfrak{H}}$ satisfies the above equation the body is elastic, and we can find $\mathfrak{g}$ directly in terms of the derivatives of $\hat{W}$ (see [23]).

where the scalar functions α_i, $i = 0, 1, 2, ..., 8$ depend on the invariants (see [202]):

$$\rho, \quad I_1 = \mathrm{tr}\,\mathbf{T}, \quad I_2 = \mathrm{tr}\,(\mathbf{T}^2), \quad I_3 = \mathrm{tr}\,(\mathbf{T}^3), \quad I_4 = \mathrm{tr}\,\mathbf{B}, \quad I_5 = \mathrm{tr}\,(\mathbf{B}^2), \tag{10}$$

$$I_6 = \mathrm{tr}\,(\mathbf{B}^3), \quad I_7 = \mathrm{tr}\,(\mathbf{T}\mathbf{B}), \quad I_8 = \mathrm{tr}\,(\mathbf{T}^2\mathbf{B}), \quad I_9 = \mathrm{tr}\,(\mathbf{B}^2\mathbf{T}), \quad I_{10} = \mathrm{tr}\,(\mathbf{T}^2\mathbf{B}^2). \tag{11}$$

In the particular case that $\alpha_l = 0$, $l = 4, 5, 6, 7, 8$ and α_m, $m = 0, 1, 2, 3$ do not depend on the invariants I_n, $n = 4, 5, 6, 7, 8, 9, 10$, from (9) we have the subclass $\mathbf{B} = \mathfrak{g}(\rho, \mathbf{T})$ whose explicit expression is [154]

$$\mathbf{B} = \bar{\alpha}_0\mathbf{I} + \bar{\alpha}_1\mathbf{T} + \bar{\alpha}_2\mathbf{T}^2, \tag{12}$$

where the functions $\bar{\alpha}_j$, $j = 0, 1, 2$ depend on the invariants I_1, I_2, I_3 and ρ from (10). Dependence on ρ in virtue of the balance of mass implies that the material functions $\bar{\alpha}_j$ depend on the determinant of the deformation gradient and hence the model (12) is yet implicit. If we make the further assumption that the material functions do not depend on the density ρ, then (12) would be an explicit relationship for $\mathbf{B}$ in terms of the stress $\mathbf{T}$.

Another type of implicit constitutive relation can be derived directly from $(6)_1$ in the following manner (see Eq. (3.3) of [155] and Chapter 6.2 of [80]). Let us rewrite $(6)_1$ as $S_{\alpha\beta}\mathrm{d}E_{\alpha\beta} = \frac{\partial\Pi}{\partial S_{\alpha\beta}}\mathrm{d}S_{\alpha\beta} + \frac{\partial\Pi}{\partial E_{\alpha\beta}}\mathrm{d}E_{\alpha\beta}$, where we have defined $\Pi = \rho_{\mathrm{r}}U$. Let us take the derivative of the above expression in $\mathbf{S}$, we obtain in indicial notation and Cartesian coordinates (see [80])

$$\mathscr{A}_{\gamma\delta\alpha\beta}\mathrm{d}S_{\alpha\beta} + \mathscr{B}_{\gamma\delta\alpha\beta}\mathrm{d}E_{\alpha\beta} = 0, \tag{13}$$

where we have defined

$$\mathscr{A}_{\gamma\delta\alpha\beta}(\mathbf{S},\mathbf{E}) = \frac{1}{4}\left(\frac{\partial^2\Pi}{\partial S_{\alpha\beta}\partial S_{\gamma\delta}} + \frac{\partial^2\Pi}{\partial S_{\alpha\beta}\partial S_{\delta\gamma}} + \frac{\partial^2\Pi}{\partial S_{\beta\alpha}\partial S_{\gamma\delta}} + \frac{\partial^2\Pi}{\partial S_{\beta\alpha}\partial S_{\delta\gamma}}\right), \tag{14}$$

$$\mathscr{B}_{\gamma\delta\alpha\beta}(\mathbf{S},\mathbf{E}) = \frac{1}{4}\left(\frac{\partial^2\Pi}{\partial E_{\alpha\beta}\partial S_{\gamma\delta}} + \frac{\partial^2\Pi}{\partial E_{\alpha\beta}\partial S_{\delta\gamma}} + \frac{\partial^2\Pi}{\partial E_{\beta\alpha}\partial S_{\gamma\delta}} + \frac{\partial^2\Pi}{\partial E_{\beta\alpha}\partial S_{\delta\gamma}}\right) - \frac{1}{2}(\delta_{\alpha\delta}\delta_{\delta\beta} + \delta_{\beta\gamma}\delta_{\alpha\delta}), \tag{15}$$

and where we have used the fact that $\mathbf{S}$ is a symmetric tensor. The above implicit relation (13) is of the form (7) and its parts have been defined explicitly in terms of $\Pi(\mathbf{S},\mathbf{E})$. If $\boldsymbol{\mathscr{A}}$ has an inverse, from (13) we obtain the subclass of implicit relations (see [80, 155])

$$\mathrm{d}S_{\epsilon\zeta} = -\overset{-1}{\mathscr{A}}_{\epsilon\zeta\gamma\delta}\,\mathscr{B}_{\gamma\delta\alpha\beta}\mathrm{d}E_{\alpha\beta},$$

where $\overset{-1}{\mathscr{A}}_{\epsilon\zeta\gamma\delta}$ are the components of $\boldsymbol{\mathscr{A}}^{-1}$.

The implicit relation (13) can be re-written in free-index notation as

$$\left[\frac{1}{2}\left(\frac{\partial^2\Pi}{\partial\mathbf{S}\partial\mathbf{E}}+\frac{\partial^2\Pi}{\partial\mathbf{E}\partial\mathbf{S}}\right)-\boldsymbol{\mathscr{I}}\right]\cdot\dot{\mathbf{E}}+\frac{\partial^2\Pi}{\partial\mathbf{S}\partial\mathbf{S}}\cdot\dot{\mathbf{S}}=\mathbf{0}, \tag{16}$$

where from (14), (15) we can define $\boldsymbol{\mathscr{A}}=\frac{\partial^2\Pi}{\partial\mathbf{S}\partial\mathbf{S}}$ and $\boldsymbol{\mathscr{B}}=\frac{1}{2}\left(\frac{\partial^2\Pi}{\partial\mathbf{S}\partial\mathbf{E}}+\frac{\partial^2\Pi}{\partial\mathbf{E}\partial\mathbf{S}}\right)-\boldsymbol{\mathscr{I}}$. In (16) the fourth order tensor $\boldsymbol{\mathscr{I}}$ in index notation and Cartesian coordinates is given by

$$\mathscr{I}_{\alpha\beta\gamma\delta}=\frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}).$$

3.2.1 An implicit relation in terms of the Lagrange Saint-Venant strain tensor and the 2nd Piola-Kirchhoff stress tensor in terms of the classical invariants by Spencer

Let us show an explicit expression for (16) if $\Pi=\Pi(\mathbf{S},\mathbf{E})$ is isotropic. From [202] we have $\Pi=\Pi(I_1,I_2,...,I_{10})$, where $I_1=\operatorname{tr}\mathbf{S}$, $I_2=\frac{1}{2}\operatorname{tr}(\mathbf{S}^2)$, $I_3=\frac{1}{3}\operatorname{tr}(\mathbf{S}^3)$, $I_4=\operatorname{tr}\mathbf{E}$, $I_5=\frac{1}{2}\operatorname{tr}(\mathbf{E}^2)$, $I_6=\frac{1}{3}\operatorname{tr}(\mathbf{E}^3)$, $I_7=\operatorname{tr}(\mathbf{SE})$, $I_8=\operatorname{tr}(\mathbf{S}^2\mathbf{E})$, $I_9=\operatorname{tr}(\mathbf{SE}^2)$ and $I_{10}=\operatorname{tr}(\mathbf{S}^2\mathbf{E}^2)$. Using the notation $\Pi_i=\frac{\partial\Pi}{\partial I_i}$ and $\Pi_{i,j}=\frac{\partial^2\Pi}{\partial I_i\partial I_j}$ we have

$$\frac{\partial\Pi}{\partial\mathbf{S}}=\Pi_1\mathbf{I}+\Pi_2\mathbf{S}+\Pi_3\mathbf{S}^2+\Pi_7\mathbf{E}+\Pi_8(\mathbf{ES}+\mathbf{SE})+\Pi_9\mathbf{E}^2+\Pi_{10}(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2),$$

thus

$$\begin{aligned}\boldsymbol{\mathscr{A}}=\frac{\partial^2\Pi}{\partial\mathbf{S}\partial\mathbf{S}}=&\ \Pi_{1,1}\boldsymbol{\mathscr{A}}^{(1)}+\Pi_{1,2}\boldsymbol{\mathscr{A}}^{(2)}+\Pi_{1,3}\boldsymbol{\mathscr{A}}^{(3)}+\Pi_{1,7}\boldsymbol{\mathscr{A}}^{(4)}+\Pi_{1,8}\boldsymbol{\mathscr{A}}^{(5)}+\Pi_{1,9}\boldsymbol{\mathscr{A}}^{(6)}\\&+\Pi_{1,10}\boldsymbol{\mathscr{A}}^{(7)}+\Pi_{2,2}\boldsymbol{\mathscr{A}}^{(8)}+\Pi_{2,3}\boldsymbol{\mathscr{A}}^{(9)}+\Pi_{2,7}\boldsymbol{\mathscr{A}}^{(10)}+\Pi_{2,8}\boldsymbol{\mathscr{A}}^{(11)}\\&+\Pi_{2,9}\boldsymbol{\mathscr{A}}^{(12)}+\Pi_{2,10}\boldsymbol{\mathscr{A}}^{(13)}+\Pi_{3,3}\boldsymbol{\mathscr{A}}^{(14)}+\Pi_{3,7}\boldsymbol{\mathscr{A}}^{(15)}+\Pi_{3,8}\boldsymbol{\mathscr{A}}^{(16)}\\&+\Pi_{3,9}\boldsymbol{\mathscr{A}}^{(17)}+\Pi_{3,10}\boldsymbol{\mathscr{A}}^{(18)}+\Pi_{7,7}\boldsymbol{\mathscr{A}}^{(19)}+\Pi_{7,8}\boldsymbol{\mathscr{A}}^{(20)}+\Pi_{7,9}\boldsymbol{\mathscr{A}}^{(21)}\\&+\Pi_{7,10}\boldsymbol{\mathscr{A}}^{(22)}+\Pi_{8,8}\boldsymbol{\mathscr{A}}^{(23)}+\Pi_{8,9}\boldsymbol{\mathscr{A}}^{(24)}+\Pi_{8,10}\boldsymbol{\mathscr{A}}^{(25)}+\Pi_{9,9}\boldsymbol{\mathscr{A}}^{(26)}\\&+\Pi_{9,10}\boldsymbol{\mathscr{A}}^{(27)}+\Pi_{10,10}\boldsymbol{\mathscr{A}}^{(28)}+\Pi_2\boldsymbol{\mathscr{I}}+\Pi_3\boldsymbol{\mathscr{S}}^{(1)}+\Pi_8\boldsymbol{\mathscr{E}}^{(1)}+\Pi_{10}\boldsymbol{\mathscr{E}}^{(1)},\end{aligned}$$

where we have defined $\boldsymbol{\mathscr{A}}^{(1)}=\mathbf{I}\otimes\mathbf{I}$, $\boldsymbol{\mathscr{A}}^{(2)}=\mathbf{I}\otimes\mathbf{S}+\mathbf{S}\otimes\mathbf{I}$, $\boldsymbol{\mathscr{A}}^{(3)}=\mathbf{I}\otimes\mathbf{S}^2+\mathbf{S}^2\otimes\mathbf{I}$, $\boldsymbol{\mathscr{A}}^{(4)}=\mathbf{I}\otimes\mathbf{E}+\mathbf{E}\otimes\mathbf{I}$, $\boldsymbol{\mathscr{A}}^{(5)}=\mathbf{I}\otimes(\mathbf{ES}+\mathbf{SE})+(\mathbf{ES}+\mathbf{SE})\otimes\mathbf{I}$, $\boldsymbol{\mathscr{A}}^{(6)}=\mathbf{I}\otimes\mathbf{E}^2+\mathbf{E}^2\otimes\mathbf{I}$, $\boldsymbol{\mathscr{A}}^{(7)}=\mathbf{I}\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)+(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)\otimes\mathbf{I}$, $\boldsymbol{\mathscr{A}}^{(8)}=\mathbf{S}\otimes\mathbf{S}$, $\boldsymbol{\mathscr{A}}^{(9)}=\mathbf{S}\otimes\mathbf{S}^2+\mathbf{S}^2\otimes\mathbf{S}$, $\boldsymbol{\mathscr{A}}^{(10)}=\mathbf{S}\otimes\mathbf{E}+\mathbf{E}\otimes\mathbf{S}$, $\boldsymbol{\mathscr{A}}^{(11)}=\mathbf{S}\otimes(\mathbf{ES}+\mathbf{SE})+(\mathbf{ES}+\mathbf{SE})\otimes\mathbf{S}$,

$\mathscr{A}^{(12)} = \mathbf{S}\otimes\mathbf{E}^2+\mathbf{E}^2\otimes\mathbf{S}$, $\mathscr{A}^{(13)} = \mathbf{S}\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{S}\mathbf{E}^2)+(\mathbf{E}^2\mathbf{S}+\mathbf{S}\mathbf{E}^2)\otimes\mathbf{S}$, $\mathscr{A}^{(14)} = \mathbf{S}^2\otimes\mathbf{S}^2$, $\mathscr{A}^{(15)} = \mathbf{S}^2\otimes\mathbf{E}+\mathbf{E}\otimes\mathbf{S}^2$, $\mathscr{A}^{(16)} = \mathbf{S}^2\otimes(\mathbf{E}\mathbf{S}+\mathbf{S}\mathbf{E})+(\mathbf{E}\mathbf{S}+\mathbf{S}\mathbf{E})\otimes\mathbf{S}^2$, $\mathscr{A}^{(17)} = \mathbf{S}^2\otimes\mathbf{E}^2+\mathbf{E}^2\otimes\mathbf{S}^2$, $\mathscr{A}^{(18)} = \mathbf{S}^2\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{S}\mathbf{E}^2)+(\mathbf{E}^2\mathbf{S}+\mathbf{S}\mathbf{E}^2)\otimes\mathbf{S}^2$, $\mathscr{A}^{(19)} = \mathbf{E}\otimes\mathbf{E}$, $\mathscr{A}^{(20)} = \mathbf{E}\otimes(\mathbf{E}\mathbf{S}+\mathbf{S}\mathbf{E})+(\mathbf{E}\mathbf{S}+\mathbf{S}\mathbf{E})\otimes\mathbf{E}$, $\mathscr{A}^{(21)} = \mathbf{E}\otimes\mathbf{E}^2+\mathbf{E}^2\otimes\mathbf{E}$, $\mathscr{A}^{(22)} = \mathbf{E}\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{S}\mathbf{E}^2)+(\mathbf{E}^2\mathbf{S}+\mathbf{S}\mathbf{E}^2)\otimes\mathbf{E}$, $\mathscr{A}^{(23)} = (\mathbf{E}\mathbf{S}+\mathbf{S}\mathbf{E})\otimes(\mathbf{E}\mathbf{S}+\mathbf{S}\mathbf{E})$, $\mathscr{A}^{(24)} = (\mathbf{E}\mathbf{S}+\mathbf{S}\mathbf{E})\otimes\mathbf{E}^2+\mathbf{E}^2\otimes(\mathbf{E}\mathbf{S}+\mathbf{S}\mathbf{E})$, $\mathscr{A}^{(25)} = (\mathbf{E}\mathbf{S}+\mathbf{S}\mathbf{E})\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{S}\mathbf{E}^2)+(\mathbf{E}^2\mathbf{S}+\mathbf{S}\mathbf{E}^2)\otimes(\mathbf{E}\mathbf{S}+\mathbf{S}\mathbf{E})$, $\mathscr{A}^{(26)} = \mathbf{E}^2\otimes\mathbf{E}^2$, $\mathscr{A}^{(27)} = \mathbf{E}^2\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{S}\mathbf{E}^2)+(\mathbf{E}^2\mathbf{S}+\mathbf{S}\mathbf{E}^2)\otimes\mathbf{E}^2$, $\mathscr{A}^{(28)} = (\mathbf{E}^2\mathbf{S}+\mathbf{S}\mathbf{E}^2)\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{S}\mathbf{E}^2)$, and the tensor $\mathscr{I}$ has been defined previously. The fourth order tensor $\mathscr{S}^{(1)}$, $\mathscr{E}^{(1)}$ and $\mathscr{E}^{(1)}$ are given in index notation (in Cartesian coordinates) as $\mathscr{S}^{(1)}_{ijkl} = \frac{1}{2}(\delta_{ik}S_{lj}+S_{ik}\delta_{jl}+\delta_{jk}S_{li}+S_{jk}\delta_{il})$, $\mathscr{E}^{(1)}_{ijkl} = \frac{1}{2}(E_{ik}\delta_{jl}+E_{il}\delta_{jk}+\delta_{ik}E_{lj}+\delta_{il}E_{kj})$ and $\mathscr{E}^{(2)}_{ijkl} = \frac{1}{2}(E_{im}E_{mk}\delta_{jl}+E_{im}E_{ml}\delta_{jk}+\delta_{ik}E_{lm}E_{mj}+\delta_{il}E_{km}E_{mj})$, respectively.

In the case of $\mathscr{B}$, considering the definition given in the previous section $\mathscr{B} = \frac{1}{2}\left(\frac{\partial^2\Pi}{\partial\mathbf{S}\partial\mathbf{E}}+\frac{\partial^2\Pi}{\partial\mathbf{E}\partial\mathbf{S}}\right)-\mathscr{I}$, using $\Pi = \Pi(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_{10})$ after several steps we obtain

$$\begin{aligned}
\mathscr{B} = \frac{1}{2}[&2\Pi_{1,4}\mathscr{A}^{(1)}+(\Pi_{1,7}+\Pi_{2,4})\mathscr{A}^{(2)}+\Pi_{1,8}\mathscr{A}^{(3)}+(\Pi_{1,5}+\Pi_{4,9})\mathscr{A}^{(4)}\\
&+(\Pi_{1,9}+\Pi_{4,8})\mathscr{A}^{(5)}+(\Pi_{1,6}+\Pi_{4,9})\mathscr{A}^{(6)}+\Pi_{4,10}\mathscr{A}^{(7)}+2\Pi_{2,7}\mathscr{A}^{(8)}\\
&+(\Pi_{2,8}+\Pi_{3,7})\mathscr{A}^{(9)}+\Pi_{2,5}\mathscr{A}^{(10)}+(\Pi_{2,9}+\Pi_{7,8})\mathscr{A}^{(11)}+(\Pi_{2,6}+\Pi_{7,9})\mathscr{A}^{(12)}\\
&+\Pi_{7,10}\mathscr{A}^{(13)}+2\Pi_{3,8}\mathscr{A}^{(14)}+\Pi_{3,5}\mathscr{A}^{(15)}+(\Pi_{3,9}+\Pi_{8,8})\mathscr{A}^{(16)}+\Pi_{7,8}\mathscr{A}^{(17)}\\
&+\Pi_{8,10}\mathscr{A}^{(18)}+2\Pi_{5,7}\mathscr{A}^{(19)}+(\Pi_{5,8}+\Pi_{7,9})\mathscr{A}^{(20)}+(\Pi_{5,9}+\Pi_{6,9})\mathscr{A}^{(21)}\\
&+\Pi_{5,10}\mathscr{A}^{(22)}+\Pi_{8,9}\mathscr{A}^{(23)}+(\Pi_{6,8}+\Pi_{9,9})\mathscr{A}^{(24)}+\Pi_{9,10}\mathscr{A}^{(25)}+2\Pi_{6,9}\mathscr{A}^{(26)}\\
&+\Pi_{6,10}\mathscr{A}^{(27)}+\Pi_{1,10}\mathscr{B}^{(1)}+\Pi_{3,10}\mathscr{B}^{(2)}+\Pi_{2,10}\mathscr{B}^{(3)}+\Pi_{8,9}\mathscr{B}^{(4)}+\Pi_{8,10}\mathscr{B}^{(5)}\\
&+\Pi_{9,10}\mathscr{B}^{(6)}+\Pi_{10,10}\mathscr{B}^{(7)}+2\Pi_7\mathscr{I}+2\Pi_8\mathscr{S}^{(1)}+2\Pi_9\mathscr{E}^{(1)}+\Pi_{10}\mathscr{F}]-\mathscr{I},
\end{aligned}$$

where the $\mathscr{A}^{(i)}$, $i = 1, 2, ..., 27$, $\mathscr{S}^{(1)}$ and $\mathscr{E}^{(1)}$ have been defined previously, and $\mathscr{B}^{(1)} = \mathbf{I}\otimes(\mathbf{E}\mathbf{S}^2+\mathbf{S}^2\mathbf{E})+(\mathbf{E}\mathbf{S}^2+\mathbf{S}^2\mathbf{E})\otimes\mathbf{I}$, $\mathscr{B}^{(2)} = (\mathbf{S}^2\mathbf{E}+\mathbf{E}\mathbf{S}^2)\otimes\mathbf{S}^2+\mathbf{S}^2\otimes(\mathbf{S}^2\mathbf{E}+\mathbf{E}\mathbf{S}^2)$, $\mathscr{B}^{(3)} = (\mathbf{S}^2\mathbf{E}+\mathbf{E}\mathbf{S}^2)\otimes\mathbf{S}+\mathbf{S}\otimes(\mathbf{S}^2\mathbf{E}+\mathbf{E}\mathbf{S}^2)$, $\mathscr{B}^{(4)} = \mathbf{E}^2\otimes\mathbf{S}^2+\mathbf{S}^2\otimes\mathbf{E}^2$, $\mathscr{B}^{(5)} = (\mathbf{E}\mathbf{S}+\mathbf{S}\mathbf{E})\otimes(\mathbf{S}^2\mathbf{E}+\mathbf{E}\mathbf{S}^2)+(\mathbf{S}^2\mathbf{E}+\mathbf{E}\mathbf{S}^2)\otimes(\mathbf{E}\mathbf{S}+\mathbf{S}\mathbf{E})$, $\mathscr{B}^{(6)} = (\mathbf{S}^2\mathbf{E}+\mathbf{E}\mathbf{S}^2)\otimes\mathbf{E}^2+\mathbf{E}^2\otimes(\mathbf{S}^2\mathbf{E}+\mathbf{E}\mathbf{S}^2)$, $\mathscr{B}^{(7)} = (\mathbf{E}^2\mathbf{S}+\mathbf{S}\mathbf{E}^2)\otimes(\mathbf{S}^2\mathbf{E}+\mathbf{E}\mathbf{S}^2)+(\mathbf{S}^2\mathbf{E}+\mathbf{E}\mathbf{S}^2)\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{S}\mathbf{E}^2)$, and the components of the fourth order tensor $\mathscr{F}$ are given (in Cartesian coordinates) as $\mathscr{F}_{ijkl} = \frac{1}{2}(\delta_{ik}E_{lm}S_{mj}+2E_{ik}S_{lj}+2S_{ik}E_{lj}+S_{im}E_{mk}\delta_{jl}+\delta_{il}E_{km}S_{mj}+2E_{il}S_{kj}+2S_{il}E_{kj}+S_{im}E_{ml}\delta_{jk}+\delta_{ik}S_{lm}E_{lj}+E_{im}S_{mk}\delta_{jl}+\delta_{il}S_{km}E_{mj}+E_{im}S_{ml}\delta_{jk})$.

3.2.2 An implicit relation in terms of the Lagrange Saint-Venant strain tensor and the 2nd Piola-Kirchhoff stress tensor, and the use of spectral invariants.

In [198] an explicit expression for Π in (16) has been proposed, but using the spectral invariants formulated by Shariff [191, 195]. Let s_i, $\mathbf{a}^{(i)}$, $i = 1, 2, 3$ denote the eigenvalues and eigenvectors of the second Piola-Kirchhoff stress tensor $\mathbf{S}$, then we have the spectral representation $\mathbf{S} = \sum_{i=1}^{3} s_i \mathbf{a}^{(i)} \otimes \mathbf{a}^{(i)}$. If ϑ_i, $i = 1, 2, 3$ represent the eigenvalues of the Lagrange-Saint Venant strain tensor, then $\mathbf{E} = \sum_{i=1}^{3} \vartheta_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)} = \sum_{i=1}^{3} \frac{1}{2}(\lambda_i^2 - 1)\mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}$, where λ_i are the principal stretches of $\mathbf{U}$, and $\mathbf{u}^{(i)}$ are the eigenvectors of $\mathbf{U}$, where $\mathbf{C} = \mathbf{U}^2$.

For $\Pi = \Pi(\mathbf{E}, \mathbf{S})$ in (16) we consider the case Π depends on two unit vector fields $\mathbf{d}_0$ and $\mathbf{f}_0$, then:

$$\Pi = \Pi(\mathbf{E}, \mathbf{S}, \mathbf{d}_0, \mathbf{f}_0) = \Pi(s_{1,2,3}, \vartheta_{1,2,3}, \varrho_{1,2,3}^{(k)}, \gamma_{1,2,3}, \zeta_{1,2,3}), \tag{17}$$

where $s_{1,2,3}$ represents the set $\{s_1, s_2, s_3\}$ and the same notation is valid for the other variables in (17). In (17) we have defined $\gamma_i = (\mathbf{a}_i \cdot \mathbf{d}_0)^2$, $\zeta_i = (\mathbf{a}_i \cdot \mathbf{f}_0)^2$ and $\varrho_i^{(k)} = \mathbf{a}^{(i)} \cdot \mathbf{u}^{(k)}$. In (17) not all the invariants are independent, and it is shown in [198] that only 13 of them are independent.

In [198] the following generalized strain and stress functions are proposed: $I_\alpha = \sum_{i=1}^{3} \xi_i^{(\alpha)} f_\alpha(s_i)$, where $\xi_i = \mathbf{a}^{(i)} \cdot (\mathbf{T}_{(\alpha)} \mathbf{a}^{(i)})$, where $\mathbf{T}_{(\alpha)}$ are given in terms of $\mathbf{u}^{(i)}$, $\mathbf{d}_0$, $\mathbf{f}_0$ and $\mathbf{E}_{(\alpha)} = \sum_{i=1}^{3} g_\beta(\vartheta_i)\mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}$.

In [198] using the above invariants the fourth order tensors $\mathscr{A}$, $\mathscr{B}$ (which have been defined in the previous section, see paragraph after Eq. (16)) are calculated. For the sake of brevity such explicit expressions are not shown here.

3.2.3 Some other implicit relations in terms of the Cauchy stress tensor and the left Cauchy-Green tensor

In this section two classes of implicit constitutive relations (16), where $\mathfrak{G}$ is anisotropic, namely, where $\mathfrak{G}$ can depend on one unit vector $\mathbf{a}$, and two unit vector fields $\mathbf{a}$, $\mathbf{b}$.

On a class of transversely isotropic solid: In [175] the following implicit relation is considered:

$$\mathfrak{G}(\mathbf{T}, \mathbf{B}, \mathbf{a}) = \mathbf{0}, \tag{18}$$

where $\mathbf{a} = \mathbf{F}\mathbf{a}_0$ and $\mathbf{a}_0$ is a unit vector field. From [202] the expression for (18) is:

$$\begin{aligned}\alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{T}^2 + \alpha_3\mathbf{B} + \alpha_4\mathbf{B}^2 + \alpha_5(\mathbf{BT} + \mathbf{TB}) + \alpha_6(\mathbf{B}^2\mathbf{T} + \mathbf{TB}^2)\\ +\alpha_7(\mathbf{BT}^2 + \mathbf{T}^2\mathbf{B}) + +\alpha_8(\mathbf{B}^2\mathbf{T}^2 + \mathbf{T}^2\mathbf{B}^2) + \alpha_9\mathbf{a}\otimes\mathbf{a}\\ +\alpha_{10}[\mathbf{a}\otimes(\mathbf{Ta}) + (\mathbf{Ta})\otimes\mathbf{a}] + \alpha_{11}[\mathbf{a}\otimes(\mathbf{T}^2\mathbf{a}) + (\mathbf{T}^2\mathbf{a})\otimes\mathbf{a}]\\ +\alpha_{12}[\mathbf{a}\otimes(\mathbf{Ba}) + (\mathbf{Ba})\otimes\mathbf{a}] + \alpha_{13}[\mathbf{a}\otimes(\mathbf{B}^2\mathbf{a}) + (\mathbf{B}^2\mathbf{a})\otimes\mathbf{a}] = \mathbf{0}, \quad (19)\end{aligned}$$

where α_i, $i = 0, 1, 2, ..., 16$ depend on 18 invariants (see [202]) I_j, $j = 1, 2, ..., 18$, which for the sake of brevity are shown, as special cases, in the next section for the problem of two directions of anisotropy.

On a class of two directions implicit constitutive relation: Here we show the full expression for the implicit constitutive relation

$$\mathfrak{G}(\mathbf{T}, \mathbf{B}, \mathbf{a}, \mathbf{b}) = \mathbf{0}, \quad (20)$$

which is a generalization of (18), (19), where $\mathbf{b} = \mathbf{F}\mathbf{b}_0$ and $\mathbf{b}_0$ is unit vector field. From [202] for (20) we obtain:

$$\begin{aligned}\alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{T}^2 + \alpha_3\mathbf{B} + \alpha_4\mathbf{B}^2 + \alpha_5(\mathbf{BT} + \mathbf{TB}) + \alpha_6(\mathbf{B}^2\mathbf{T} + \mathbf{TB}^2)\\ +\alpha_7(\mathbf{BT}^2 + \mathbf{T}^2\mathbf{B}) + +\alpha_8(\mathbf{B}^2\mathbf{T}^2 + \mathbf{T}^2\mathbf{B}^2) + \alpha_9\mathbf{a}\otimes\mathbf{a}\\ +\alpha_{10}[\mathbf{a}\otimes(\mathbf{Ta}) + (\mathbf{Ta})\otimes\mathbf{a}] + \alpha_{11}[\mathbf{a}\otimes(\mathbf{T}^2\mathbf{a}) + (\mathbf{T}^2\mathbf{a})\otimes\mathbf{a}]\\ +\alpha_{12}[\mathbf{a}\otimes(\mathbf{Ba}) + (\mathbf{Ba})\otimes\mathbf{a}] + \alpha_{13}[\mathbf{a}\otimes(\mathbf{B}^2\mathbf{a}) + (\mathbf{B}^2\mathbf{a})\otimes\mathbf{a}] + \alpha_{14}\mathbf{b}\otimes\mathbf{b}\\ +\alpha_{15}[\mathbf{b}\otimes(\mathbf{Tb}) + (\mathbf{Tb})\otimes\mathbf{b}] + \alpha_{16}[\mathbf{b}\otimes(\mathbf{T}^2\mathbf{b}) + (\mathbf{T}^2\mathbf{b})\otimes\mathbf{b}]\\ +\alpha_{17}[\mathbf{b}\otimes(\mathbf{Bb}) + (\mathbf{Bb})\otimes\mathbf{b}] + \alpha_{18}[\mathbf{b}\otimes(\mathbf{B}^2\mathbf{b}) + (\mathbf{B}^2\mathbf{b})\otimes\mathbf{b}]\\ +\alpha_{19}(\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a}) + \alpha_{20}(\mathbf{a}\cdot\mathbf{b})[\mathbf{a}\otimes(\mathbf{Tb}) + (\mathbf{Tb})\otimes\mathbf{a} + \mathbf{b}\otimes(\mathbf{Ta})\\ +(\mathbf{Ta})\otimes\mathbf{b}] + \alpha_{21}(\mathbf{a}\cdot\mathbf{b})[\mathbf{a}\otimes(\mathbf{Bb}) + (\mathbf{Bb})\otimes\mathbf{a} + \mathbf{b}\otimes(\mathbf{Ba}) + (\mathbf{Ba})\otimes\mathbf{b}] = \mathbf{0},\end{aligned}$$

where the functions α_i, $i = 0, 1, 2, ..., 21$ depend on the invariants (here there are some differences with the results shown in [175]): $I_1 = \operatorname{tr}\mathbf{T}$, $I_2 = \operatorname{tr}(\mathbf{T}^2)$, $I_3 = \operatorname{tr}(\mathbf{T}^3)$, $I_4 = \operatorname{tr}\mathbf{B}$, $I_5 = \operatorname{tr}(\mathbf{B}^2)$, $I_6 = \operatorname{tr}(\mathbf{B}^3)$, $I_7 = \operatorname{tr}(\mathbf{TB})$, $I_8 = \operatorname{tr}(\mathbf{T}^2\mathbf{B})$, $I_9 = \operatorname{tr}(\mathbf{TB}^2)$, $I_{10} = \operatorname{tr}(\mathbf{T}^2\mathbf{B}^2)$, $I_{11} = \mathbf{a}\cdot(\mathbf{Ta})$, $I_{12} = \mathbf{a}\cdot(\mathbf{T}^2\mathbf{a})$, $I_{13} = \mathbf{a}\cdot(\mathbf{Ba})$, $I_{14} = \mathbf{a}\cdot(\mathbf{B}^2\mathbf{a})$, $I_{15} = \mathbf{a}\cdot[(\mathbf{TB} + \mathbf{BT})\mathbf{a}]$, $I_{16} = \mathbf{a}\cdot[(\mathbf{T}^2\mathbf{B} + \mathbf{BT}^2)\mathbf{a}]$, $I_{17} = \mathbf{a}\cdot[(\mathbf{TB}^2 + \mathbf{B}^2\mathbf{T})\mathbf{a}]$, $I_{18} = \mathbf{a}\cdot[(\mathbf{T}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{T}^2)\mathbf{a}]$, $I_{19} = \mathbf{b}\cdot(\mathbf{Tb})$, $I_{20} = \mathbf{b}\cdot(\mathbf{T}^2\mathbf{b})$, $I_{21} = \mathbf{b}\cdot(\mathbf{Bb})$, $I_{22} = \mathbf{b}\cdot(\mathbf{B}^2\mathbf{b})$, $I_{23} = \mathbf{b}\cdot[(\mathbf{TB} + \mathbf{BT})\mathbf{b}]$, $I_{24} = \mathbf{b}\cdot[(\mathbf{T}^2\mathbf{B} + \mathbf{BT}^2)\mathbf{b}]$, $I_{25} = \mathbf{b}\cdot[(\mathbf{TB}^2 + \mathbf{B}^2\mathbf{T})\mathbf{b}]$, $I_{26} = \mathbf{b}\cdot[(\mathbf{T}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{T}^2)\mathbf{b}]$, $I_{27} = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}\cdot(\mathbf{Tb})$, $I_{28} = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}\cdot(\mathbf{T}^2\mathbf{b})$, $I_{29} = \mathbf{a}\cdot(\mathbf{Bb})$, $I_{30} = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}\cdot(\mathbf{B}^2\mathbf{b})$, $I_{31} = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}\cdot[(\mathbf{TB} + \mathbf{BT})\mathbf{b}]$, $I_{32} = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}\cdot[(\mathbf{T}^2\mathbf{B} + \mathbf{BT}^2)\mathbf{b}]$, $I_{33} = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}\cdot[(\mathbf{TB}^2 + \mathbf{B}^2\mathbf{T})\mathbf{b}]$, $I_{34} = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}\cdot[(\mathbf{T}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{T}^2)\mathbf{b}]$.

3.3 Some other classes of nonlinear elastic bodies, wherein some measure for the strains are functions of some measures for the stresses

In the previous section we showed some expressions for the implicit constitutive relations, considering either $\mathbf{S}$, $\mathbf{E}$, or $\mathbf{T}$, $\mathbf{B}$ as the main variables. In this section we are interested in presenting some constitutive equations, wherein some measure of the strain is a function of some measure of the stress, using the first law of thermodynamics and the Gibss potential. In such a way we obtain expressions that are similar to (12).

3.3.1 Constitutive equation for isotropic bodies, wherein the Hencky strain tensor is a function of the Kirchhoff stress tensor

A model similar to (12) has been proposed in [203], where the basic variables are the Cauchy stress tensor, the tensor $\mathbf{V}$ from the polar decomposition $\mathbf{F} = \mathbf{V}\mathbf{R}$, where $\mathbf{V}^2 = \mathbf{B}$, and the complementary or Gibbs energy function $\mathcal{G}$. In [36, 43, 45, 147, 203] the following constitutive equation based on the complementary function $\mathcal{G}$ is proposed[5]

$$\boldsymbol{\eta} = \frac{\partial \mathcal{G}}{\partial \boldsymbol{\tau}}, \quad \boldsymbol{\eta} = \ln \mathbf{V} = \frac{1}{2} \ln \mathbf{B}, \tag{21}$$

where $\boldsymbol{\tau} = J\mathbf{T}$ is the Kirchhoff stress tensor and $\boldsymbol{\eta}$ is the Hencky strain tensor. Eq. (21) is actually an implicit relation, since $\mathcal{G}$ depends on the Cauchy stress and J. The above relation is only valid for isotropic bodies (see, for example, [79] and Eq. 18 in [182]). For such isotropic bodies $\mathcal{G}$ depends on three invariants. On possible set is (see [203]) $\{a_0, a_1, a_3\}$, where $a_0 = \frac{\mathrm{tr}\,(\boldsymbol{\tau})}{3}$, $a_1 = \sqrt{\mathrm{tr}\,(\boldsymbol{\tau}_{\mathrm{D}}\boldsymbol{\tau}_{\mathrm{D}}^{\mathrm{T}})}$, where $\boldsymbol{\tau}_{\mathrm{D}} = \boldsymbol{\tau} - a_0\mathbf{I}$, and $a_2 = \det \mathbf{N}$, where $\mathbf{N} = \frac{1}{a_1}\boldsymbol{\tau}_{\mathrm{D}}$. In this case from (21) we have the representation

$$\mathrm{tr}\,\boldsymbol{\eta} = \ln J = \frac{\partial \mathcal{G}}{\partial a_0}, \quad \mathrm{dev}\,(\boldsymbol{\eta}) = \frac{\partial \mathcal{G}}{\partial \boldsymbol{\tau}_{\mathrm{D}}}, \tag{22}$$

where $\mathrm{dev}\,(\boldsymbol{\eta})$ is the deviatoric part of the tensor $\boldsymbol{\eta}$. In the case of incompressible bodies, it has been shown that $\mathcal{G}$ would only depend on $\mathbf{T}$ since $J = 1$ always.

[5]In an earlier work [64], starting from an implicit relation of the form (9), in Eq. (3.1) therein Criscione and Rajagopal already studied the use of the Hencky strain tensor and the Kirchhoff stress tensor, for the modelling of rubber, using a constitutive equation very similar to (23) shown in this section.

In [36, 43] a different set of invariants was proposed for $\mathcal{G}$, namely $\mathcal{G} = \mathcal{G}(I_1, I_2, I_3)$, where $I_1 = \mathrm{tr}\,\boldsymbol{\tau}$, $I_2 = \frac{1}{2}\mathrm{tr}\,(\boldsymbol{\tau}^2)$ and $I_3 = \frac{1}{3}\mathrm{tr}\,(\boldsymbol{\tau}^3)$ and from (21) we get

$$\boldsymbol{\eta} = \mathcal{G}_1\mathbf{I} + \mathcal{G}_2\boldsymbol{\tau} + \mathcal{G}_3\boldsymbol{\tau}^2, \tag{23}$$

where $\mathcal{G}_i = \frac{\partial \mathcal{G}}{\partial I_i}$, $i = 1, 2, 3$.

Finally, in [45] the authors considered an expression for $\mathcal{G}$ in terms of the principal values of $\boldsymbol{\tau}$, which are denoted τ_i, i.e., $\mathcal{G} = \mathcal{G}(\tau_1, \tau_2, \tau_3)$, where $\mathcal{G}(\tau_1, \tau_2, \tau_3) = \mathcal{G}(\tau_2, \tau_1, \tau_3) = \mathcal{G}(\tau_1, \tau_3, \tau_2) = \mathcal{G}(\tau_3, \tau_2, \tau_1)$. In such a case if λ_i are the principal stretches, from (21) it is shown that

$$\ln \lambda_i = \frac{\partial \mathcal{G}}{\partial \tau_i}, \quad i = 1, 2, 3. \tag{24}$$

For incompressible isotropic solid in [8] Eqs. (21), (23) and (24) have been used for the fitting for rubber using the experimental data from Treloar [206], see Section 7.2.2 and 7.2.3.

3.3.2 A constitutive equation for the Hencky strain tensor as a function of the Cauchy stress tensor by Prusa et al.

In [147] a constitutive equation for the Hencky strain tensor as a function of the Cauchy stress tensor has been proposed, using the Gibbs potential mentioned in the previous section. Using the definition $\boldsymbol{\eta} = \ln \mathbf{V} = \frac{1}{2} \ln \mathbf{B}$ in [147] it is shown that $\mathbf{T} \cdot \mathbf{D} = \mathbf{T} \cdot \dot{\boldsymbol{\eta}}$. On the other hand defining $\bar{\mathbf{B}} = \mathbf{B}/(J^{2/3})$ and $\bar{\boldsymbol{\eta}} = \frac{1}{2} \ln \bar{\mathbf{B}}$, then if $\boldsymbol{\eta}_{\mathrm{D}} = \boldsymbol{\eta} - \frac{1}{3}(\mathrm{tr}\,\boldsymbol{\eta})\mathbf{I}$, it is possible to prove that $\boldsymbol{\eta}_{\mathrm{D}} = \bar{\boldsymbol{\eta}}$. The first law of thermodynamics (without internal sources of heat) is

$$\rho \dot{U} = \mathbf{T} \cdot \mathbf{D} - \mathrm{div}\,\mathbf{Q}, \tag{25}$$

where $\mathbf{Q}$ is the heat flux in the reference configuration. If for the specific internal energy we assume $U = U(s, J, \bar{\boldsymbol{\eta}})$ (where s is the entropy) we have for the absolute temperature $\theta = \frac{\partial U}{\partial s}$. On the other hand $\mathbf{T} \cdot \mathbf{D} = \mathbf{T}_{\mathrm{D}} \cdot \dot{\bar{\boldsymbol{\eta}}} + m\, \dot{\overline{\mathrm{tr}\,\boldsymbol{\eta}}}$, where $m = \frac{1}{3}\mathrm{tr}\,\mathbf{T}$, $\mathbf{T}_{\mathrm{D}} = \mathbf{T} - m\mathbf{I}$ and $\dot{\overline{\mathrm{tr}\,\boldsymbol{\eta}}} = \dot{J}/J$. Using the above results and relations in (25) we obtain

$$\rho \dot{s} + \mathrm{div}\left(\frac{1}{\theta}\mathbf{Q}\right) = \frac{1}{\theta}\left\{\frac{1}{J}(m + p_{\mathrm{th}})\dot{J} + [\mathbf{T}_{\mathrm{D}} - (\mathbf{T}_{\mathrm{th}})_{\mathrm{D}}] \cdot \dot{\bar{\boldsymbol{\eta}}}\right\} - \mathbf{Q} \cdot \frac{1}{\theta^2}\nabla\theta, \tag{26}$$

where for a second order tensor $\mathbf{A}$ we define $\mathbf{A}_{\mathrm{D}} = \mathbf{A} - \frac{1}{2}(\mathrm{tr}\,\mathbf{A})\mathbf{I}$. From (26) Prusa et al. have defined $p_{\mathrm{th}} = -\rho J \frac{\partial U}{\partial J}$ and $\frac{(\mathbf{T}_{\mathrm{th}})_{\mathrm{D}}}{\rho} = \left(\frac{\partial U}{\partial \bar{\boldsymbol{\eta}}}\right)_{\mathrm{D}}$. In (26) the authors assume that the solid does not produce any entropy in purely mechanical processes, then

$$m = -p_{\rm th}, \quad \mathbf{T}_{\rm D} = (\mathbf{T}_{\rm th})_{\rm D}.$$

In [147] the following Gibbs potential is introduced

$$\mathcal{G}(\theta, p_{\rm th}, (\mathbf{T}_{{\rm th},\rho})_{\rm D}) = U(s, J, \bar{\boldsymbol{\eta}}) - \theta s + \frac{p_{\rm th}}{\rho} - (\mathbf{T}_{{\rm th},\rho})_{\rm D} \cdot \bar{\boldsymbol{\eta}},$$

where $(\mathbf{T}_{{\rm th},\rho})_{\rm D} = \frac{(\mathbf{T}_{\rm th})_{\rm D}}{\rho}$, then from (26) we obtain

$$\frac{\partial \mathcal{G}}{\partial \theta} = -s, \quad \frac{\partial \mathcal{G}}{\partial p_{\rm th}} = \frac{J}{\rho_{\rm r}} \left(\frac{\partial \mathcal{G}}{\partial (\mathbf{T}_{{\rm th},\rho})_{\rm D}} \right)_{\rm D} = -\bar{\boldsymbol{\eta}}. \tag{27}$$

All these results are valid for $\mathcal{G}$ isotropic, and in [147] the authors assume $\mathcal{G} = \mathcal{G}(\theta, p_{\rm th}, K_2, K_3)$, where $K_2 = \frac{1}{2}\{[{\rm tr}((\mathbf{T}_{{\rm th},\rho})_{\rm D})]^2 - {\rm tr}[(\mathbf{T}_{{\rm th},\rho})^2_{\rm D}]\}$ and $K_3 = \det(\mathbf{T}_{{\rm th},\rho})_{\rm D}$, and from (27) they get

$$\boldsymbol{\eta} = \bar{\boldsymbol{\eta}} + \frac{1}{2} \ln \left(\rho_{\rm r} \frac{\partial \mathcal{G}}{\partial p_{\rm th}} \right) \mathbf{I}, \tag{28}$$

where

$$\bar{\boldsymbol{\eta}} = -\frac{2}{3} K_2 \frac{\partial \mathcal{G}}{\partial K_3} \mathbf{I} + \frac{\partial \mathcal{G}}{\partial K_2} (\mathbf{T}_{{\rm th},\rho})_{\rm D} - \frac{\partial \mathcal{G}}{\partial K_3} [(\mathbf{T}_{{\rm th},\rho})_{\rm D}]^2.$$

As a result of the above relation we have $\frac{\partial \mathcal{G}}{\partial p_{\rm th}} = \frac{1}{\rho}$, then $\mathcal{G}$ should be such that the above derivative is always positive since $\rho > 0$, which is a restriction on $\mathcal{G}$.

Several examples are presented in [147] for $\mathcal{G}$, one of them is $\mathcal{G}(\theta, p_{\rm th}, K_2) = -c_{v_{\rm r}} \theta \left[\ln\left(\frac{\theta}{\theta_{\rm r}}\right) - 1 \right] - \frac{k_{\rm r}}{\rho_{\rm r}} e^{-\frac{p_{\rm th}}{k_{\rm r}}} + \frac{\rho_{\rm r}}{2\mu_{\rm r}} K_2$, where $c_{v_{\rm r}}$ is the specific heat at constant volume in the reference configuration, $\theta_{\rm r}$ is the reference temperature, $k_{\rm r}$ and $\mu_{\rm r}$ are the bulk and shear moduli in the reference configuration. From (28) and the above particular expression for the Gibbs potential we obtain $\boldsymbol{\eta} = -\frac{p_{\rm th}}{3k_{\rm r}} \mathbf{I} + \frac{1}{2\mu_{\rm r}} e^{-\frac{p_{\rm th}}{k_{\rm r}}} (\mathbf{T}_{\rm th})_{\rm D}$. In [147] other examples for $\mathcal{G}$ are proposed, wherein the bulk modulus, the shear modulus, the Young's modulus, and the Poisson ration can depends on the density of the body (see, Section 5.1.1 for some constitutive relations for bodies, whose mechanical properties can depend on mass density, for the special case the gradient of the displacement field is small).

3.3.3 A constitutive equation for the Lagrange Saint-Venant strain tensor as a function of the second Piola-Kirchhoff stress tensor

The Gibbs potential introduced in the previous section can also be expressed as a function of the second Piola-Kirchhoff stress tensor, then from the first law of thermodynamics (isothermal processes) for such a class of elastic bodies we have $\mathcal{G} = \mathcal{G}(\mathbf{S})$, as a result (see [47]):

$$\mathbf{E} = \frac{\partial \mathcal{G}}{\partial \mathbf{S}}. \tag{29}$$

Unlike (21) the constitutive relations studied in Sections 3.3.1, 3.3.2, here (29) is valid for both isotropic and anisotropic bodies. Since only the Cauchy stress has a physical meaning, and considering (3), we see that (29) is actually an implicit constitutive relation if we were going to use the Cauchy stress tensor as the main variable through (3).

Let us show an explicit expression for (29) in the case $\mathcal{G}$ is a transversely isotropic function, i.e., $\mathcal{G} = \mathcal{G}(\mathbf{S}, \mathbf{a}_0)$, where $\mathbf{a}_0$ is a unit vector field. In the case we use the classical invariants by Spencer [202], we have $\mathcal{G} = \mathcal{G}(I_1, I_2, I_3, I_4, I_5)$, where, for example, $I_1 = \mathrm{tr}\,\mathbf{S}$, $I_2 = \frac{1}{2}\mathrm{tr}\,(\mathbf{S}^2)$, $I_3 = \frac{1}{3}\mathrm{tr}\,(\mathbf{S}^3)$, $I_4 = \mathbf{a}_0 \cdot (\mathbf{S}\mathbf{a}_0)$ and $I_5 = \mathbf{a}_0 \cdot (\mathbf{S}^2\mathbf{a}_0)$, then from (29) we get:

$$\mathbf{E} = \mathcal{G}_1\mathbf{I} + \mathcal{G}_2\mathbf{S} + \mathcal{G}_3\mathbf{S}^2 + \mathcal{G}_4\mathbf{a}_0 \otimes \mathbf{a}_0 + \mathcal{G}_5[\mathbf{a}_0 \otimes (\mathbf{S}\mathbf{a}_0) + (\mathbf{S}\mathbf{a}_0) \otimes \mathbf{a}_0], \tag{30}$$

where $\mathcal{G}_i = \frac{\partial \mathcal{G}}{\partial I_i}$, $i = 1, 2, 3, 4, 5$. The case of isotropic $\mathcal{G}$ is a special subclass of the above, where $\mathcal{G} = \mathcal{G}(I_1, I_2, I_3)$.

For the above we can also use the spectral invariants [191, 192, 194, 195], but for the sake of brevity such equivalent expressions for $\mathcal{G}$ are not shown here.

4 Electro- and magneto-elastic bodies, thermoelastic bodies, and residually stressed bodies described by implicit constitutive relations

The implicit constitutive relations and their subclasses shown in the previous section, have been extended to other problems, where we have electroand magneto-sensitive solids, thermoelastic solids, and in the modelling of residually stressed bodies. In this section we show some of such constitutive relations and equations proposed in the literature. The meaning of the different variables mentioned here can be found in the Appendix.

4.1 Constitutive equations and relations for a first class of electro- and magneto-elastic bodies. Cauchy-like bodies

The classical approach used to model the behaviour of electro- and magneto-elastic bodies has been to assume that the stresses and one of the electric or magnetic variables, are expressed as functions of the strain (or deformation gradient) and the independent electric or magnetic variable. For example, the classical linearized electro-elastic model is based on considering $\mathbb{P}$ (the electric polarization) as the electric independent variable, and the constitutive equations are of the form (in indicial notation and Cartesian coordinates) $\tau_{ij} = \mathscr{C}_{ijkl}\varepsilon_{kl} + \mathscr{K}_{ijk}\mathbb{P}_k$, $\mathbb{E}_i = \mathscr{K}_{ijk}\varepsilon_{jk} + \mathscr{D}_{ij}\mathbb{P}_j$, where the electric displacement $\mathbb{D}$ would be found from $\mathbb{P} = \mathbb{D} - \epsilon_o\mathbb{E}$ and $\mathbb{E}$ is the electric field. In the above constitutive equation $\boldsymbol{\tau}$ is the total stress tensor that is composed of the Cauchy stress tensor plus some electric contributions, and it should not be confused with the Kirchhoff stress tensor used in the previous section. In electro- and magneto-elasticity in this review the Cauchy stress tensor is denoted $\boldsymbol{\sigma}$. In the nonlinear case, for electro-elastic bodies the usual constitutive assumptions are of the form[6] $\boldsymbol{\tau} = \mathfrak{f}(\mathbf{F},\mathbb{P})$, $\mathbb{E} = \mathfrak{q}(\mathbf{F},\mathbb{P})$, where $\mathfrak{q}$ is a vector function. For nonlinear magneto-elastic bodies, assuming the magnetization as the independent magnetic variable, we have[7] $\boldsymbol{\tau} = \mathfrak{f}(\mathbf{F},\mathbb{M})$, $\mathbb{H} = \mathfrak{q}(\mathbf{F},\mathbb{M})$. For example, for nonlinear electro-elastic bodies in [69] it has been proposed the constitutive equations $\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}}$, $\mathbb{D}_{\mathrm{L}} = -\frac{\partial\Omega}{\partial\mathbb{E}_{\mathrm{L}}}$, where $\Omega = \Omega(\mathbf{F},\mathbb{E}_{\mathrm{L}})$ is called the total energy function, the electric field has been chosen as the independent electric variable, and $\mathbb{D}_{\mathrm{L}} = J\mathbf{F}^{-1}\mathbb{D}$, $\mathbb{E}_{\mathrm{L}} = \mathbf{F}^{\mathrm{T}}\mathbb{E}$.

In view of what has been presented in Section 3.2, a natural extension of the above constitutive equations is to assume implicit relations that for

[6] See, for example, Section 3(a) in [21] for a brief survey of the constitutive equations for electroelastic bodies.

[7] See Section 3 of [24] for a short review of the constitutive equations in magnetoelasticity.

electro-elastic bodies would be of the form (see [21])

$$\mathfrak{G}(\boldsymbol{\tau}, \mathbf{B}, \mathbb{E}, \mathbb{D}) = \mathbf{0}, \quad \mathfrak{l}(\boldsymbol{\tau}, \mathbf{B}, \mathbb{E}, \mathbb{D}) = \mathbf{0}, \tag{31}$$

where $\mathfrak{G}$ is a tensorial implicit relation, and $\mathfrak{l}$ would be a vector implicit relation. They are generalizations of the functions $\mathfrak{f}$, $\mathfrak{q}$ mentioned previously. In (31) we have chosen $\mathbb{E}$ and $\mathbb{D}$ as the electric variables for the problem and from $\mathbb{P} = \mathbb{D} - \epsilon_o \mathbb{E}$ we can obtain $\mathbb{P}$. In the case of magneto-elastic bodies we would have $\mathfrak{G}(\boldsymbol{\tau}, \mathbf{B}, \mathbb{H}, \mathbb{B}) = \mathbf{0}$, $\mathfrak{l}(\boldsymbol{\tau}, \mathbf{B}, \mathbb{H}, \mathbb{B}) = \mathbf{0}$, where $\mathbb{M}$ can be found from $\mathbb{B} = \mu_o(\mathbb{H} + \mathbb{M})$.

For electro-elastic bodies, in the case $\mathfrak{G}$ and $\mathfrak{l}$ are isotropic relations (31) becomes, respectively (see, for example, [202, 225]):

$$\begin{aligned}
&\gamma_0 \mathbf{I} + \gamma_1 \boldsymbol{\tau} + \gamma_2 \mathbf{B} + \gamma_3 \boldsymbol{\tau}^2 + \gamma_4 \mathbf{B}^2 + \gamma_5 (\boldsymbol{\tau}\mathbf{B} + \mathbf{B}\boldsymbol{\tau}) + \gamma_6 (\boldsymbol{\tau}^2\mathbf{B} + \mathbf{B}\boldsymbol{\tau}^2) + \gamma_7 (\boldsymbol{\tau}\mathbf{B}^2 + \mathbf{B}^2\boldsymbol{\tau}) \\
&+\gamma_8 (\boldsymbol{\tau}^2\mathbf{B}^2 + \mathbf{B}^2\boldsymbol{\tau}^2) + \gamma_9 \mathbb{E} \otimes \mathbb{E} + \gamma_{10} [\mathbb{E} \otimes (\boldsymbol{\tau}\mathbb{E}) + (\boldsymbol{\tau}\mathbb{E}) \otimes \mathbb{E}] + \gamma_{11} [\mathbb{E} \otimes (\mathbf{B}\mathbb{E}) + (\mathbf{B}\mathbb{E}) \otimes \mathbb{E}] \\
&+\gamma_{12} [\mathbb{E} \otimes (\boldsymbol{\tau}^2\mathbb{E}) + (\boldsymbol{\tau}^2\mathbb{E}) \otimes \mathbb{E}] + \gamma_{13} [\mathbb{E} \otimes (\mathbf{B}^2\mathbb{E}) + (\mathbf{B}^2\mathbb{E}) \otimes \mathbb{E}] + \gamma_{14} [\mathbb{E} \otimes (\boldsymbol{\tau}\mathbf{B}\mathbb{E}) \\
&+(\boldsymbol{\tau}\mathbf{B}\mathbb{E}) \otimes \mathbb{E}] + \gamma_{15} [\mathbb{E} \otimes (\mathbf{B}\boldsymbol{\tau}\mathbb{E}) + (\mathbf{B}\boldsymbol{\tau}\mathbb{E}) \otimes \mathbb{E}] + \gamma_{16} [(\boldsymbol{\tau}\mathbb{E}) \otimes (\mathbf{B}\mathbb{E}) + (\mathbf{B}\mathbb{E}) \otimes (\boldsymbol{\tau}\mathbb{E})] \\
&+\gamma_{17} \mathbb{D} \otimes \mathbb{D} + \gamma_{18} [\mathbb{D} \otimes (\boldsymbol{\tau}\mathbb{D}) + (\boldsymbol{\tau}\mathbb{D}) \otimes \mathbb{D}] + \gamma_{19} [\mathbb{D} \otimes (\mathbf{B}\mathbb{D}) + (\mathbf{B}\mathbb{D}) \otimes \mathbb{D}] \\
&+\gamma_{20} [\mathbb{D} \otimes (\boldsymbol{\tau}^2\mathbb{D}) + (\boldsymbol{\tau}^2\mathbb{D}) \otimes \mathbb{D}] + \gamma_{21} [\mathbb{D} \otimes (\mathbf{B}^2\mathbb{D}) + (\mathbf{B}^2\mathbb{D}) \otimes \mathbb{D}] + \gamma_{22} [\mathbb{D} \otimes (\boldsymbol{\tau}\mathbf{B}\mathbb{D}) \\
&+(\boldsymbol{\tau}\mathbf{B}\mathbb{D}) \otimes \mathbb{D}] + \gamma_{23} [\mathbb{D} \otimes (\mathbf{B}\boldsymbol{\tau}\mathbb{D}) + (\mathbf{B}\boldsymbol{\tau}\mathbb{D}) \otimes \mathbb{D}] + \gamma_{24} [(\boldsymbol{\tau}\mathbb{D}) \otimes (\mathbf{B}\mathbb{D}) + (\mathbf{B}\mathbb{D}) \otimes (\boldsymbol{\tau}\mathbb{D})] \\
&\gamma_{25} (\mathbb{E} \cdot \mathbb{D})(\mathbb{D} \otimes \mathbb{E} + \mathbb{E} \otimes \mathbb{D}) + \gamma_{26} (\mathbb{E} \cdot \mathbb{D})[\mathbb{D} \otimes (\boldsymbol{\tau}\mathbb{E}) + (\boldsymbol{\tau}\mathbb{E}) \otimes \mathbb{D} + (\boldsymbol{\tau}\mathbb{D}) \otimes \mathbb{E} + \mathbb{E} \otimes (\boldsymbol{\tau}\mathbb{D})] \\
&+\gamma_{27} (\mathbb{E} \cdot \mathbb{D})[\mathbb{D} \otimes (\mathbf{B}\mathbb{E}) + (\mathbf{B}\mathbb{E}) \otimes \mathbb{D} + (\mathbf{B}\mathbb{D}) \otimes \mathbb{E} + \mathbb{E} \otimes (\mathbf{B}\mathbb{D})] + \gamma_{28} (\mathbb{E} \cdot \mathbb{D})[\mathbb{D} \otimes (\boldsymbol{\tau}^2\mathbb{E}) \\
&+(\boldsymbol{\tau}^2\mathbb{E}) \otimes \mathbb{D} + (\boldsymbol{\tau}^2\mathbb{D}) \otimes \mathbb{E} + \mathbb{E} \otimes (\boldsymbol{\tau}^2\mathbb{D})] + \gamma_{29} (\mathbb{E} \cdot \mathbb{D})[\mathbb{D} \otimes (\mathbf{B}^2\mathbb{E}) + (\mathbf{B}^2\mathbb{E}) \otimes \mathbb{D} \\
&+(\mathbf{B}^2\mathbb{D}) \otimes \mathbb{E} + \mathbb{E} \otimes (\mathbf{B}^2\mathbb{D})] + \gamma_{30} (\mathbb{E} \cdot \mathbb{D})[\mathbb{E} \otimes (\boldsymbol{\tau}\mathbf{B}\mathbb{D}) + (\boldsymbol{\tau}\mathbf{B}\mathbb{D}) \otimes \mathbb{E} + \mathbb{D} \otimes (\boldsymbol{\tau}\mathbf{B}\mathbb{E}) \\
&+(\boldsymbol{\tau}\mathbf{B}\mathbb{E}) \otimes \mathbb{D}] + \gamma_{31} (\mathbb{E} \cdot \mathbb{D})[\mathbb{E} \otimes (\mathbf{B}\boldsymbol{\tau}\mathbb{D}) + (\mathbf{B}\boldsymbol{\tau}\mathbb{D}) \otimes \mathbb{E} + \mathbb{D} \otimes (\mathbf{B}\boldsymbol{\tau}\mathbb{E}) + (\mathbf{B}\boldsymbol{\tau}\mathbb{E}) \otimes \mathbb{D}] \\
&+\gamma_{32} (\mathbb{E} \cdot \mathbb{D})[(\boldsymbol{\tau}\mathbb{E}) \otimes (\mathbf{B}\mathbb{D}) + (\mathbf{B}\mathbb{D}) \otimes (\boldsymbol{\tau}\mathbb{E}) + (\boldsymbol{\tau}\mathbb{D}) \otimes (\mathbf{B}\mathbb{E}) + (\mathbf{B}\mathbb{E}) \otimes (\boldsymbol{\tau}\mathbb{D})] = \mathbf{0},
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
&\psi_1 \mathbb{E} + \psi_2 \boldsymbol{\tau}\mathbb{E} + \psi_3 \boldsymbol{\tau}^2\mathbb{E} + \psi_4 \mathbf{B}\mathbb{E} + \psi_5 \mathbf{B}^2\mathbb{E} + \psi_6 \mathbb{D} + \psi_7 \boldsymbol{\tau}\mathbb{D} + \psi_8 \boldsymbol{\tau}^2\mathbb{D} + \psi_9 \mathbf{B}\mathbb{D} \\
&+\psi_{10} \mathbf{B}^2\mathbb{D} + \psi_{11} (\boldsymbol{\tau}\mathbf{B} + \mathbf{B}\boldsymbol{\tau})\mathbb{E} + \psi_{12} (\boldsymbol{\tau}\mathbf{B} + \mathbf{B}\boldsymbol{\tau})\mathbb{D} + \psi_{13} (\boldsymbol{\tau}^2\mathbf{B} + \mathbf{B}\boldsymbol{\tau}^2)\mathbb{E} \\
&+\psi_{14} (\boldsymbol{\tau}^2\mathbf{B} + \mathbf{B}\boldsymbol{\tau}^2)\mathbb{D} + \psi_{15} (\boldsymbol{\tau}\mathbf{B}^2 + \mathbf{B}^2\boldsymbol{\tau})\mathbb{E} + \psi_{16} (\boldsymbol{\tau}\mathbf{B}^2 + \mathbf{B}^2\boldsymbol{\tau})\mathbb{D} + \psi_{17} (\boldsymbol{\tau}^2\mathbf{B}^2 \\
&+\mathbf{B}^2\boldsymbol{\tau}^2)\mathbb{E} + \psi_{18} (\boldsymbol{\tau}^2\mathbf{B}^2 + \mathbf{B}^2\boldsymbol{\tau}^2)\mathbb{D} = \mathbf{0},
\end{aligned} \tag{33}$$

where the scalar functions γ_i, ψ_j, $i = 0, 1, ..., 32$, $j = 1, 2, .., 21$ depend on invariants defined in terms of $\boldsymbol{\tau}$, $\mathbf{B}$, $\mathbb{E}$ and $\mathbb{D}$, which for the sake of brevity are not listed here.

4.2 On a class of implicit constitutive relation for electro-elastic and thermo-electro-elastic bodies based on the use of the first law of thermodynamics

The constitutive relations presented in the previous section do not satisfy automatically the laws of thermodynamics. In this section we present a theory originally developed in [37, 56], where implicit constitutive relations are proposed for electro-elastic solids and thermo-electro-elastic solids, based on the use of the 1st law of thermodynamics, following the method presented in [80, 155].

4.2.1 Electro-elastic bodies

Let us start with the isothermal case, where there is no heat transfer and dissipation of energy, then the 1st law of thermodynamics is the same as the 2nd law, and for electro-elastic solids we have

$$-\rho_{\mathrm{r}}\dot{\psi} - \dot{\mathbb{E}}_{\mathrm{L}} \cdot \dot{\mathbb{P}}_{\mathrm{L}} + \operatorname{tr}(\mathbf{S}\dot{\mathbf{E}}) = 0, \tag{34}$$

where ψ is the Helmohltz free energy, $\mathbb{E}_{\mathrm{L}}$ is the Lagrangian electric field defined in the previous section, $\mathbb{P}_{\mathrm{L}}$ is the Lagrangian polarization field, defined as (see, for example, [69]) $\mathbb{P}_{\mathrm{L}} = J\mathbf{F}^{-1}\mathbb{P}$, and $\mathbf{S}$ is a second Piola-Kirchhoff stress tensor defined as $\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-\mathrm{T}}$, where in this section $\boldsymbol{\tau}$ is a symmetric stress tensor defined as[8] (see [78]) $\boldsymbol{\tau} = \boldsymbol{\sigma} + \mathbb{P}\otimes\mathbb{E}$, where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, which for electro-elastic bodies is in general non-symmetric.

Following the procedure presented in [80], assuming that $\psi = \psi(\mathbf{S}, \mathbf{E}, \mathbb{E}_{\mathrm{L}}, \mathbb{P}_{\mathrm{L}})$ and taking the derivative of (34) in $\mathbf{S}$ and $\mathbb{P}_{\mathrm{L}}$ we obtain (in index notation and Cartesian coordinates)

$$\mathscr{A}_{\alpha\beta\gamma\delta}\dot{S}_{\alpha\beta} + \mathscr{B}_{\alpha\beta\gamma\delta}\dot{E}_{\alpha\beta} + \mathscr{C}_{\gamma\delta\alpha}\dot{\mathbb{E}}_{\mathrm{L}_\alpha} + \mathscr{D}_{\gamma\delta\alpha}\dot{\mathbb{P}}_{\mathrm{L}_\alpha} = 0, \tag{35}$$

$$\mathscr{D}_{\alpha\beta\gamma}\dot{S}_{\alpha\beta} + \mathscr{Q}_{\gamma\alpha\beta}\dot{E}_{\alpha\beta} + \mathscr{R}_{\gamma\alpha}\dot{\mathbb{E}}_{\mathrm{L}_\alpha} + \mathscr{S}_{\gamma\alpha}\dot{\mathbb{P}}_{\mathrm{L}_\alpha} = 0, \tag{36}$$

where we have defined $\boldsymbol{\mathscr{A}} = \rho_{\mathrm{r}}\frac{\partial^2\psi}{\partial\mathbf{S}\partial\mathbf{S}}$, $\boldsymbol{\mathscr{B}} = \frac{\rho_{\mathrm{r}}}{2}\left(\frac{\partial^2\psi}{\partial\mathbf{E}\partial\mathbf{S}} + \frac{\partial^2\psi}{\partial\mathbf{S}\partial\mathbf{E}}\right) - \boldsymbol{\mathscr{I}}$, $\boldsymbol{\mathscr{C}} = \frac{\rho_{\mathrm{r}}}{2}\left(\frac{\partial^2\psi}{\partial\mathbb{E}_{\mathrm{L}}\partial\mathbf{S}} + \frac{\partial^2\psi}{\partial\mathbf{S}\partial\mathbb{E}_{\mathrm{L}}}\right)$, $\boldsymbol{\mathscr{D}} = \frac{\rho_{\mathrm{r}}}{2}\left(\frac{\partial^2\psi}{\partial\mathbb{P}_{\mathrm{L}}\partial\mathbf{S}} + \frac{\partial^2\psi}{\partial\mathbf{S}\partial\mathbb{P}_{\mathrm{L}}}\right)$, $\boldsymbol{\mathscr{Q}} = \frac{\rho_{\mathrm{r}}}{2}\left(\frac{\partial^2\psi}{\partial\mathbb{P}_{\mathrm{L}}\partial\mathbf{E}} + \frac{\partial^2\psi}{\partial\mathbf{E}}\partial\mathbb{P}_{\mathrm{L}}\right)$, $\boldsymbol{\mathscr{R}} = \mathbf{I} + \frac{\rho_{\mathrm{r}}}{2}\left(\frac{\partial^2\psi}{\partial\mathbb{E}_{\mathrm{L}}\partial\mathbb{P}_{\mathrm{L}}} + \frac{\partial^2\psi}{\partial\mathbb{P}_{\mathrm{L}}\partial\mathbb{E}_{\mathrm{L}}}\right)$ and $\boldsymbol{\mathscr{S}} = \rho_{\mathrm{r}}\frac{\partial^2\psi}{\partial\mathbb{P}_{\mathrm{L}}\partial\mathbb{P}_{\mathrm{L}}}$.

The special case of assuming small gradient of the displacement field is treated in Section 5.3.

[8] This stress is not the same as the total stress $\boldsymbol{\tau}$ presented in the previous section, and it should not be confused with the Kirchhoff stress tensor either.

4.2.2 Thermo-electro-elastic bodies

In this section we expand the results presented previously, for the case of considering heat transfer, but we do not take into account for other sources of dissipation of energy such as electrical hysteresis (see [37, 56]). In terms of the same variables defined in the previous section the 1st and 2nd laws of thermodynamics are:

$$\rho_{\mathrm{r}}(\dot{\psi}+\dot{\theta}s+\theta\dot{s})-\operatorname{tr}(\mathbf{S}\dot{\mathbf{E}})+\operatorname{Div}\mathbf{Q}_{\mathrm{L}}-\rho_{\mathrm{r}}+\dot{\mathbb{E}}_{\mathrm{L}}\cdot\dot{\mathbb{P}}_{\mathrm{L}}=0, \tag{37}$$

$$-\rho_{\mathrm{r}}\dot{\psi}-\rho_{\mathrm{r}}\dot{\theta}s-\dot{\mathbb{E}}_{\mathrm{L}}\cdot\dot{\mathbb{P}}_{\mathrm{L}}+\operatorname{tr}(\mathbf{S}\dot{\mathbf{E}})-\frac{1}{\theta}\mathrm{Grad}\theta\cdot\mathbf{Q}_{\mathrm{L}}\geqslant 0, \tag{38}$$

where θ is the absolute temperature, s is the entropy, r is the internal heat production and $\mathbf{Q}_{\mathrm{L}}$ is the heat flux in the reference configuration.

If for the Helmohltz free energy we assume $\psi=\psi(\mathbf{S},\mathbf{E},\mathbb{E}_{\mathrm{L}},\mathbb{P}_{\mathrm{L}},\theta)$ replacing this in (38), it is possible to show that the second law of thermodynamics is satisfied if $s=-\frac{\partial\psi}{\partial\theta}$ and

$$-\rho_{\mathrm{r}}\frac{\partial\psi}{\partial S_{\alpha\beta}}\dot{S}_{\alpha\beta}+\left(S_{\alpha\beta}-\rho_{\mathrm{r}}\frac{\partial\psi}{\partial E_{\alpha\beta}}\right)\dot{E}_{\alpha\beta}-\left(\mathbb{P}_{\mathrm{L}_\alpha}+\rho_{\mathrm{r}}\frac{\partial\psi}{\partial\mathbb{E}_{\mathrm{L}_\alpha}}\right)\dot{\mathbb{E}}_{\mathrm{L}_\alpha}-\rho_{\mathrm{r}}\frac{\partial\psi}{\partial\mathbb{P}_{\mathrm{L}_\alpha}}\dot{\mathbb{P}}_{\mathrm{L}_\alpha}=0, \tag{39}$$

$$-\frac{1}{\theta}\frac{\partial\theta}{\partial X_\alpha}Q_{\mathrm{L}_\alpha}\geqslant 0,$$

where we have used index notation and Cartesian coordinates. Eq.(39) must be satisfied for any $\dot{\mathbf{S}}$, $\dot{\mathbf{E}}$, $\dot{\mathbb{E}}_{\mathrm{L}}$ and $\dot{\mathbb{P}}_{\mathrm{L}}$.

From (39) taking the derivative in $\dot{\mathbf{S}}$ and $\dot{\mathbb{P}}_{\mathrm{L}}$ we obtain the same implicit relations (35), (36) but now the tensors $\boldsymbol{\mathscr{A}}$, $\boldsymbol{\mathscr{B}}$, $\boldsymbol{\mathscr{C}}$, $\boldsymbol{\mathscr{D}}$, $\boldsymbol{\mathscr{Q}}$, $\boldsymbol{\mathscr{R}}$ and $\boldsymbol{\mathscr{S}}$ defined there also depend on θ.

Regarding the 1st law (37), it can be used to obtain θ, for which we need an additional relation that can be considered a generalization of the Fourier's constitutive model for heat transfer (see Eq. (46) in the next section), and the following vector implicit relation has been proposed

$$\boldsymbol{\mathfrak{H}}(\mathbf{S},\mathbf{E},\mathbb{E}_{\mathrm{L}},\mathbb{P}_{\mathrm{L}},\theta,\mathrm{Grad}\theta,\mathbf{Q}_{\mathrm{L}},\dot{\mathbf{Q}}_{\mathrm{L}})=\mathbf{0}. \tag{40}$$

A special simpler subclass of the above implicit constitutive relation can also be proposed (that is a modification of a relation presented in [37]):

$$\mathbf{Q}_{\mathrm{L}}+\boldsymbol{\mathfrak{S}}(\mathbf{S},\mathbb{E}_{\mathrm{L}},\theta)\dot{\mathbf{Q}}_{\mathrm{L}}=\boldsymbol{\mathfrak{R}}(\mathbf{S},\mathbb{E}_{\mathrm{L}},\theta,\mathrm{Grad}\theta).$$

4.3 Thermo-elastic bodies

In this section we show implicit constitutive equations for some classes of thermo-elastic bodies [31]. Let us consider as basic variables the second Piola-Krichhoff stress tensor $\mathbf{S}$, the Lagrange-Saint Venant strain tensor $\mathbf{E}$ and the absolute temperature θ. The following implicit constitutive relation for thermo-elastic bodies has been proposed (see (5)):

$$\boldsymbol{\mathfrak{F}}(\mathbf{S}, \mathbf{E}, \theta) = 0. \tag{41}$$

The first law of thermodynamics is the statement

$$\rho_{\mathrm{r}} \dot{U} = \omega + \operatorname{Div} \mathbf{Q}_{\mathrm{L}} + \rho_{\mathrm{r}} r, \tag{42}$$

where we have defined $\omega = \operatorname{tr}(\mathbf{S}\dot{\mathbf{E}})$, Div is the divergence operator with respect to the reference configuration, $\mathbf{Q}_{\mathrm{L}}$ is the heat flux and r is the internal source of heat per unit of mass, both in the reference configuration.

If we define the dissipation δ as $\delta = \theta\dot{s} - \dot{U} + \frac{\omega}{\rho_{\mathrm{r}}}$, where s is the entropy, and $\boldsymbol{\gamma} = \theta \operatorname{Grad}\left(\frac{1}{\theta}\right)$, the dissipation must satisfy the inequality $\delta \geqslant 0$, while $\mathbf{Q}_{\mathrm{L}}$ must satisfy the so called Fourier inequality $-\mathbf{Q}_{\mathrm{L}} \cdot \boldsymbol{\gamma} \geqslant 0$. Both inequalities together lead to the Clausius-Duhem inequality

$$\rho_{\mathrm{r}} \left(\theta \dot{s} - \dot{U} + \frac{\omega}{\rho_{\mathrm{r}}} \right) - \mathbf{Q}_{\mathrm{L}} \cdot \boldsymbol{\gamma} \geqslant 0. \tag{43}$$

Now, if the Helmholtz potential ψ is defined as $\psi = U - \theta s$, using this and (42), it is possible to write (43) in terms of ψ. For implicit thermo-elastic bodies it is assumed now that $\psi = \psi(\mathbf{S}, \mathbf{E}, \theta)$, and (43) becomes (in index notation and Cartesian coordinates)

$$-\frac{\partial \psi}{\partial S_{\alpha\beta}} \dot{S}_{\alpha\beta} - \frac{\partial \psi}{\partial E_{\alpha\beta}} \dot{E}_{\alpha\beta} + \frac{1}{\rho_{\mathrm{r}}} S_{\alpha\beta} \dot{E}_{\alpha\beta} - Q_{\mathrm{L}_\alpha} \gamma_\alpha \geqslant 0, \tag{44}$$

and taking the derivative of (41) in time we obtain

$$\frac{\partial \mathfrak{F}_{\gamma\zeta}}{\partial S_{\alpha\beta}} \dot{S}_{\alpha\beta} + \frac{\partial \mathfrak{F}_{\gamma\zeta}}{\partial E_{\alpha\beta}} \dot{E}_{\alpha\beta} + \frac{\partial \mathfrak{F}_{\gamma\zeta}}{\partial \theta} \dot{\theta} = 0. \tag{45}$$

For a body to be thermo-elastic, $\boldsymbol{\mathfrak{F}}$ and ψ must satisfy (44) and (45) for any $\dot{\mathbf{S}}$, $\dot{\mathbf{E}}$ and $\dot{\theta}$.

An additional relation that is necessary is a generalization of Fourier's model for heat transfer (see [31]), namely (compare with (40) and using the notation $\nabla_{\mathbf{X}}$ for Grad)

$$\mathfrak{m}(\mathbf{S}, \mathbf{E}, \mathbf{Q}_{\mathrm{L}}, \dot{\mathbf{Q}}_{\mathrm{L}}, \nabla_{\mathbf{X}}\theta, \theta) = \mathbf{0}, \tag{46}$$

where $\mathfrak{m}$ is a vector implicit relation.

In the case of $\mathfrak{F}$ being an isotropic relation, its representation would be exactly as (9), interchanging $\mathbf{T}$ by $\mathbf{S}$, and $\mathbf{B}$ by $\mathbf{E}$ (the same for the invariants defined in (10), (11)), adding to the list of invariants the temperature θ. In the case of the vector implicit relation (46), if $\mathfrak{m}$ is isotropic, it becomes

$$\begin{aligned}
&\beta_1\mathbf{h}_{\mathrm{r}} + \beta_2\mathbf{S}\mathbf{Q}_{\mathrm{L}} + \beta_3\mathbf{S}^2\mathbf{Q}_{\mathrm{L}} + \beta_4\mathbf{E}\mathbf{Q}_{\mathrm{L}} + \beta_5\mathbf{E}^2\mathbf{Q}_{\mathrm{L}} + \beta_6\dot{\mathbf{Q}}_{\mathrm{L}} + \beta_7\mathbf{S}\dot{\mathbf{Q}}_{\mathrm{L}} + \beta_8\mathbf{S}^2\dot{\mathbf{Q}}_{\mathrm{L}} + \beta_9\mathbf{E}\dot{\mathbf{Q}}_{\mathrm{L}}\\
&+\beta_{10}\mathbf{E}^2\dot{\mathbf{Q}}_{\mathrm{L}} + \beta_{11}\nabla_{\mathbf{X}}\theta + \beta_{12}\mathbf{S}\nabla_{\mathbf{X}}\theta + \beta_{13}\mathbf{S}^2\nabla_{\mathbf{X}}\theta + \beta_{14}\mathbf{E}\nabla_{\mathbf{X}}\theta + \beta_{15}\mathbf{E}^2\nabla_{\mathbf{X}}\theta\\
&+\beta_{16}(\mathbf{SE} + \mathbf{ES})\mathbf{Q}_{\mathrm{L}} + \beta_{17}(\mathbf{S}^2\mathbf{E} + \mathbf{E}\mathbf{S}^2)\mathbf{Q}_{\mathrm{L}} + \beta_{18}(\mathbf{S}\mathbf{E}^2 + \mathbf{E}^2\mathbf{S})\mathbf{Q}_{\mathrm{L}} + \beta_{19}(\mathbf{S}^2\mathbf{E}^2 + \mathbf{E}^2\mathbf{S}^2)\mathbf{Q}_{\mathrm{L}}\\
&+\beta_{20}(\mathbf{SE} + \mathbf{ES})\dot{\mathbf{Q}}_{\mathrm{L}} + \beta_{21}(\mathbf{S}^2\mathbf{E} + \mathbf{E}\mathbf{S}^2)\dot{\mathbf{Q}}_{\mathrm{L}} + \beta_{22}(\mathbf{S}\mathbf{E}^2 + \mathbf{E}^2\mathbf{S})\dot{\mathbf{Q}}_{\mathrm{L}} + \beta_{23}(\mathbf{S}^2\mathbf{E}^2 + \mathbf{E}^2\mathbf{S}^2)\dot{\mathbf{Q}}_{\mathrm{L}}\\
&+\beta_{24}(\mathbf{SE} + \mathbf{ES})\nabla_{\mathbf{X}}\theta + \beta_{25}(\mathbf{S}^2\mathbf{E} + \mathbf{E}\mathbf{S}^2)\nabla_{\mathbf{X}}\theta + \beta_{26}(\mathbf{S}\mathbf{E}^2 + \mathbf{E}^2\mathbf{S})\nabla_{\mathbf{X}}\theta + \beta_{27}(\mathbf{S}^2\mathbf{E}^2\\
&+\mathbf{E}^2\mathbf{S}^2)\nabla_{\mathbf{X}}\theta = \mathbf{0},
\end{aligned}$$

where the scalar functions β_i, $i = 1, 2, ..., 27$ depend on the invariants formulated in terms of $\mathbf{S}$, $\mathbf{E}$, $\mathbf{Q}_{\mathrm{L}}$, $\dot{\mathbf{Q}}_{\mathrm{L}}$ and $\mathrm{Grad}\,\theta$, which for brevity are not shown here.

In Section 6.3 a constitutive relation for thermo-elastic incompressible isotropic bodies is presented, which was proposed in [60].

4.4 Residual stresses and initial stresses in elastic bodies

Implicit constitutive relations and some of their subclasses, wherein some measure of the strains are functions of some measures of the stress, have been used for the analysis of bodies with residual and initial stresses. In the following sections we can see that such new constitutive relations and equations can be indeed useful and interesting for such problems. It is important to indicate here that by residually stressed bodies we understand bodies that in the reference configuration have stresses, when there is not external load. As for initial stressed bodies, it is assumed that for the reference configuration we have stresses, but we can have external loads too.

4.4.1 On a class of residually stressed bodies

In this section some results presented in [59] are listed, wherein we work with a Gibbs potential $\mathcal{G} = \mathcal{G}(\mathbf{S})$ such that $\mathbf{E} = \mathfrak{f}(\mathbf{S}) = \frac{\partial \mathcal{G}}{\partial \mathbf{S}}$, for the case $\mathcal{G}$ is a transversely isotropic function $\mathcal{G} = \mathcal{G}(\mathbf{S}, \mathrm{a}_0)$, where $\mathbf{a}_0$ is a unit vector field.

In the reference configuration there is no deformation, thus $\mathbf{E} = \mathbf{0}$, but it is assumed there are residual stresses, therefore $\mathbf{S} = \mathbf{T}_\mathrm{R}$, where $\mathbf{T}_\mathrm{R}$ represents such residual stresses. The residual stresses must satisfy

$$\mathbf{0} = \mathfrak{f}(\mathbf{T}_\mathrm{R}), \quad \mathrm{Div}\,\mathbf{T}_\mathrm{R} + \rho_\mathrm{r}\mathbf{b} = \mathbf{0}, \quad \mathbf{T}_\mathrm{R}\mathbf{N} = \mathbf{0} \quad \mathbf{X} \in \partial\boldsymbol{\kappa}_\mathrm{r}(\mathcal{B}), \tag{47}$$

where the last restriction is very important and implies that in the reference residually stressed configuration there is no external load yet the stresses are not zero.

From [59] (see also [94, 95]) the residual stresses $\mathbf{T}_\mathrm{R}$ must satisfy the restriction

$$\mathbf{H}\mathbf{T}_\mathrm{R} = \mathbf{T}_\mathrm{R}\mathbf{H}, \tag{48}$$

where $\mathbf{H}$ belongs to the symmetry group for the body. In the case $\mathcal{G}$ is a transversely isotropic function $\mathcal{G} = \mathcal{G}(\mathbf{S}, \mathbf{a}_0)$, where $\mathbf{a}_0$ is a unit vector field, from [94, 95] (and the references therein) we have that for $\mathbf{H}$ belonging to that symmetry group Eq. (48) is satisfied if $\mathbf{T}_\mathrm{R} = \alpha\mathbf{I} + \beta\mathbf{a}_0 \otimes \mathbf{a}_0$.

In the case $\mathcal{G} = \mathcal{G}(\mathbf{S}, \mathbf{a}_0)$ and using the classical invariants by Spencer [202] we have $\mathcal{G} = \mathcal{G}(I_1, I_2, I_3, I_4, I_5)$ (where the I_i, $i = 1, 2, 3, 4, 5$ have been defined in Section 3.3.3), and the expression for $\mathbf{E}$ is given as in (30). Using that Eq. (30) in $(47)_1$ considering the above expression for $\mathbf{T}_\mathrm{R}$ we obtain

$$\mathcal{G}_1 + (\alpha + \beta)\mathcal{G}_2 + [\alpha^2 + \beta(2\alpha + \beta)]\mathcal{G}_3 + \mathcal{G}_4 + 2(\alpha + \beta)\mathcal{G}_5 = 0, \tag{49}$$

where $\mathcal{G}_i$, $i = 1, 2, 3, 4, 5$ above has been defined in Section 3.3.3, and in this case such functions are evaluated with $\mathbf{T}_\mathrm{R} = \alpha\mathbf{I} + \beta\mathbf{a}_0 \otimes \mathbf{a}_0$ instead of $\mathbf{S}$.

In [59] one boundary value problem was studied, namely the behaviour of a residually stressed hollow sphere, where it is assumed that $\mathbf{a} = \mathbf{E}_R$, thus $\mathbf{T}_\mathrm{R} = \alpha(\mathbf{E}_R \otimes \mathbf{E}_R + \mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + \mathbf{E}_\Phi \otimes \mathbf{E}_\Phi) + \beta\mathbf{E}_R \otimes \mathbf{E}_R$, where $\alpha = \alpha(R)$ and $\beta = \beta(R)$, and $\mathbf{E}_R$, $\mathbf{E}_\Theta$ and $\mathbf{E}_\Phi$ are the unit vectors in spherical coordinates and in the reference configuration. In [59] the authors considered the special case for $\mathcal{G}(I_1, I_2, I_3, I_4, I_5) = \sum_{j=1}^{5} c_i I_i$, where c_i can depend on R (the body is assumed to be inhomogeneous). Using the above expression for the residual stresses and $\mathcal{G}$ in $(47)_2$ and in (49) (assuming $\mathbf{b} = \mathbf{0}$) some relations for α, β and c_i, $i = 1, 2, 3, 4, 5$ are obtained, which for the sake of brevity are not listed here.

4.4.2 An implicit constitutive relation for an elastic body with initial stresses

In [136] an implicit constitutive relation has been proposed for elastic bodies with initial stresses. Such initial stress tensor is denoted $\mathbf{T}^{\mathrm{i}}$ in the reference configuration $\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})$. Those initial stresses can be produced by some initial external load, but if there is no external traction in $\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})$ such initial stresses become the residual stresses mentioned in the previous section $\mathbf{T}_{\mathrm{R}}$.

The body deforms from $\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})$ to $\boldsymbol{\kappa}_{\mathrm{t}}(\mathscr{B})$ and the deformation gradient is $\mathbf{F}$. Second order tensors that are defined in the reference configuration can be brought to the current configuration through the push forward operations $\mathbf{F}(\quad)\mathbf{F}^{\mathrm{T}}$ or $\mathbf{F}^{-\mathrm{T}}(\quad)\mathbf{F}^{-1}$. The following implicit constitutive relation is proposed for initially stressed elastic bodies:

$$\mathfrak{F}(\mathbf{B}, \mathbf{T}, \mathbf{F}\mathbf{T}^{\mathrm{i}}\mathbf{F}^{\mathrm{T}}, \mathbf{F}(\mathbf{T}^{\mathrm{i}})^2\mathbf{F}^{\mathrm{T}}, ...) = \mathbf{0}. \tag{50}$$

In [136] the author focus on the subclass of (50):

$$\mathbf{B} = \mathfrak{g}(\mathbf{T}, \mathbf{F}\mathbf{T}^{\mathrm{i}}\mathbf{F}^{\mathrm{T}}, \mathbf{F}(\mathbf{T}^{\mathrm{i}})^2\mathbf{F}^{\mathrm{T}}), \tag{51}$$

which for the case $\mathfrak{g}$ is isotropic becomes

$$\begin{aligned}\mathbf{B} = {} & \alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{T}^2 + \alpha_3\mathbf{F}\mathbf{T}^{\mathrm{i}}\mathbf{F}^{\mathrm{T}} + \alpha_4(\mathbf{T}\mathbf{F}\mathbf{T}^{\mathrm{i}}\mathbf{F}^{\mathrm{T}} + \mathbf{F}\mathbf{T}^{\mathrm{i}}\mathbf{F}^{\mathrm{T}}\mathbf{T}) \\ & + \alpha_5\mathbf{F}(\mathbf{T}^{\mathrm{i}})^2\mathbf{F}^{\mathrm{T}} + \alpha_6[\mathbf{T}\mathbf{F}(\mathbf{T}^{\mathrm{i}})^2\mathbf{F}^{\mathrm{T}} + \mathbf{F}(\mathbf{T}^{\mathrm{i}})^2\mathbf{F}^{\mathrm{T}}\mathbf{T}]. \end{aligned} \tag{52}$$

In the case that $\mathbf{F} = \mathbf{I}$ this implies that $\mathbf{B} = \mathbf{I}$ and we have that $\boldsymbol{\kappa}_{\mathrm{t}}(\mathscr{B}) = \boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})$ and $\mathbf{T} = \mathbf{T}^{\mathrm{i}}$, then from (52) we have $\mathbf{I} = \alpha_0'\mathbf{I} + (\alpha_1' + \alpha_3')\mathbf{T}^{\mathrm{i}} + (\alpha_2' + 2\alpha_4' + \alpha_5')(\mathbf{T}^{\mathrm{i}})^2$, where $\alpha_i' := \alpha_i|_{\mathbf{F}=\mathbf{I},\mathbf{T}=\mathbf{T}^{\mathrm{i}}}$, and the above relation is satisfied if $\alpha_0' = 1$, $\alpha_1' + \alpha_3' = 0$ and $\alpha_2' + 2\alpha_4' + \alpha_5' = 0$.

In [136] the author also proposed an alternative expression for (50), where all the variables are written in the reference configuration, taking a push backward of $\mathbf{T}$, the alternative expression for (50) is $\tilde{\mathfrak{F}}(\mathbf{C}, \mathbf{T}^{\mathrm{i}}, \mathbf{F}\mathbf{T}\mathbf{F}^{\mathrm{T}}, \mathbf{F}\mathbf{T}^2\mathbf{F}^{\mathrm{T}}, ...) = \mathbf{0}$. In (50) the initial stress $\mathbf{T}^{\mathrm{i}}$ must satisfy the same restriction mentioned in Section 4.4.1 (see (48)), i.e., $\mathbf{T}^{\mathrm{i}}\mathbf{H} = \mathbf{H}\mathbf{T}^{\mathrm{i}}$ for all $\mathbf{H}$ in the symmetry group of $\mathfrak{F}$ (see, for example, [94, 95]).

In [136] a special subclass of (52) is obtained in the following way. The author assumed the existence of a function $\mathfrak{f} = \mathfrak{f}(\mathbf{T})$ such that (see, for example, (12)) $\mathbf{B} = \mathfrak{f}(\mathbf{T})$. Then it is assumed there exist a virtual free stress configuration $\boldsymbol{\kappa}_0(\mathscr{B})$, where $\mathbf{I} = \mathfrak{f}(\mathbf{0})$ (it is important to notice that that configuration may not even exist, and that is the reason is called 'virtual').

Then, we have three configurations $\boldsymbol{\kappa}_0(\mathscr{B})$, $\boldsymbol{\kappa}_{\rm r}(\mathscr{B})$ and $\boldsymbol{\kappa}_{\rm t}(\mathscr{B})$. From $\boldsymbol{\kappa}_0(\mathscr{B})$ to $\boldsymbol{\kappa}_{\rm r}(\mathscr{B})$ the deformation gradient is $\tilde{\mathbf{F}}$, from $\boldsymbol{\kappa}_{\rm r}(\mathscr{B})$ to $\boldsymbol{\kappa}_{\rm t}(\mathscr{B})$ the deformation gradient is $\mathbf{F}$, finally from $\boldsymbol{\kappa}_0(\mathscr{B})$ to $\boldsymbol{\kappa}_{\rm t}(\mathscr{B})$ is $\mathbf{F}_0$, then $\mathbf{F}_0 = \mathbf{F}\tilde{\mathbf{F}}$. Defining $\mathbf{B}_0 = \mathbf{F}_0\mathbf{F}_0^{\rm T}$ from $\mathbf{B} = \mathfrak{f}(\mathbf{T})$ in the case the function is isotropic we have

$$\mathbf{B}_0 = \kappa_0(\mathbf{T})\mathbf{I} + \kappa_1(\mathbf{T})\mathbf{T} + \kappa_2(\mathbf{T})\mathbf{T}^2.$$

Replacing the above $\mathbf{T}$ by $\mathbf{T}^{\rm i}$ and $\mathbf{B}$ by $\tilde{\mathbf{B}} = \tilde{\mathbf{F}}\tilde{\mathbf{F}}^{\rm T}$ after several manipulations the author obtained a constitutive equation of the form (12), namely

$$\mathbf{B} = \frac{\kappa_0(\mathbf{T})}{\kappa_0(\mathbf{T}^{\rm i})}\mathbf{I} + \frac{\kappa_1(\mathbf{T})}{\kappa_0(\mathbf{T}^{\rm i})}\mathbf{T} + \frac{\kappa_2(\mathbf{T})}{\kappa_0(\mathbf{T}^{\rm i})}\mathbf{T}^2 - \frac{\kappa_1(\mathbf{T}^{\rm i})}{\kappa_0(\mathbf{T}^{\rm i})}\mathbf{F}\mathbf{T}^{\rm i}\mathbf{F}^{\rm T} - \frac{\kappa_2(\mathbf{T}^{\rm i})}{\kappa_0(\mathbf{T}^{\rm i})}\mathbf{F}(\mathbf{T}^{\rm i})^2\mathbf{F}^{\rm T}. \quad (53)$$

If $\mathbf{T} = \mathbf{T}^{\rm i}$, i.e., $\mathbf{F} = \mathbf{I}$ and $\boldsymbol{\kappa}_{\rm t}(\mathscr{B}) = \boldsymbol{\kappa}_{\rm r}(\mathscr{B})$ from (53) we have that $\mathbf{B} = \mathbf{I}$, which is correct from the above analysis.

Finally, in [136] the case the gradient of the displacement field is small is considered for (53), and for brevity such subclasses of bodies are not presented here.

5 Constitutive relations and equations when the norm of the gradient of the displacement field is small

In this section we study (8) (see also (16)) in the case $|\nabla_{\mathbf{X}}\mathbf{u}| \sim O(\delta)$, $\delta \ll 1$, i.e. when the norm of the gradient of the displacement field is very small. That means that $\mathbf{B}$ (or $\mathbf{E}$) can be written directly in terms of the linearized strain tensor $\boldsymbol{\varepsilon}$ as $\mathbf{B} \approx 2\boldsymbol{\varepsilon} + \mathbf{I}$ and $\mathbf{E} \approx \boldsymbol{\varepsilon}$, and for example, the implicit relation (8) becomes:

$$\mathfrak{G}(\rho, \mathbf{T}, \boldsymbol{\varepsilon}, \mathbf{X}) = \mathbf{0}. \quad (54)$$

The above implicit relations can have many applications in the modelling of porous solids, and for materials that behave nonlinearly in the range of small elastic strains, such as bone, rock, concrete and some metal alloys. Some subclasses of (54) have been proposed in the literature for the modelling of strain limiting deformations, which is important for stress concentration analysis and fracture mechanics (see Section 7.3 for such applications and the references mentioned therein, see also [93, 123, 124, 138, 142, 146] for experimental data for different materials, whose mechanical properties depend on density). In the next sections we study different particular subclasses of (54).

5.1 Some classes of implicit constitutive relations in terms of the linearized strain and the Cauchy stress tensor

In this section some explicit expressions for (54) are presented for some particular cases for $\boldsymbol{\mathfrak{G}}$, in particular if $\boldsymbol{\mathfrak{G}}$ does not depend explicitly on $\mathbf{X}$ (homogeneous body), where $\mathbf{T}$ appears linearly in the implicit relation (one case where $\mathbf{T}$ appears nonlinearly is also considered), and because $|\boldsymbol{\varepsilon}|$ is small that variable also appears linearly in (54). The density ρ can be replaced by $\operatorname{tr}\boldsymbol{\varepsilon}$.

5.1.1 Implicit constitutive relations for bodies whose mechanical properties depend on the density and the mechanical pressure. The case the implicit relation is isotropic

In [171] the general constitutive relation (54) is used for the analysis of porous elastic solids, whose material moduli depend on density. For the particular case of homogeneous bodies where $\boldsymbol{\mathfrak{G}}$ is isotropic and $\mathbf{T}$ appears linearly, the following subclass of implicit relation is obtained from (54) (see, for example, [202]):

$$\begin{aligned}\boldsymbol{\varepsilon} + A_1\mathbf{T} + A_2(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{T} + A_3(\operatorname{tr}\mathbf{T})\mathbf{I} + A_4(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{I} + A_5(\operatorname{tr}\mathbf{T})\boldsymbol{\varepsilon}\\ +A_6(\operatorname{tr}\mathbf{T})(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{I} + A_7(\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) + A_8\operatorname{tr}(\boldsymbol{\varepsilon}\mathbf{T})\mathbf{I} = \mathbf{0},\end{aligned} \quad (55)$$

where A_i, $i = 1, 2, 3, ..., 8$ are constants. The density ρ has an influence in the behaviour of the body through $\operatorname{tr}\boldsymbol{\varepsilon}$ since $\rho_{\mathrm{r}} = \rho = J\rho \approx (1+\operatorname{tr}\boldsymbol{\varepsilon})$ because $J \approx 1 + \operatorname{tr}\boldsymbol{\varepsilon}$.

In [171] a detailed study of the case $\mathbf{T} = \sigma\mathbf{e}_1 \otimes \mathbf{e}_1$ and $\boldsymbol{\varepsilon} = \varepsilon\mathbf{e}_1 \otimes \mathbf{e}_1$ is considered, as well as this, two special subclasses of (55) are analyzed: $\boldsymbol{\varepsilon} = B_1[1+\lambda_2(\operatorname{tr}\boldsymbol{\varepsilon})]\mathbf{T}+B_2[1+\lambda_3(\operatorname{tr}\boldsymbol{\varepsilon})](\operatorname{tr}\mathbf{T})\mathbf{I}$ and $\boldsymbol{\varepsilon} = B_1[(1+\lambda_2)-\lambda_2\rho/\rho_{\mathrm{r}}]\mathbf{T}+ B_2[(1 + \lambda_3) - \lambda_3\rho/\rho_{\mathrm{r}}](\operatorname{tr}\mathbf{T})\mathbf{I}$, where B_1, B_2, λ_2 and λ_3 are constants. Here and thereafter λ_i should not be confused with the principal stretches. These and other subclasses of (55) are shown in the following list.

Some constitutive relations proposed by Alagappan et al.: In [1] some subclasses of (55) are proposed, where the material moduli depend on the density and the mechanical *pressure* applied on the solids, with potential applications to porous elastic solids such as bone, concrete and rock. As mentioned previously if we have small deformations the density is proportional to $\mathrm{tr}\,\boldsymbol{\varepsilon}$, and for an homogeneous body (54) becomes $\boldsymbol{\mathfrak{G}}(\mathbf{T},\boldsymbol{\varepsilon}) = \mathbf{0}$. When $\boldsymbol{\mathfrak{G}}$ is isotropic and $\mathbf{T}$ appears linearly[9] we have (compare with (55)):

$$\boldsymbol{\varepsilon}[1 + \lambda_1(\mathrm{tr}\,\mathbf{T})] = B_1[1 + \lambda_2(\mathrm{tr}\,\boldsymbol{\varepsilon})]\mathbf{T} + B_2[1 + \lambda_3(\mathrm{tr}\,\boldsymbol{\varepsilon})](\mathrm{tr}\,\mathbf{T})\mathbf{I}, \tag{56}$$

where B_1, B_2, λ_i, $i = 1, 2, 3$ are material constants. If $\lambda_i = 0$ and $B_1 = (1 + \nu)/E$ and $B_2 = -\nu/E$ from (56) we recover the linearized elastic constitutive equation.

In [1] the subclass $\lambda_1 = \lambda$, $\lambda_2 = \lambda_3 = \gamma$, with $\lambda < 0$, $\gamma > 0$ is also considered, where from (56)

$$\boldsymbol{\varepsilon}[1 + \lambda(\mathrm{tr}\,\mathbf{T})] = B_1[1 + \gamma(\mathrm{tr}\,\boldsymbol{\varepsilon})]\mathbf{T} + B_2[1 + \gamma(\mathrm{tr}\,\boldsymbol{\varepsilon})](\mathrm{tr}\,\mathbf{T})\mathbf{I} \tag{57}$$

In [1] some numerical values have been proposed for λ, γ, B_1, B_2 but are not connected with actual experimental data for materials. In [211] a simpler subclass of (57) has been considered, namely (here we use a different notation for some material constants):

$$\boldsymbol{\varepsilon} = B_1[1 + \gamma_1(\mathrm{tr}\,\boldsymbol{\varepsilon})]\mathbf{T} + B_2[1 + \gamma_2(\mathrm{tr}\,\boldsymbol{\varepsilon})](\mathrm{tr}\,\mathbf{T})\mathbf{I}. \tag{58}$$

In [211] the authors use the notation E_1, λ, E_2 and λ_2 instead B_1, γ_1, B_2 and γ_2, respectively. In that paper some boundary value problems are solved using the finite element method (using the program Comsol), such as the analysis of the behaviour of a square plate with a rigid elliptical inclusion in its center. In [212] the same implicit relation (58) is used for the analysis of another boundary value problems, namely the study of a plate with an elliptical hole in its centre under tension.

The implicit relation (57) is used in [140] for the analysis of stresses for a plate with a circular hole. In that work, it is assumed that $\gamma_1 = \gamma_2 = \gamma$ (which in [140] are called λ_1, λ_2, respectively). Defining $B_1 = (1 + \nu)/E$, $B_2 = -\nu/E$ and from (57) in [140] they obtain (compare with (60) below):

$$\boldsymbol{\varepsilon} = \frac{(1 + \nu)}{E(1 - \gamma\,\mathrm{tr}\,\boldsymbol{\varepsilon})}\mathbf{T} - \frac{\nu}{E(1 - \gamma\,\mathrm{tr}\,\boldsymbol{\varepsilon})}(\mathrm{tr}\,\mathbf{T})\mathbf{I}. \tag{59}$$

[9]This constitutive relation (56) has been also considered in [211], where the only difference is the name given to some material constants, where λ_1, B_1, λ_2, B_2, λ_3 are replaced by τ, E_1, λ_1, E_2 and λ_2, respectively.

In [221] Eq. (57) is used for the analysis of some boundary value problems as well. The particular expression used in that paper is:

$$\boldsymbol{\varepsilon} = \frac{(1+\nu)(1+\beta\,\mathrm{tr}\,\boldsymbol{\varepsilon})}{E}\mathbf{T} - \frac{\nu(1+\beta\,\mathrm{tr}\,\boldsymbol{\varepsilon})}{E}\mathrm{tr}\,(\mathbf{T})\mathbf{I}, \tag{60}$$

where β is a material constant, and ν, E are ground values for the Poisson ratio and the Young's modulus. In order to solve boundary value problems with the finite element method in [221] Eq. (60) is inverted to obtain the stress as a function of the linearized strain (this is just a mathematical device, which from the physical point of view cannot be done, see [169]), and then a weak formulation is proposed in terms of the displacement field. Some problems with plane plates with cracks are analyzed under traction and shear, where the cracks are located in the middle or on the edges of such plates. There is an analysis of the effect on the stresses of the material parameter β.

A constitutive relation by Bustamante and Rajagopal: In [49] taking as basis (54), Bustamante and Rajagopal proposed the following constitutive relation for rock and concrete, where the density (conected with $\mathrm{tr}\,\boldsymbol{\varepsilon}$) and the spherical part of the stress play an important role:

$$\begin{aligned}\boldsymbol{\varepsilon} + q^{(0)}(\mathrm{tr}\,\mathbf{T})\boldsymbol{\varepsilon} + h^{(0)}(\mathbf{T}\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}\mathbf{T}) + f^{(1)}(\mathrm{tr}\,\boldsymbol{\varepsilon})\mathbf{I} + f^{(2)}(\mathrm{tr}\,\mathbf{T})(\mathrm{tr}\,\boldsymbol{\varepsilon})\mathbf{I}\\ + q^{(0)}\mathrm{tr}\,(\mathbf{T}\boldsymbol{\varepsilon})\mathbf{I} + q^{(0)}(\mathrm{tr}\,\boldsymbol{\varepsilon})\mathbf{T} + \sum_{i=1}^{3}\frac{\partial \mathfrak{p}}{\partial \sigma_i}\mathbf{t}^{(i)} \otimes \mathbf{t}^{(i)} = \mathbf{0}.\end{aligned} \tag{61}$$

In (61) above $f^{(1)}$, $f^{(2)}$, $h^{(0)}$ and $q^{(0)}$ are material constants, where the function $\mathfrak{p}$ should satisfy $\mathfrak{p} = \mathfrak{p}(\sigma_1,\sigma_2,\sigma_3) = \mathfrak{p}(\sigma_2,\sigma_1,\sigma_3) = \mathfrak{p}(\sigma_3,\sigma_2,\sigma_1) = \mathfrak{p}(\sigma_1,\sigma_3,\sigma_2)$, where σ_i, $\mathbf{t}^{(i)}$ are the principal stresses and principal directions of $\mathbf{T}$, respectively. The following expression was proposed for $\mathfrak{p}$

$$\mathfrak{p}(\sigma_1,\sigma_2,\sigma_3) = \mathfrak{f}(\sigma_1) + \mathfrak{f}(\sigma_2) + \mathfrak{f}(\sigma_3) + \mathfrak{g}((\sigma_1+\sigma_2+\sigma_3)/3).$$

In [49] three boundary value problems were considered, namely the uniform tension-compression of a cylinder without lateral constraints, the uniform tension-compression with lateral load, and the simple shear of a slab. In the particular case of the tension-compression of a cylinder without lateral constraints, if σ_z, $\varepsilon_z = \hat{\varepsilon}_z(\sigma_z)$ and $\varepsilon_r = \hat{\varepsilon}_r(\sigma_z)$ denote the axial stress, axial and radial strains, respectively, which are obtained from experiments, from

(61) it is possible to show that:

$$\mathfrak{f}'(x) = \hat{\varepsilon}_r(x) - q^{(0)}\hat{\varepsilon}_r(x)x - \hat{\varepsilon}_z(x)[1 + 2(h^{(0)} + q^{(0)})x], \tag{62}$$

$$\mathfrak{g}'(x) = -3\{\hat{\varepsilon}_r(3x)[1 + 2f^{(1)} + 3(2f^{(2)} + q^{(0)})x] + \hat{\varepsilon}_z(3x)[f^{(1)} + 3(f^{(2)} + q^{(0)})x]\}. \tag{63}$$

Specific applications of (61)-(63) for rock and concrete are shown in Sections 7.3.4 and 7.3.5, respectively.

Some constitutive relations by Itou et al: In [103] an implicit constitutive relation for porous solids has been proposed that is of the form:

$$\boldsymbol{\varepsilon}_{\mathrm{D}} = E_1\mathbf{T}_{\mathrm{D}}, \quad \frac{1}{3}\mathrm{tr}\,\boldsymbol{\varepsilon} = \frac{E_3}{[1 - \lambda_4\mathfrak{G}(\mathrm{tr}\,\boldsymbol{\varepsilon})]}\mathrm{tr}\,\mathbf{T}, \tag{64}$$

In [103] some expressions for $\mathfrak{G} = \mathfrak{G}(\mathrm{tr}\,\boldsymbol{\varepsilon})$ are proposed such that $\mathrm{tr}\,\boldsymbol{\varepsilon}$ is limited (bounded). The above implicit relation (64) has been used to analyze the behaviour of bodies with cracks, in particular the authors have studied some restrictions on $\mathfrak{G}$ such that there exists a unique solution for the boundary value problem.

In [102] Itou et al. proposed a variation of the implicit relation (56) of the form (in [102] the names of some constants are different):

$$\boldsymbol{\varepsilon}[1 + \lambda_3(\mathrm{tr}\,\mathbf{T})] = E_1[1 + \lambda_1(\mathrm{tr}\,\boldsymbol{\varepsilon})]\mathbf{T}_{\mathrm{D}} + E_4[1 + \lambda_4(\mathrm{tr}\,\boldsymbol{\varepsilon})](\mathrm{tr}\,\mathbf{T})\mathbf{I}, \tag{65}$$

where $\mathbf{T}_{\mathrm{D}} = \mathbf{T} - \frac{1}{3}(\mathrm{tr}\,\mathbf{T})\mathbf{I}$ is the deviatoric stress (see Section 3.3.2). Some of the above constants are defined as $E_1 = (1+\nu)/E$, $E_4 = E_1/3 + E_2$, where $E_2 = -\nu/E$ and $\lambda_4 = \frac{(E_1/3)\lambda_1 + E_2\lambda_2}{E_4}$, where E and ν are the Young's modulus and the Poisson ratio in the reference configuration, respectively. From (65) after several manipulations we can obtain an expression for the linearized strain as a function of the stress of the form (see Section 5.2):

$$\boldsymbol{\varepsilon} = E_1A_1(\mathrm{tr}\,\mathbf{T})A_2(\mathrm{tr}\,\mathbf{T})\mathbf{T}_{\mathrm{D}} + E_2A_2(\mathrm{tr}\,\mathbf{T})(\mathrm{tr}\,\mathbf{T})\mathbf{I}, \tag{66}$$

where the functions A_1 and A_2 are defined as $A_1(\mathrm{tr}\,\mathbf{T}) = 1 + \frac{3E_4(\lambda_1-\lambda_4)}{1/\mathrm{tr}\,\mathbf{T} - 1/\tau_{\mathrm{cr}_1}}$, $A_2(\mathrm{tr}\,\mathbf{T}) = \frac{1}{1-\mathrm{tr}\,\mathbf{T}/\tau_{\mathrm{cr}_2}}$, where $\tau_{\mathrm{cr}_1} = -1/\lambda_3$ and $\tau_{\mathrm{cr}_2} = \frac{1}{(3E_4\lambda_4 - \lambda_3)}$. In [102] special attention is given to a subclass of (66), wherein it is assumed that $\lambda_1 = \lambda_3 = 0$, such that (66) becomes $\boldsymbol{\varepsilon} = E_1\mathbf{T} + \frac{E_2[1+3E_4(\lambda_2-\lambda_4)\mathrm{tr}\,\mathbf{T}]}{(1-3E_4\lambda_4\mathrm{tr}\,\mathbf{T})}(\mathrm{tr}\,\mathbf{T})\mathbf{I}$, from where after some manipulations we have

$$\boldsymbol{\varepsilon}_{\mathrm{D}} = E_1\mathbf{T}_{\mathrm{D}}, \quad \frac{1}{3}\mathrm{tr}\,\boldsymbol{\varepsilon} = E_4B(p), \tag{67}$$

where $\boldsymbol{\varepsilon}_{\rm D}$ is the deviatoric linearized strain tensor defined as $\boldsymbol{\varepsilon}_{\rm D} = \boldsymbol{\varepsilon} - \frac{1}{3}(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{I}$, and $B(p) = \frac{p}{1-p/\tau_{\rm cr}}$ (where $p = \operatorname{tr}\mathbf{T}$ and $\tau_{\rm cr} = \frac{1}{3E_4\lambda_4}$). In [102] it is shown that the above constitutive equation (67) is continuous, coersive and bounded. In that paper it is also shown the well-posedness of the regularized problem, and a variational formulation is proposed in terms of $(\mathbf{u}, \mathbf{T}_{\rm D}, p)$.

A constitutive relation by Prusa and Trnka: In [149] a constitutive relation has been proposed, which is of the form (54), but which for practical reason is written as (59). Here the focus is on defining a Young's modulus that depends on density, $E(\rho) \approx E_{\rm r}(1 - n\operatorname{tr}\boldsymbol{\varepsilon})$ (where $E_{\rm r}$ is the Young's modulus in the referenc configuration). The constitutive relation is (compare with (59)):

$$\boldsymbol{\varepsilon} = \frac{(1+\nu_{\rm r})}{E_{\rm r}(1 - n\operatorname{tr}\boldsymbol{\varepsilon})}\mathbf{T} - \frac{\nu_{\rm r}}{E_{\rm r}(1 - n\operatorname{tr}\boldsymbol{\varepsilon})}(\operatorname{tr}\mathbf{T})\mathbf{I}, \tag{68}$$

where $\nu_{\rm r}$ is the Poisson ratio in the reference configuration[10]. Considering that $|\boldsymbol{\varepsilon}|$ is very small, from a Taylor expansion on the right side of (68) we have:

$$\boldsymbol{\varepsilon} \approx \frac{1}{E_{\rm r}}(1 + n\operatorname{tr}\boldsymbol{\varepsilon})[(1+\nu_{\rm r})\mathbf{T} - \nu_{\rm r}(\operatorname{tr}\mathbf{T})\mathbf{I}]. \tag{69}$$

In [149] there is a study on the influence of the parameter n on the behaviour of the body. As well as this, some boundary value problems are solved using the finite element method, such as the extension of a circular cylinder, the deflection of a thin plate, and the bending of a beam (see Section 8). In order to use a standard finite element formulation in [149] Eq. (69) is inverted to express the stress as a function of the linearized strain. We need to remark again that such an inversion is only a mathematical device that from the physical point of view cannot be done, see [169] for comments on that. Such an inverse expression is: $\mathbf{T} = \alpha_{\rm r}(1 - n\operatorname{tr}\boldsymbol{\varepsilon})(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{I} + 2\beta_{\rm r}(1 - n\operatorname{tr}\boldsymbol{\varepsilon})\boldsymbol{\varepsilon}$, where $\alpha_{\rm r} = \frac{\nu_{\rm r}E_{\rm r}}{(1+\nu_{\rm r})(1-2\nu_{\rm r})}$ and $\beta_{\rm r} = \frac{E_{\rm r}}{2(1+\nu_{\rm r})}$. The results of such boundary value problems have been compared with the predictions of the classical linearized theory of elasticity, and the behaviour of such solid is very different compared with such linearized elastic solids.

A constitutive relation by Rajagopal and Saccomandi: In [172] a constitutive relation of the form (8) has been proposed for the particular case

[10] This constitutive equation (68) comes from the implicit relation $\boldsymbol{\varepsilon}E_{\rm r}(1 - n\operatorname{tr}\boldsymbol{\varepsilon}) - (1+\nu_{\rm r})\mathbf{T} + \nu_{\rm r}(\operatorname{tr}\mathbf{T})\mathbf{I} = \mathbf{0}$.

the mechanical properties depend on density $\rho = \rho_r/J$. In [172] the following particular constitutive relation has been considered (where ρ is replaced by J):

$$\mathbf{B} = \left[1 + \frac{(\nu_r^2 - 1)}{E_r} f(J)\mathrm{tr}\,\mathbf{T}\right]\mathbf{I} - \frac{(\nu_r + 1)(\nu_r^2 - 1)}{E_r} f(J)\mathbf{T} + \frac{(1 - \nu_r^2)}{2E_r}(\mathbf{TB} + \mathbf{BT}), \tag{70}$$

where $f(J) = \frac{J^{\frac{2\nu_r}{(2\nu_r - 1)}} - 1}{(\nu_r + 1)J^{\frac{2\nu_r}{(2\nu_r - 1)}} - \nu_r J^{\frac{2\nu_r + 2}{(2\nu_r - 1)}} - 1}$ and E_r, ν_r are material constants. In [172] the particular case of small gradient of the displacement field is small is assumed $|\nabla \mathbf{u}| \sim O(\delta)$, $\delta \ll 1$, from where $\mathbf{B} \approx \mathbf{I} + 2\boldsymbol{\varepsilon}$ and after several manipulations from (70) we obtain:

$$\boldsymbol{\varepsilon} = -\left[\frac{\nu_r}{E_r}(1 + \Gamma_r \mathrm{tr}\boldsymbol{\varepsilon})\mathrm{tr}\,\mathbf{T}\right]\mathbf{I} + \frac{(1 + \nu_r)}{E_r}(1 + \Gamma_r \mathrm{tr}\boldsymbol{\varepsilon})\mathbf{T} + \Gamma_1(\mathbf{T}\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}\mathbf{T}),$$

where Γ_r is a constant[11].

5.1.2 Two directions elasticity and a transversely isotropic body

In many applications with porous materials, it is necessary to consider anisotropic constitutive relations, due to the nature of some of such materials, such as bone (see Section 7.3.6 and the references mentioned therein). Therefore, there is a need for expressions for (54) when $\mathfrak{G}$ is anisotropic. In this section we show the same, for two cases, where $\mathfrak{G}$ depends on one or two unit vectors fields, which represent the directions the body is anisotropic.

In the case $|\nabla_{\mathbf{X}} \mathbf{u}| \sim O(\delta)$, $\delta \ll 1$ from (54) we obtain (see [52]):

$$\begin{aligned}
&\gamma_0\mathbf{I} + \gamma_1\mathbf{T} + \gamma_2\mathbf{T}^2 + \gamma_3\boldsymbol{\varepsilon} + \gamma_4(\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) + \gamma_5(\boldsymbol{\varepsilon}\mathbf{T}^2 + \mathbf{T}^2\boldsymbol{\varepsilon}) + \gamma_6\mathbf{a}\otimes\mathbf{a} \\
&+\gamma_7[\mathbf{a}\otimes(\mathbf{Ta}) + (\mathbf{Ta})\otimes\mathbf{a}] + \gamma_8\mathbf{b}\otimes\mathbf{b} + \gamma_9[\mathbf{a}\otimes(\boldsymbol{\varepsilon}\mathbf{a}) + (\boldsymbol{\varepsilon}\mathbf{a})\otimes\mathbf{a}] \\
&+\gamma_{10}[\mathbf{b}\otimes(\mathbf{Tb}) + (\mathbf{Tb})\otimes\mathbf{b}] + \gamma_{11}[\mathbf{b}\otimes(\boldsymbol{\varepsilon}\mathbf{b}) + (\boldsymbol{\varepsilon}\mathbf{b})\otimes\mathbf{b}] + \gamma_{12}(\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\otimes\mathbf{b} \\
&+\mathbf{b}\otimes\mathbf{a}) + \gamma_{13}(\mathbf{a}\cdot\mathbf{b})[\mathbf{a}\otimes(\mathbf{Tb}) + (\mathbf{Tb})\otimes\mathbf{a} + \mathbf{b}\otimes(\mathbf{Ta}) + (\mathbf{Ta})\otimes\mathbf{b}] \\
&+\gamma_{13}(\mathbf{a}\cdot\mathbf{b})[\mathbf{a}\otimes(\boldsymbol{\varepsilon}\mathbf{b}) + (\boldsymbol{\varepsilon}\mathbf{b})\otimes\mathbf{a} + \mathbf{b}\otimes(\boldsymbol{\varepsilon}\mathbf{a}) + (\boldsymbol{\varepsilon}\mathbf{a})\otimes\mathbf{b}] = \mathbf{0},
\end{aligned} \tag{71}$$

[11] In [172] there is a study of plane wave propagation considering (70) (see Section 8.2.1), where it is assumed that $x = X + u(X,t)$, $y = Y + v(X,t)$ and $z = Z$ (using the notation x, y, z for x_i, $i = 1,2,3$ and X, Y, Z for X_α, $\alpha = 1,2,3$). In such a case the equations of motion $(4)_1$ (without body forces in the case that $\mathbf{T} = T_{11}\mathbf{e}_1\otimes\mathbf{e}_1 + T_{12}(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1)$) become $\rho_r\ddot{u} = \frac{\partial T_{11}}{\partial X}$ and $\rho_r\ddot{v} = \frac{\partial T_{12}}{\partial X}$. In that paper the authors obtain some properties for the above partial differential equations, and one solution is found, which for the sake of brevity is not shown here.

where γ_i, $i = 0, 1, 2, ..., 13$ can depend on the stress $\mathbf{T}$, γ_k, $k = 0, 1, 2, 6, 7, 8, 10, 12, 13$ can only depend linearly in $\boldsymbol{\varepsilon}$, while γ_j, $j = 3, 4, 5, 9, 11, 13$ do not depend on $\boldsymbol{\varepsilon}$.

The case the implicit relation is transversely isotropic is a special subclass of (71), and if we assume as well that the stress can only appears linearly, we have

$$\vartheta_0\mathbf{I} + \vartheta_1\mathbf{T} + \vartheta_2\boldsymbol{\varepsilon} + \vartheta_3(\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) + \vartheta_4\mathbf{a}\otimes\mathbf{a} + \vartheta_5[\mathbf{a}\otimes(\mathbf{Ta}) + (\mathbf{Ta})\otimes\mathbf{a}] + \vartheta_6[\mathbf{a}\otimes(\boldsymbol{\varepsilon}\mathbf{a}) + (\boldsymbol{\varepsilon}\mathbf{a})\otimes\mathbf{a}] = \mathbf{0}, \quad (72)$$

where (see Section 4.3 in [52]) $\vartheta_i = \eta^{(0)}_{o_i} + \eta^{(1)}_{o_i}\operatorname{tr}\boldsymbol{\varepsilon} + \eta^{(2)}_{o_i}\mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a})$, $i = 1, 5$, $\vartheta_k = \eta^{(0)}_{o_k} + \eta^{(1)}_{o_k}\operatorname{tr}\mathbf{T} + \eta^{(2)}_{o_i}\mathbf{a}\cdot(\mathbf{Ta})$, $k = 2, 6$, $\vartheta_l = \eta^{(1)}_{o_l}\operatorname{tr}\boldsymbol{\varepsilon} + \eta^{(2)}_{o_l}\mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a}) + \eta^{(3)}_{o_l}\operatorname{tr}\mathbf{T} + \eta^{(4)}_{o_l}\mathbf{a}\cdot(\mathbf{Ta})$, $l = 0, 4$, where $\eta^{(n)}_{o_m}$ are material constants.

From Eqs. (15) and (16) in [175] some implicit constitutive relations for bodies whose mechanical properties depend on the spherical stress and density have been proposed, which are special cases of (72) and (3), respectively. In the case of a body with two directions of anisotropy from (71) after different manipulations (which for the sake of brevity are not shown here) and keeping only linear terms in the stress we have:

$$(1 + \lambda\operatorname{tr}\mathbf{T})\boldsymbol{\varepsilon} + \beta_1(1 + \gamma\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{T} + \beta_2(1 + \gamma\operatorname{tr}\boldsymbol{\varepsilon})(\operatorname{tr}\mathbf{T})\mathbf{I} + \beta_3\mathbf{a}\otimes\mathbf{a} + \beta_4[\mathbf{a}\otimes(\mathbf{Ta}) + (\mathbf{Ta})\otimes\mathbf{a}] + \beta_5[\mathbf{a}\otimes(\boldsymbol{\varepsilon}\mathbf{a}) + (\boldsymbol{\varepsilon}\mathbf{a})\otimes\mathbf{a}] + \beta_6\mathbf{b}\otimes\mathbf{b} + \beta_7[\mathbf{b}\otimes(\mathbf{Tb}) + (\mathbf{Tb})\otimes\mathbf{b}] + \beta_8[\mathbf{b}\otimes(\boldsymbol{\varepsilon}\mathbf{b}) + (\boldsymbol{\varepsilon}\mathbf{b})\otimes\mathbf{b}] = \mathbf{0},$$

while for transversely isotropic bodies from (72) we obtain

$$(1 + \lambda\operatorname{tr}\mathbf{T})\boldsymbol{\varepsilon} + \beta_1(1 + \gamma\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{T} + \beta_2(1 + \gamma\operatorname{tr}\boldsymbol{\varepsilon})(\operatorname{tr}\mathbf{T})\mathbf{I} + \beta_3[\lambda\mathbf{a}\cdot(\mathbf{Ta}) + \gamma\mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a})]\mathbf{a}\otimes\mathbf{a} + \beta_4[\mathbf{a}\otimes(\mathbf{Ta}) + (\mathbf{Ta})\otimes\mathbf{a}] + \beta_5[\mathbf{a}\otimes(\boldsymbol{\varepsilon}\mathbf{a}) + (\boldsymbol{\varepsilon}\mathbf{a})\otimes\mathbf{a}] = \mathbf{0},$$

where β_i, $i = 1, 2, ..., 8$, λ and γ are material constants. In [175] some boundary value problems are studied considering homogeneous distributions for the stresses and strains, and also the problem of torsion of a cylinder, where the displacement field is assumed to be $\mathbf{u} = u_r(r)\mathbf{e}_r + krz\mathbf{e}_\theta + (\lambda_z - 1)z\mathbf{e}_z$ is studied. In that last problem some nonlinear ordinary differential equations for u_r and some components of the stress tensor are obtained, which for the sake of brevity are not reported here.

5.2 Constitutive equations for the linearized strain as a function of the Cauchy stress for purely elastic deformations

In this section we study the case of (9) when $|\nabla_{\mathbf{X}}\mathbf{u}| \sim O(\delta)$, $\delta \ll 1$ (see [17, 18, 157, 161]), where our aim is to obtain an explicit expression for $\boldsymbol{\varepsilon}$ versus $\mathbf{T}$. In such a situation we have $\mathbf{B} \approx \mathbf{I} + 2\boldsymbol{\varepsilon}$, replacing this in (9), and appealing to the approximation $\alpha_i = \alpha_i(\mathbf{T}, \mathbf{B}) \approx \alpha_i(\mathbf{T}, \boldsymbol{\varepsilon}) \approx \alpha_i(\mathbf{T}, \mathbf{0}) + \frac{\partial \alpha_i}{\partial \boldsymbol{\varepsilon}}(\mathbf{T}, \mathbf{0}) \cdot \boldsymbol{\varepsilon}$, $i = 0, 1, 2, ..., 8$, using the notation $\alpha_i^{(0)} = \alpha_i(\mathbf{T}, \mathbf{0})$ and $\boldsymbol{\zeta}_i = \frac{\partial \alpha_i}{\partial \boldsymbol{\varepsilon}}(\mathbf{T}, \mathbf{0})$ and neglecting terms of order δ^2 or higher, (9) becomes (see, for example, [145]):

$$\boldsymbol{\varpi}_1(\mathbf{T}) + \varpi_2(\mathbf{T})\boldsymbol{\varepsilon} + [\boldsymbol{\varpi}_3(\mathbf{T}) \cdot \boldsymbol{\varepsilon}]\mathbf{T} + [\boldsymbol{\varpi}_4(\mathbf{T}) \cdot \boldsymbol{\varepsilon}]\mathbf{T}^2 + \varpi_5(\mathbf{T})(\mathbf{T}\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}\mathbf{T}) + \varpi_6(\mathbf{T})(\mathbf{T}^2\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}\mathbf{T}^2) = \mathbf{0}, \tag{73}$$

where $\boldsymbol{\varpi}_i(\mathbf{T})$, $i = 1, 3, 4$ and $\varpi_j(\mathbf{T})$, $j = 2, 5, 6$ are vector and scalar functions, respectively, which only depend on the stress tensor and are defined in terms of $\alpha_k^{(0)}$ and $\boldsymbol{\zeta}_k$. We notice from equation (73) that the strains always appear as linear expressions, whereas the stresses can be arbitrarily large and can appear in a nonlinear manner. Eq. (73) can be rewritten as (in indicial notation and Cartesian coordinates)

$$\mathscr{A}_{ijkl}(\mathbf{T})\varepsilon_{kl} = \mathscr{B}_{ij}(\mathbf{T}). \tag{74}$$

If the fourth order tensor function $\boldsymbol{\mathscr{A}}$ has inverse then $\varepsilon_{ij} = \overset{-1}{\mathcal{A}}_{ijkl}(\mathbf{T})\mathscr{B}_{kl}(\mathbf{T})$, where $\overset{-1}{\mathscr{A}}_{ijkl}$ are the components i, j, k, l of $\boldsymbol{\mathscr{A}}^{-1}$. From (74) we obtain (see, for example, (66))

$$\boldsymbol{\varepsilon} = \mathfrak{h}(\mathbf{T}). \tag{75}$$

This subclass of constitutive equations is very useful in describing metallic alloys and even materials like concrete. Its applications are studied in more detail in Section 7.3 (see [157, 161]).

Another way to obtain (75) is to consider the subclass of constitutive equation (12): $\mathbf{B} = \bar{\alpha}_0\mathbf{I} + \bar{\alpha}_1\mathbf{T} + \bar{\alpha}_2\mathbf{T}^2$, where $\bar{\alpha}_i$ are only functions of the invariants of the stress (and do not depend on the density) and further, assuming that $|\nabla_{\mathbf{X}}\mathbf{u}| \sim O(\delta)$, $\delta \ll 1$ we once again obtain (75).

5.2.1 Isotropic Bodies

In the case $\mathfrak{h}(\mathbf{T})$ is an isotropic function we have the representation [18]

$$\boldsymbol{\varepsilon} = \vartheta_0 \mathbf{I} + \vartheta_1 \mathbf{T} + \vartheta_2 \mathbf{T}^2, \tag{76}$$

where the scalar functions ϑ_i, $i = 0, 1, 2$, depend on the invariants $I_1 = \operatorname{tr} \mathbf{T}$, $I_2 = \frac{1}{2}\operatorname{tr}(\mathbf{T}^2)$ and $I_3 = \frac{1}{3}\operatorname{tr}(\mathbf{T}^3)$.

Let us consider a subclass of (76), wherein we assume that there exists a Gibbs potential $\mathcal{G} = \mathcal{G}(\mathbf{T})$ such that[12] [17] $\mathfrak{h}(\mathbf{T}) = \frac{\partial \mathcal{G}}{\partial \mathbf{T}}$. In the case $\mathcal{G}(\mathbf{T})$ is an isotropic function $\mathcal{G}(\mathbf{T}) = \mathcal{G}(I_1, I_2, I_3)$, where these invariants have been defined above, we obtain for $\boldsymbol{\varepsilon}$ the expression (compare with (76))

$$\boldsymbol{\varepsilon} = \mathcal{G}_1 \mathbf{I} + \mathcal{G}_2 \mathbf{T} + \mathcal{G}_3 \mathbf{T}^2, \tag{77}$$

where $\mathcal{G}_i = \frac{\partial \mathcal{G}}{\partial I_i}$, $i = 1, 2, 3$. If instead of using the classical invariants I_i, $i = 1, 2, 3$ we use the principal stresses σ_i, $i = 1, 2, 3$ of $\mathbf{T}$, then for $\mathcal{G}(\mathbf{T})$ isotropic we have the alternative representation (see, for example, [32]) $\mathcal{G}(\mathbf{T}) = \mathcal{G}(\sigma_1, \sigma_2, \sigma_3)$ that must satisfy the symmetry conditions $\mathcal{G}(\sigma_1, \sigma_2, \sigma_3) = \mathcal{G}(\sigma_2, \sigma_1, \sigma_3) = \mathcal{G}(\sigma_1, \sigma_3, \sigma_2)$, and for $\boldsymbol{\varepsilon}$ we have the expression (see [32]):

$$\boldsymbol{\varepsilon} = \sum_{i=1}^{3} \frac{\partial \mathcal{G}}{\partial \sigma_i} \mathbf{t}^{(i)} \otimes \mathbf{t}^{(i)}, \tag{78}$$

where $\mathbf{t}^{(i)}$, $i = 1, 2, 3$ are the eigenvectors (principal directions) of $\mathbf{T}$. Finally, if ε_i, $i = 1, 2, 3$ are the principal strains of $\boldsymbol{\varepsilon}$ it is possible to show that

$$\varepsilon_i = \frac{\partial \mathcal{G}}{\partial \sigma_i}, \quad i = 1, 2, 3. \tag{79}$$

5.2.2 Transversely isotropic bodies

The case of a transversely isotropic function $\mathfrak{h}(\mathbf{T})$ has been briefly considered in [25] within the context of analyzing inextensible bodies, and has been also analyzed for the case of transversely isotropic rock (see Section 7.3.4). If ones uses the classical results of Rivlin and Spencer (see, for example, [202]), in the case $\mathfrak{h}(\mathbf{T})$ is a transversely isotropic function and if we assume the existence of a Gibbs potential $\mathcal{G} = \mathcal{G}(\mathbf{T}, \mathbf{a})$ where $\mathfrak{h}(\mathbf{T}) = \frac{\partial \mathcal{G}}{\partial \mathbf{T}}$, where $\mathbf{a}$ is the

[12] See [23] for a more detailed study on the nature of $\mathcal{G}$ for these problems.

preferred direction with respect to which the body is transversely isotropic, we have that $\mathcal{G}(\mathbf{T},\mathbf{a}) = \mathcal{G}(I_1, I_2, I_3, I_4, I_5)$, where I_i, $i = 1, 2, 3$ have been defined in Section 5.2.1, and $I_4 = \mathbf{a}\cdot(\mathbf{Ta})$, $I_5 = \mathbf{a}\cdot(\mathbf{T}^2\mathbf{a})$. It then follows that (compare with (30)):

$$\boldsymbol{\varepsilon} = \mathcal{G}_1\mathbf{I} + \mathcal{G}_2\mathbf{T} + \mathcal{G}_3\mathbf{T}^2 + \mathcal{G}_4\mathbf{a}\otimes\mathbf{a} + \mathcal{G}_5[\mathbf{a}\otimes(\mathbf{Ta}) + (\mathbf{Ta})\otimes\mathbf{a}], \tag{80}$$

where $\mathcal{G}_i = \frac{\partial\mathcal{G}}{\partial I_i}$, $i = 1, 2, 3, 4, 5$.

Shariff [191] has proposed a different sets of invariants for anisotropic functions of tensors and vectors. In the case of a function of the form $\mathcal{G}(\mathbf{T},\mathbf{a})$ he suggested the use of the spectral invariants σ_i, ζ_i, $i = 1, 2, 3$, where σ_i are the principal stresses of $\mathbf{T}$ and $\zeta_i = [\mathbf{a}\cdot\mathbf{t}^{(i)}]^2$, where $\mathbf{t}^{(i)}$, $i = 1, 2, 3$ are the eigenvectors of $\mathbf{T}$. One of the invariants ζ_i, $i = 1, 2, 3$ is not independent since $\zeta_1 + \zeta_2 + \zeta_3 = 1$, so for the set σ_i, ζ_i, $i = 1, 2, 3$ we would have five independent invariants. We have $\mathcal{G}(\mathbf{T}) = \mathcal{G}(\sigma_1, \sigma_2, \sigma_3, \zeta_1, \zeta_2, \zeta_3)$, which must satisfy the symmetry conditions $\mathcal{G}(\sigma_1, \sigma_2, \sigma_3, \zeta_1, \zeta_2, \zeta_3) = \mathcal{G}(\sigma_2, \sigma_1, \sigma_3, \zeta_2, \zeta_1, \zeta_3) = \mathcal{G}(\sigma_1, \sigma_3, \sigma_2, \zeta_1, \zeta_3, \zeta_2)$ and which must be independent of ζ_i if the principal values of $\mathbf{T}$ are all the same. For the components of the strain tensor we have (see [191])

$$\varepsilon_{ii} = \frac{\partial\mathcal{G}}{\partial\sigma_i}, \quad i = 1, 2, 3, \tag{81}$$

$$\varepsilon_{ij} = \left(\frac{\partial\mathcal{G}}{\partial\zeta_i} - \frac{\partial\mathcal{G}}{\partial\zeta_j}\right)\frac{\mathbf{t}^{(i)}\cdot(\mathbf{At}^{(j)})}{(\sigma_i - \sigma_j)}, \quad i \neq j, \quad i, j = 1, 2, \tag{82}$$

$$\varepsilon_{k3} = \frac{\partial\mathcal{G}}{\partial\zeta_k}\frac{\mathbf{t}^{(k)}\cdot(\mathbf{At}^{(3)})}{(\sigma_k - \sigma_3)}, \quad k = 1, 2, \tag{83}$$

where we have defined $\mathbf{A} = \mathbf{a}\otimes\mathbf{a}$, and where $\varepsilon_{ii} = \mathbf{t}^{(i)}\cdot(\boldsymbol{\varepsilon}\mathbf{t}^{(i)})$, $\varepsilon_{ij} = \mathbf{t}^{(i)}\cdot(\boldsymbol{\varepsilon}\mathbf{t}^{(j)})$, where in all the above expressions there is no sum with respect to the repeated index. In the case of (82), (83) we assume that as $\sigma_i \to \sigma_j$ and $\sigma_k \to \sigma_3$ the function $\mathcal{G}$ is regular enough such that the limits exist.

5.2.3 Bodies with two directions of anisotropy

For the case of a body with two directions of anisotropy, when $\mathfrak{h} = \frac{\partial\mathcal{G}}{\partial\mathbf{T}}$ and $\mathcal{G} = \mathcal{G}(\mathbf{T},\mathbf{a},\mathbf{b})$, where $\mathbf{b}$ is a unit tensor field, which must not be confused with the body force (see $(4)_1$). From [202] in the case of using the classical invariants of Rivlin and Spencer $\mathcal{G} = \mathcal{G}(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9)$, where I_1, I_2 and I_3 have been defined in Section 5.2.1, I_4, I_5 are defined in the previous

Section 5.2.2 and $I_6 = \mathbf{b} \cdot (\mathbf{Tb})$, $I_7 = \mathbf{b} \cdot (\mathbf{T}^2\mathbf{b})$, $I_8 = (\mathbf{a} \cdot \mathbf{b})\mathbf{a} \cdot (\mathbf{Tb})$ and $I_9 = (\mathbf{a} \cdot \mathbf{b})\mathbf{a} \cdot (\mathbf{T}^2\mathbf{b})$. Therefore, we have:

$$\begin{aligned}\boldsymbol{\varepsilon} &= \mathcal{G}_1\mathbf{I} + \mathcal{G}_2\mathbf{T} + \mathcal{G}_3\mathbf{T}^2 + \mathcal{G}_4\mathbf{a}\otimes\mathbf{a} + \mathcal{G}_5[\mathbf{a}\otimes(\mathbf{Ta}) + (\mathbf{Ta})\otimes\mathbf{a}] + \mathcal{G}_4\mathbf{b}\otimes\mathbf{b} \\ &\quad + \mathcal{G}_7[\mathbf{b}\otimes(\mathbf{Tb}) + (\mathbf{Tb})\otimes\mathbf{b}] + \mathcal{G}_8\frac{1}{2}(\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a}) \\ &\quad + \mathcal{G}_9\frac{1}{2}(\mathbf{a}\cdot\mathbf{b})[\mathbf{a}\otimes(\mathbf{Tb}) + (\mathbf{Tb})\otimes\mathbf{a} + \mathbf{b}\otimes(\mathbf{Ta}) + (\mathbf{Ta})\otimes\mathbf{b}],\end{aligned}$$

where $\mathcal{G}_i = \frac{\partial\mathcal{G}}{\partial I_i}$, $i = 1, 2, ..., 9$.

In the case of using the spectral invariants of Shariff [193] we have $\mathcal{G} = \mathcal{G}(\sigma_{1,2,3}, \zeta_{1,2,3}, \xi_{1,2,3}, \chi_{1,2,3})$, where $\sigma_{1,2,3}$ means the set $\sigma_1, \sigma_2, \sigma_3$ and the same notation is used for the other variables. In the expressions above we have defined $\zeta_i = \left(\mathbf{a}\cdot\mathbf{t}^{(i)}\right)^2$, $\xi_i = \left(\mathbf{b}\cdot\mathbf{t}^{(i)}\right)^2$, $\chi_i = (\mathbf{a}\cdot\mathbf{b})\left(\mathbf{a}\cdot\mathbf{t}^{(i)}\right)\left(\mathbf{b}\cdot\mathbf{t}^{(i)}\right)$ (above there is no sum in i). For $\mathcal{G}$ we have $3\times 4 = 12$ invariants, but not all of them are independent, since from [193] we have the conections $\zeta_1 + \zeta_2 + \zeta_3 = 1$, $\xi_1 + \xi_2 + \xi_3 = 1$, $\chi_1 + \chi_2 + \chi_3 = (\mathbf{a}\cdot\mathbf{b})^2$, $\chi_1^2 = (\mathbf{a}\cdot\mathbf{b})^2\zeta_1\xi_1$, $\chi_2^2 = (\mathbf{a}\cdot\mathbf{b})^2\zeta_2\xi_2$ and $\zeta_2\xi_2\zeta_3\xi_3(8\zeta_1\xi_1 + 4c)^2 = [c^2 + 4(\zeta_2\xi_2\zeta_3\xi_3 - \zeta_1\xi_1\zeta_2\xi_2 - \zeta_1\xi_1\zeta_3\xi_3)]^2$, where $c = (\mathbf{a}\cdot\mathbf{b})^2 - \sum_{i=1}^3 \zeta_i\xi_i$, therefore, we have $12 - 6 = 6$ independent variables. Using the above invariants in $\boldsymbol{\varepsilon} = \frac{\partial\mathcal{G}}{\partial\mathbf{T}}$ we obtain:

$$\varepsilon_{ii} = \frac{\partial\mathcal{G}}{\partial\sigma_i}, \quad i = 1, 2, 3, \tag{84}$$

$$\begin{aligned}\varepsilon_{ij} &= \left(\frac{\partial\mathcal{G}}{\partial\zeta_i} - \frac{\partial\mathcal{G}}{\partial\zeta_j}\right)\frac{\mathbf{t}^{(i)}\cdot\left(\mathbf{At}^{(j)}\right)}{(\sigma_i - \sigma_j)} + \left(\frac{\partial\mathcal{G}}{\partial\xi_i} - \frac{\partial\mathcal{G}}{\partial\xi_j}\right)\frac{\mathbf{t}^{(i)}\cdot\left(\mathbf{Bt}^{(j)}\right)}{(\sigma_i - \sigma_j)} \\ &\quad + \frac{1}{(\sigma_i - \sigma_j)}\left(\frac{\partial\mathcal{G}}{\partial\chi_i} - \frac{\partial\mathcal{G}}{\partial\chi_j}\right)\left[\mathbf{t}^{(i)}\cdot\left(\mathbf{ABt}^{(j)}\right) + \mathbf{t}^{(j)}\cdot\left(\mathbf{ABt}^{(i)}\right)\right],\end{aligned} \tag{85}$$

where $\mathbf{A} = \mathbf{a}\otimes\mathbf{a}$ and $\mathbf{B} = \mathbf{b}\otimes\mathbf{b}$ (this last tensor must not be confused with the left Cauchy-Green tensor and in (84), (85) the repetition of indices i, j does not mean sum in such indices).

5.2.4 Residually stressed bodies

In the case of residually stressed bodies from [33] the counterpart of the relations and equations presented in Section 4.4 are:

$$\boldsymbol{\varepsilon} = \mathfrak{f}(\mathbf{T}) \quad \text{s.t.} \quad \mathbf{0} = \mathfrak{f}(\mathbf{T}_\mathrm{R}), \quad \text{where} \quad \operatorname{div}\mathbf{T}_\mathrm{R} + \rho\mathbf{b} = \mathbf{0}, \tag{86}$$

$$\mathbf{T}_\mathrm{R}\mathbf{n} = \mathbf{0}, \quad \mathbf{x}\in\partial\boldsymbol{\kappa}_\mathrm{r}(\mathcal{B}). \tag{87}$$

Following [59] we consider the constitutive equation for $\mathcal{G}$ transversely isotropic (80), where from [94, 95] we have to consider $\mathbf{T}_{\rm R} = \alpha\mathbf{I} + \beta\mathbf{a}\otimes\mathbf{a}$ (see Section 4.4.1). Here we assume that in general $\mathcal{G} = \mathcal{G}(\mathbf{T}, \mathbf{x})$, $\alpha = \alpha(\mathbf{x})$, $\beta = \beta(\mathbf{x})$ and $\mathbf{a} = \mathbf{a}(\mathbf{x})$, i.e., the body can be inhomogeneous. Replacing (80) in $(86)_2$, using the notation $\bar{\mathcal{G}} = \mathcal{G}(\mathbf{T}_{\rm R}, \mathbf{x})$ we obtain

$$\mathbf{0} = \bar{\mathcal{G}}_1\mathbf{I} + \bar{\mathcal{G}}_2\mathbf{T}_{\rm R} + \bar{\mathcal{G}}_3\mathbf{T}_{\rm R}^2 + \bar{\mathcal{G}}_4\mathbf{a}\otimes\mathbf{a} + \bar{\mathcal{G}}_5[\mathbf{a}\otimes(\mathbf{T}_{\rm R}\mathbf{a}) + (\mathbf{T}_{\rm R}\mathbf{a})\otimes\mathbf{a}],$$

and on using $\mathbf{T}_{\rm R} = \alpha\mathbf{I} + \beta\mathbf{a}\otimes\mathbf{a}$ the above equation becomes

$$\mathbf{0} = \bar{\mathcal{G}}_1\mathbf{I} + \bar{\mathcal{G}}_2(\alpha\mathbf{I} + \beta\mathbf{a}\otimes\mathbf{a}) + \bar{\mathcal{G}}_3[\alpha^2\mathbf{I} + (2\alpha+\beta)\beta\mathbf{a}\otimes\mathbf{a}] + \bar{\mathcal{G}}_4\mathbf{a}\otimes\mathbf{a} + \bar{\mathcal{G}}_5 2(\alpha+\beta)\mathbf{a}\otimes\mathbf{a}.$$

If we demand that the above relation to hold for any vector $\mathbf{a}$ we obtain

$$\bar{\mathcal{G}}_1 + \alpha\bar{\mathcal{G}}_2 + \alpha^2\bar{\mathcal{G}}_3 = 0, \quad \beta\bar{\mathcal{G}}_2 + (2\alpha+\beta)\beta\bar{\mathcal{G}}_3 + \bar{\mathcal{G}}_4 + \bar{\mathcal{G}}_5 2(\alpha+\beta)\mathbf{a}\otimes\mathbf{a} = 0. \quad (88)$$

Finally using the above residual stress in $(86)_3$ assuming no body force we get $\mathrm{div}(\alpha\mathbf{I} + \beta\mathbf{a}\otimes\mathbf{a}) = \mathbf{0}$ that is equivalent to:

$$\mathrm{grad}\,\alpha + (\mathbf{a}\cdot\mathrm{grad}\,\beta)\mathbf{a} + (\mathrm{div}\,\mathbf{a})\beta\mathbf{a} = \mathbf{0}. \quad (89)$$

In (88) and (89) we have in general 5 equations, which can be used to find, for example, α, β and the 3 components of $\mathbf{a}$. Regarding the boundary conditions $\mathbf{T}_{\rm R}\mathbf{n} = \mathbf{0}$ we obtain

$$\alpha\mathbf{n} + \beta(\mathbf{a}\cdot\mathbf{n})\mathbf{a} = \mathbf{0}.$$

5.3 Electro-elastic bodies

In this section we study the constitutive relations proposed in Sections 4.1 and 4.2. In Section 5.3.1 we consider the relations proposed in [21] (see also the corrections presented in [34]), which were obtained not taking into account directly the laws of thermodynamic, while in Section 5.3.2 we start with (35), (36), where using the first law of thermodynamic, we obtain a class of electro-elastic solid, which satisfies automatically such laws.

5.3.1 On a first class of electro-elastic bodies

In the case of electro-elastic bodies from (32), (33) if we assume that $|\nabla_{\mathbf{X}}\mathbf{u}| \sim O(\delta)$, $\delta \ll 1$, following a procedure similar to that outlined in Section 5.2, it is possible to obtain the subclass (see [21])

$$\boldsymbol{\varepsilon} = \mathfrak{h}(\boldsymbol{\tau}, \mathbb{E}, \mathbb{D}), \quad \mathfrak{l}(\boldsymbol{\tau}, \mathbb{E}, \mathbb{D}) = \mathbf{0}, \quad (90)$$

where $(90)_2$ is an implicit vector relation but that does not depend on the strains.

If D_o is used to denote some characteristic value for the electric displacement, for example, the magnitude of the electric displacement near the electric saturation point for some materials, then assuming further that $\frac{1}{D_o}|\mathbb{D}| \sim O(\delta)$, $\delta \ll 1$, from $(90)_2$ we obtain

$$\mathbb{D} = \mathfrak{r}(\boldsymbol{\tau}, \mathbb{E}),$$

and as a result $\boldsymbol{\varepsilon} = \mathfrak{h}(\boldsymbol{\tau}, \mathbb{E})$.

Let us study in more detail the cases $|\nabla_{\mathbf{X}}\mathbf{u}| \sim O(\delta)$ and $\frac{1}{D_o}|\mathbb{D}| \sim O(\delta)$ where, $\delta \ll 1$, when the functions $\mathfrak{h}$ and $\mathfrak{r}$ are transversely isotropic, and where $\mathbf{a}$ is a preferred direction for the body, i.e., $\boldsymbol{\varepsilon} = \mathfrak{h}(\boldsymbol{\tau}, \mathbb{E}, \mathbf{a})$, $\mathbb{D} = \mathfrak{r}(\boldsymbol{\tau}, \mathbb{E}, \mathbf{a})$. This case is relevant to many technical problems since many electro-elastic materials present a direction where the behaviour is different. In this case we have the representations (in terms of the classical invariants [202]):

$$\begin{aligned}\boldsymbol{\varepsilon} = {} & \bar{\gamma}_0\mathbf{I} + \bar{\gamma}_1\boldsymbol{\tau} + \bar{\gamma}_2\boldsymbol{\tau}^2 + \bar{\gamma}_3\mathbb{E}\otimes\mathbb{E} + \bar{\gamma}_4[\mathbb{E}\otimes(\boldsymbol{\tau}\mathbb{E}) + (\boldsymbol{\tau}\mathbb{E})\otimes\mathbb{E}] \\ & + \bar{\gamma}_5[\mathbb{E}\otimes(\boldsymbol{\tau}^2\mathbb{E}) + (\boldsymbol{\tau}^2\mathbb{E})\otimes\mathbb{E}] + \bar{\gamma}_6\mathbf{a}\otimes\mathbf{a} + \bar{\gamma}_7[\mathbf{a}\otimes(\boldsymbol{\tau}\mathbf{a}) + (\boldsymbol{\tau}\mathbf{a})\otimes\mathbf{a}] \\ & + \bar{\gamma}_8[\mathbf{a}\otimes(\boldsymbol{\tau}^2\mathbf{a}) + (\boldsymbol{\tau}^2\mathbf{a})\otimes\mathbf{a}] + \bar{\gamma}_9(\mathbb{E}\cdot\mathbf{a})(\mathbf{a}\otimes\mathbb{E} + \mathbb{E}\otimes\mathbf{a}) \\ & + \bar{\gamma}_{10}[\mathbf{a}\otimes\cdot(\boldsymbol{\tau}\mathbb{E}) + (\boldsymbol{\tau}\mathbb{E})\otimes\mathbf{a} + (\boldsymbol{\tau}\mathbf{a})\otimes\mathbb{E} + \mathbb{E}\otimes(\boldsymbol{\tau}\mathbf{a})] \\ & + \bar{\gamma}_{11}[\mathbf{a}\otimes\cdot(\boldsymbol{\tau}^2\mathbb{E}) + (\boldsymbol{\tau}^2\mathbb{E})\otimes\mathbf{a} + (\boldsymbol{\tau}^2\mathbf{a})\otimes\mathbb{E} + \mathbb{E}\otimes(\boldsymbol{\tau}^2\mathbf{a})],\end{aligned} \tag{91}$$

and

$$\mathbb{D} = \bar{\psi}_1\mathbb{E} + \bar{\psi}_2\boldsymbol{\tau}\mathbb{E} + \bar{\psi}_3\boldsymbol{\tau}^2\mathbb{E} + \bar{\psi}_4\mathbf{a} + \bar{\psi}_5\boldsymbol{\tau}\mathbf{a} + \bar{\psi}_6\boldsymbol{\tau}^2\mathbf{a},$$

where the scalar functions $\bar{\gamma}_i$, $i = 0, 1, 2, ..., 11$ and $\bar{\psi}_j$, $j = 1, 2, ..., 6$ depend on the invariants obtained from $\boldsymbol{\tau}$, $\mathbb{E}$ and $\mathbf{a}$, which are not documented here but the interested reader can find in [21].

5.3.2 On a second class of electro-elastic bodies

From Section 4.2 assuming small gradient of the displacement field we have the approximations $\mathbf{S} \approx \boldsymbol{\tau}$, $\mathbf{E} \approx \boldsymbol{\varepsilon}$, $\mathbb{E}_{\mathrm{L}} \approx \mathbb{E}$ and $\mathbb{P}_{\mathrm{L}} \approx \mathbb{P}$ and from (35), (36) we obtain in index notation and Cartesian coordinates (see [37, 56]):

$$\mathrm{d}\mathbb{P}_i \approx \mathscr{M}_{0_{ijk}}\mathrm{d}\tau_{jk} + \mathscr{N}_{0_{ij}}\mathrm{d}\mathbb{E}_j, \quad \mathrm{d}\varepsilon_{ij} \approx \mathscr{T}_{0_{ijkl}}\mathrm{d}\tau_{jk} + \mathscr{U}_{0_{ijk}}\mathrm{d}\mathbb{E}_k, \tag{92}$$

where we have defined $\boldsymbol{\mathscr{M}}_0 = \boldsymbol{\mathscr{M}}(\boldsymbol{\tau}, \mathbf{0}, \mathbb{E}, \mathbb{P})$, $\boldsymbol{\mathscr{N}}_0 = \boldsymbol{\mathscr{N}}(\boldsymbol{\tau}, \mathbf{0}, \mathbb{E}, \mathbb{P})$, $\boldsymbol{\mathscr{T}}_0 = \boldsymbol{\mathscr{T}}(\boldsymbol{\tau}, \mathbf{0}, \mathbb{E}, \mathbb{P})$, $\boldsymbol{\mathscr{U}}_0 = \boldsymbol{\mathscr{U}}(\boldsymbol{\tau}, \mathbf{0}, \mathbb{E}, \mathbb{P})$, where $\boldsymbol{\mathscr{M}}$, $\boldsymbol{\mathscr{N}}$, $\boldsymbol{\mathscr{T}}$ and $\boldsymbol{\mathscr{U}}$ are defined in terms of the tensors $\boldsymbol{\mathscr{A}}$, $\boldsymbol{\mathscr{B}}$, $\boldsymbol{\mathscr{C}}$, $\boldsymbol{\mathscr{D}}$, $\boldsymbol{\mathscr{Q}}$, $\boldsymbol{\mathscr{R}}$ and $\boldsymbol{\mathscr{S}}$ presented in Section 4.2.1, and for brevity such long expressions are not shown here[13].

Let us study some subclasses of the above constitutive equations (92), and some alternative theories developed working with the case of small gradient of the displacement field from the beginning regarding the laws of thermodynamics.

The case the polarization field is 'small': Let us assume that besides the gradient of the displacement field being small, we assume that the norm of the polarization field $\mathbb{P}$ (divided by some characteristic value such that it is a dimensionless field) is very small, i.e., $|\mathbb{P}| \sim O(\delta)$, $\delta \ll 1$, then from the definition for $\boldsymbol{\tau}$ from Section 4.2.1 we have the approximation $\boldsymbol{\tau} \approx \boldsymbol{\sigma}$ (where $\boldsymbol{\sigma}$ is the notation used in this section for the Cauchy stress tensor) then from (92) we have $\mathrm{d}\mathbb{P}_i \approx \mathscr{M}_{0_{ijk}}(\boldsymbol{\sigma}, \mathbb{E}, \mathbf{0})\,\mathrm{d}\tau_{jk} + \mathscr{N}_{0_{ij}}(\boldsymbol{\sigma}, \mathbb{E}, \mathbf{0})\,\mathrm{d}\mathbb{E}_j$, $\mathrm{d}\varepsilon_{ij} \approx \mathscr{T}_{0_{ijkl}}(\boldsymbol{\sigma}, \mathbb{E}, \mathbf{0})\,\mathrm{d}\tau_{jk} + \mathscr{U}_{0_{ijk}}(\boldsymbol{\sigma}, \mathbb{E}, \mathbf{0})\,\mathrm{d}\mathbb{E}_k$. Here the Cauchy stress tensor $\boldsymbol{\sigma}$ is approximately symmetric.

An alternative formulation for the case of small gradient of the displacement field, assuming that the magnitude of the polarization can be arbitrary: Let us work with (34) from the beginning assuming that the gradient of the displacement field is small, then we replace $\mathbf{S}$ by $\boldsymbol{\tau}$, $\mathbb{E}_{\mathrm{L}}$ by $\mathbb{E}$ and $\mathbb{P}_{\mathrm{L}}$ by $\mathbb{P}$, assuming that $\psi = \psi(\boldsymbol{\tau}, \mathbb{E}, \mathbb{P})$ and (34) becomes (here $\rho_{\mathrm{r}} \approx \rho$)

$$-\rho\mathrm{tr}\left(\frac{\partial\psi}{\partial\boldsymbol{\tau}}\dot{\boldsymbol{\tau}}\right) + \mathrm{tr}\,(\boldsymbol{\tau}\dot{\boldsymbol{\varepsilon}}) - \left(\dot{\mathbb{P}} + \rho\frac{\partial\psi}{\partial\mathbb{E}}\right)\cdot\dot{\mathbb{E}} - \rho\frac{\partial\psi}{\partial\mathbb{P}}\cdot\dot{\mathbb{P}} = 0. \tag{93}$$

Let us assume that there exists a scalar potential $\Pi = \Pi(\boldsymbol{\tau}, \mathbb{E}, \mathbb{P})$ such that $\boldsymbol{\varepsilon} = \frac{\partial\Pi}{\partial\boldsymbol{\tau}}$ and replacing this in (93) we get (in index notation and Cartesian coordinates)

$$\left(\frac{\partial^2\Pi}{\partial\tau_{ij}\partial\tau_{kl}}\tau_{ij} - \rho\frac{\partial\psi}{\partial\tau_{kl}}\right)\dot{\tau}_{kl} + \left(\frac{\partial^2\Pi}{\partial\tau_{ij}\partial\mathbb{E}_k}\tau_{ij} - \mathbb{P}_k - \rho\frac{\partial\psi}{\partial\mathbb{E}_k}\right)\dot{\mathbb{E}}_k + \left(\frac{\partial^2\Pi}{\partial\tau_{ij}\partial\mathbb{P}_k}\tau_{ij} - \rho\frac{\partial\psi}{\partial\mathbb{P}_k}\right)\dot{\mathbb{P}}_k = 0.$$

[13] The interested reader can see Section 4.1 of [37] and also the corrections presented in [56], in particular regarding the use of the tensor $\boldsymbol{\tau}$ instead of $\boldsymbol{\sigma}$ for several expressions shown in Section 4.1 of [37].

If we look for Π and ψ that satisfy the above equation for any $\dot{\boldsymbol{\tau}}$, $\dot{\mathbb{E}}$ and $\dot{\mathbb{P}}$ we find constitutive relations for an electro-elastic body, wherein the gradient of the displacement field is small and where the norm of the polarization field can be arbitrarily large. Let us show an example for $\boldsymbol{\varepsilon} = \frac{\partial \Pi}{\partial \boldsymbol{\tau}}$ in the case Π is an isotropic function considering the invariants of Spencer [202], $\Pi = \Pi(I_1, I_2, ..., I_{11})$ (for the definitions of such invariants see [37, 56]) we have

$$\begin{aligned}\boldsymbol{\varepsilon} &= \Pi_1 \mathbf{I} + \Pi_2 \boldsymbol{\tau} + \Pi_3 \boldsymbol{\tau}^2 + \Pi_5 \mathbb{E} \otimes \mathbb{E} + \Pi_6 [\mathbb{E} \otimes (\boldsymbol{\tau}\mathbb{E}) + (\boldsymbol{\tau}\mathbb{E}) \otimes \mathbb{E}] \\ &\quad + \Pi_8 \mathbb{P} \otimes \mathbb{P} + \Pi_9 [\mathbb{P} \otimes (\boldsymbol{\tau}\mathbb{P}) + (\boldsymbol{\tau}\mathbb{P}) \otimes \mathbb{P}] + \Pi_{11} (\mathbb{E} \cdot \mathbb{P})(\mathbb{E} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{E}),\end{aligned}$$

where $\Pi_i = \frac{\partial \Pi}{\partial I_i}$, $i = 1, 2, ..., 11$.

A formulation for the case of small gradient of the displacement field and small polarization: Here we assume that the norms of gradient of the displacement field and the polarization (which has been divided by some characteristic value) are small, then $\boldsymbol{\tau} \approx \boldsymbol{\sigma}$ and assuming that $\psi = \psi(\boldsymbol{\sigma}, \mathbb{E})$ the first law of thermodynamics (93) becomes

$$-\rho \operatorname{tr} \left(\frac{\partial \psi}{\partial \boldsymbol{\sigma}} \dot{\boldsymbol{\sigma}} \right) + \operatorname{tr} (\boldsymbol{\sigma} \dot{\boldsymbol{\varepsilon}}) - \left(\dot{\mathbb{P}} + \rho \frac{\partial \psi}{\partial \mathbb{E}} \right) \cdot \dot{\mathbb{E}} = 0. \tag{94}$$

Let us assume there exist scalar potentials $\Pi = \Pi(\boldsymbol{\sigma}, \mathbb{E})$ and $\Upsilon = \Upsilon(\boldsymbol{\sigma}, \mathbb{E})$ such that $\boldsymbol{\varepsilon} = \frac{\partial \Pi}{\partial \boldsymbol{\sigma}}$ and $\mathbb{P} = \frac{\partial \Upsilon}{\partial \mathbb{E}}$, then replacing them into (94) we obtain the equations $\rho \psi(\boldsymbol{\sigma}, \mathbb{E}) = \operatorname{tr} \left[\boldsymbol{\sigma} \frac{\partial \Pi}{\partial \boldsymbol{\sigma}} (\boldsymbol{\sigma}, \mathbb{E}) \right] - \Pi(\boldsymbol{\sigma}, \mathbb{E}) + \wp(\mathbb{E})$ and $\Upsilon(\boldsymbol{\sigma}, \mathbb{E}) = \Pi(\boldsymbol{\sigma}, \mathbb{E}) - \ell(\boldsymbol{\sigma}) + \wp(\mathbb{E})$, from where we can express, for example, ψ and Υ in terms of Π. In the particular case that $\wp = 0$ we have that $\frac{\partial \Upsilon}{\partial \mathbb{E}} = \frac{\partial \Pi}{\partial \mathbb{E}}$.

If Π is a transversely isotropic function such that $\Pi = \Pi(\boldsymbol{\sigma}, \mathbb{E}, \mathbf{a})$, where $\mathbf{a}$ is a unit vector field, considering the invariants by Spencer we have $\Pi = \Pi(I_1, I_2, ..., I_{10})$ (those invariants can be found in Section 5 of [37] and for brevity are not presented here), assuming that $\ell = 0$ and $\wp = 0$ above, we obtain the constitutive equations (compare with (91))

$$\begin{aligned}\boldsymbol{\varepsilon} &= \Pi_1 \mathbf{I} + \Pi_2 \boldsymbol{\sigma} + \Pi_3 \boldsymbol{\sigma}^2 + \Pi_4 \mathbb{E} \otimes \mathbb{E} + \Pi_6 [\mathbb{E} \otimes (\boldsymbol{\sigma}\mathbb{E}) + (\boldsymbol{\sigma}\mathbb{E}) \otimes \mathbb{E}] + \Pi_7 \mathbf{a} \otimes \mathbf{a} \\ &\quad + \Pi_8 [\mathbf{a} \otimes (\boldsymbol{\sigma}\mathbf{a}) + (\boldsymbol{\sigma}\mathbf{a}) \otimes \mathbf{a}] + \Pi_{10} (\mathbb{E} \cdot \mathbf{a})(\mathbb{E} \otimes \mathbf{a} + \mathbf{a} \otimes \mathbb{E}),\end{aligned} \tag{95}$$

$$\mathbb{P} = 2\Pi_4 \mathbb{E} + 2\Pi_5 \boldsymbol{\sigma}\mathbb{E} + 2\Pi_6 \boldsymbol{\sigma}^2 \mathbb{E} + 2\Pi_9 (\mathbb{E} \cdot \mathbf{a})\mathbf{a} + \Pi_{10} \{[\mathbb{E} \cdot (\boldsymbol{\sigma}\mathbf{a})]\mathbf{a} + (\mathbb{E} \cdot \mathbf{a})\boldsymbol{\sigma}\mathbf{a}\}, \tag{96}$$

where $\Pi_i = \frac{\partial \Pi}{\partial I_i}$, $i = 1, 2, ..., 10$.

5.4 Thermo-elastic bodies

Here we repeat the analysis of the two previous sections in the case of thermoelastic bodies (see Section 4.3). On assuming that $|\nabla_{\mathbf{X}}\mathbf{u}| \sim O(\delta)$, $\delta \ll 1$, we have $\mathbf{E} \approx \boldsymbol{\varepsilon}$, $\mathbf{S} \approx \mathbf{T}$, $\mathbf{Q}_{\mathrm{L}} \approx \mathbf{Q}$, $\dot{\mathbf{Q}}_{\mathrm{L}} \approx \dot{\mathbf{Q}}$ and $\nabla_{\mathbf{X}}\theta \approx \nabla\theta$, where $\mathbf{Q}$ is the heat flux in the current configuration, and ∇ is the gradient operator with respect to the current configuration. From (41), (46) we obtain (see [31])

$$\boldsymbol{\varepsilon} = \mathfrak{h}(\mathbf{T}, \theta), \quad \mathfrak{m}(\mathbf{T}, \mathbf{Q}, \dot{\mathbf{Q}}, \nabla\theta, \theta) = \mathbf{0}. \tag{97}$$

In the case $\mathfrak{h}$ is an isotropic function the expression for $(97)_1$ is the same as that in (76), (77), but in this case the functions ϑ_i or $\mathcal{G}$ would also depend on θ. In the case $(97)_2$ being isotropic we have the representation

$$\breve{\beta}_1\mathbf{Q} + \breve{\beta}_2\mathbf{T}\mathbf{Q} + \breve{\beta}_3\mathbf{T}^2\mathbf{Q} + \breve{\beta}_4\dot{\mathbf{Q}} + \breve{\beta}_5\mathbf{T}\dot{\mathbf{Q}} + \breve{\beta}_6\mathbf{T}^2\dot{\mathbf{Q}} + \breve{\beta}_7\nabla\theta + \breve{\beta}_8\mathbf{T}\nabla\theta + \breve{\beta}_9\mathbf{T}^2\nabla\theta = \mathbf{0}, \tag{98}$$

where the scalar functions $\breve{\beta}_i$, $i = 1, 2, ..., 9$ depend on some of the invariants defined in terms of $\mathbf{T}$, $\mathbf{Q}$, $\dot{\mathbf{Q}}$ and $\nabla\theta$, which for the sake of brevity are not shown here. The following subclass of (97) is also interesting. Let us define Q and Q_d as characteristic values for $\mathbf{Q}$ and $\dot{\mathbf{Q}}$, respectively, let us assume that $\frac{1}{Q}|\mathbf{Q}| \sim O(\delta)$ and $\frac{1}{Q_d}|\dot{\mathbf{Q}}| \sim O(\delta)$, where $\delta \ll 1$, it is possible to show that from (98) we obtain

$$\mathbf{Q} + \alpha(\mathbf{T}, \theta)\dot{\mathbf{Q}} = -\mathfrak{q}(\mathbf{T}, \nabla\theta, \theta), \tag{99}$$

where the scalar function α may also depend on $\nabla\theta$. The above subclass of (97) is a generalization of models for heat transfer, which has been proposed in [98], which does not present some of the problems that the classical Fourier's model for heat transfer does, in particular the infinite speed of propagation for heat. The classical Fourier's model is obtained from (99) in the case $\alpha = 0$ and $\mathfrak{q} = \mathfrak{q}(\nabla\theta) = k\nabla\theta$, where k is a constant.

The new classes of constitutive relations and equations for thermo-elastic bodies (97), (98) and (99) have not been yet applied for the modelling of some real solids. In Section 9.4 one boundary value problem is shown for a class of constitutive relation considering large elastic deformations (see [60]). Different boundary value problems have been studied in that paper, but for the sake of brevity several of such results are not shown in Sections 8 and 9. For the case of small gradient of the displacement field, wherein $(97)_1$ is valid, in [224] the following expression is considered for $\mathfrak{h}$ isotropic:

$$\boldsymbol{\varepsilon} = \mathfrak{h}(\mathbf{T}, \theta) = \vartheta_0\mathbf{I} + \vartheta_1\mathbf{T} + \vartheta_2\mathbf{T}^2, \tag{100}$$

where $\vartheta_i = \vartheta_i(I_1, I_2, I_3, \theta)$, $i = 0, 1, 2$, and $I_1 = \operatorname{tr}\mathbf{T}$, $I_2 = \frac{1}{2}\operatorname{tr}(\mathbf{T}^2)$ and $I_3 = \frac{1}{3}\operatorname{tr}(\mathbf{T}^3)$. In [224] expressions for ϑ_i such that the body shows a strain limiting behaviour are studied. In the same paper (100) is inverted to obtain the stress as a function of the linearized strain and temperature (the above is just a mathematical device, and it does not have a physical meaning, see [169]), and that inverted constitutive equation is used in a weak formulation in terms of the displacement field, to use it as basis for the implementation of the finite element method. In that paper the issue of existence and uniqueness is addressed. Some boundary value problems are solved, in particular the study of the behaviour of plane plates with and without cracks. The predictions of (100) are compared with the predictions of the linearized theory of elasticity.

6 Constraints

In this section we discuss kinematic constraints within the context of the new constitutive theories reviewed in this paper. In the case of a fully implicit relation, of the form, for example, (8), it is possible to incorporate the constraint of incompressibility $\det \mathbf{F} = 1$ which is equivalent to $\det \mathbf{B} = 1$, by decomposing the stress into a spherical plus a deviatoric part as $\mathbf{T} = \frac{\operatorname{tr}\mathbf{T}}{3}\mathbf{I} + \mathbf{T}_\mathrm{D}$, where $\mathbf{T}_\mathrm{D} = \mathbf{T} - \frac{\operatorname{tr}\mathbf{T}}{3}\mathbf{I}$, requiring that $\boldsymbol{\mathfrak{G}}(\mathbf{T}, \mathbf{B})$ would not be influenced by the spherical part of the stress $\frac{\operatorname{tr}\mathbf{T}}{3}\mathbf{I}$, i.e., to impose restrictions on $\boldsymbol{\mathfrak{G}}(\mathbf{T}, \mathbf{B})$ such that (8) would be equivalent to $\boldsymbol{\mathfrak{G}}(\mathbf{T}_\mathrm{D}, \mathbf{B}) = 0$. Regarding the condition $\det \mathbf{B} = 1$, further restrictions on $\boldsymbol{\mathfrak{G}}$ could be applied such that any deformation for the class of models defined through $\boldsymbol{\mathfrak{G}}(\mathbf{T}_\mathrm{D}, \mathbf{B}) = 0$ would be isochoric[14].

In the case of large elastic deformations, where some measure of the strain is given as a function of a stress (see (12), (21) and (75)), there are two options, for example, in the case of the subclass (12), namely $\mathbf{B} = \boldsymbol{\mathfrak{g}}(\mathbf{T})$, the constraint of incompressibility can be incorporated directly into such a constitutive equation (see [29]) obtaining $\det(\boldsymbol{\mathfrak{g}}(\mathbf{T})) = 1$, which is a direct restriction on $\boldsymbol{\mathfrak{g}}$. In [29] such a restriction was studied for the particular case when $\boldsymbol{\mathfrak{g}}$ is isotropic and is obtained from some scalar potential. The expression for $\boldsymbol{\mathfrak{g}}$ obtained in such a way does not depend on a spherical stress $\sigma_\mathrm{S} = -\frac{\operatorname{tr}\mathbf{T}}{3}$, i.e., $\boldsymbol{\mathfrak{g}}(\sigma_\mathrm{S}\mathbf{I}) = \mathbf{I}$, however, it has not been possible so far to

[14]See, for example, [83], Sections 6.2.1 and 6.3.1 of [80] and Section 3 of [203] for a treatment of incompressibility within the context of a class of isotropic implicit body.

show that $\mathfrak{g}\left(\frac{\mathrm{tr}\,\mathbf{T}}{3}\mathbf{I} + \mathbf{T}_\mathrm{D}\right) = \mathfrak{g}(\mathbf{T}_\mathrm{D})$. The model proposed in [203] (see (21) and (22)) does not present such a problem, and with the use of the Kirchhoff stress $\boldsymbol{\tau}$ and the Hencky strain $\boldsymbol{\eta} = \ln \mathbf{V} = \frac{1}{2}\ln \mathbf{B}$, for the particular case of incompressible bodies and using the Gibbs potential $\mathcal{G}$, it is possible to obtain the explicit expression $\mathrm{dev}(\ln \mathbf{V}) = \frac{\partial \mathcal{G}}{\partial \mathbf{T}_\mathrm{D}}$, where the potential $\mathcal{G}$ only depends on the deviatoric part of the stress tensor (here dev means the deviatoric part of $\ln \mathbf{V}$). In Section 6.2 we show some results from [36], where the Hencky strain tensor is a function of the Cauchy stress tensor and we find some restriction of $\mathcal{G}$ such that the body is incompressible.

6.1 Incompressibility in the case of assuming small gradient of the displacement field (small strains)

In the case of small strains the incompressibility constraint reads $\mathrm{tr}\,\boldsymbol{\varepsilon} = 0$, and replacing the subclass of the model presented in (77) into that equation we obtain the first order linear partial differential equation

$$\frac{\partial \mathcal{G}}{\partial I_1} + I_1 \frac{\partial \mathcal{G}}{\partial I_2} + 2 I_2 \frac{\partial \mathcal{G}}{\partial I_3} = 0, \tag{101}$$

whose solution is of the form (see [30]) $\mathcal{G} = \bar{\mathcal{G}}(\bar{I}_1, \bar{I}_2)$, where $\bar{I}_1 = I_2 - \frac{I_1^2}{6}$, $\bar{I}_2 = I_3 + \frac{2}{27} I_1^3 - \frac{2}{3} I_1 I_2$, from which we obtain

$$\boldsymbol{\varepsilon} = \left(\mathbf{T} - \frac{I_1}{3}\mathbf{I}\right)\frac{\partial \bar{\mathcal{G}}}{\partial \bar{I}_1} + \left[2\left(\frac{I_1^2}{9} - \frac{I_2}{3}\right)\mathbf{I} - \frac{2I_1}{3}\mathbf{T} + \mathbf{T}^2\right]\frac{\partial \bar{\mathcal{G}}}{\partial \bar{I}_2}. \tag{102}$$

In [30] it was proved that if the stress is decomposed as $\mathbf{T} = \frac{(\mathrm{tr}\,\mathbf{T})}{3}\mathbf{I} + \mathbf{T}_\mathrm{D}$, where $\mathbf{T}_\mathrm{D}$ is the deviatoric stress, then $\boldsymbol{\varepsilon} = \mathfrak{h}(\mathbf{T}) = \mathfrak{h}(\mathbf{T}_\mathrm{D})$. Therefore, we have a constitutive equation that satisfies the constraint automatically, and which is not affected by the spherical part of the stress.

Let us study the constraint $\mathrm{tr}\,\boldsymbol{\varepsilon} = 0$ for the case of (78), which is an alternative representation for $\mathfrak{h}$ for isotropic bodies. In such a case $\mathrm{tr}\,\boldsymbol{\varepsilon} = 0$ becomes $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$, and substituting (79) we obtain the first order linear partial differential equation

$$\frac{\partial \mathcal{G}}{\partial \sigma_1} + \frac{\partial \mathcal{G}}{\partial \sigma_2} + \frac{\partial \mathcal{G}}{\partial \sigma_3} = 0,$$

whose solution is $\mathcal{G} = \bar{\mathcal{G}}(\sigma_1 - \sigma_2, \sigma_1 - \sigma_3) + \bar{\mathcal{G}}(\sigma_2 - \sigma_1, \sigma_2 - \sigma_3) + \bar{\mathcal{G}}(\sigma_3 - \sigma_1, \sigma_3 - \sigma_2)$, where we have the additional restriction $\bar{\mathcal{G}}(x, y) = \bar{\mathcal{G}}(y, x)$.

6.2 Incompressibility in the case of large elastic deformations

In the case of $(21)_2$ we have $\ln J = \mathrm{tr}\boldsymbol{\eta}$, then if $J = 1$ we have that $\mathrm{tr}\boldsymbol{\eta} = 0$, and from $\boldsymbol{\tau} = J\mathbf{T}$ and $(21)_1$ we have $\boldsymbol{\tau} = \mathbf{T}$ thus $\boldsymbol{\eta} = \frac{\partial \mathcal{G}}{\partial \mathbf{T}}$, as a result for incompressible solids we have the restriction $\mathrm{tr}\left(\frac{\partial \mathcal{G}}{\partial \mathbf{T}}\right) = 0$. For $\mathcal{G}$ isotropic let us choose the set of invariants $I_1 = \mathrm{tr}\,\mathbf{T}$, $I_2 = \frac{1}{2}\mathrm{tr}\,(\mathbf{T}^2)$ and $I_3 = \frac{1}{3}\mathrm{tr}\,(\mathbf{T}^3)$, such that $\mathcal{G} = \mathcal{G}(I_1, I_2, I_3)$, then $\boldsymbol{\eta} = \mathcal{G}_1\mathbf{I} + \mathcal{G}_2\mathbf{T} + \mathcal{G}_3\mathbf{T}^2$, where $\mathcal{G}_i = \frac{\partial \mathcal{G}}{\partial I_i}$, $i = 1, 2, 3$. Using the above in $\mathrm{tr}\left(\frac{\partial \mathcal{G}}{\partial \mathbf{T}}\right) = 0$ we obtain the same first order linear partial differential equation (101): $\frac{\partial \mathcal{G}}{\partial I_1} + I_1\frac{\partial \mathcal{G}}{\partial I_2} + 2I_2\frac{\partial \mathcal{G}}{\partial I_3} = 0$, whose solution is[15] (see [36])

$$\mathcal{G} = \hat{\mathcal{G}}(I_{\mathrm{D}_2}, I_{\mathrm{D}_3}), \quad \text{where} \quad I_{\mathrm{D}_2} = \frac{1}{2}\mathrm{tr}\left(\mathbf{T}_{\mathrm{D}}^2\right), \quad I_{\mathrm{D}_3} = \frac{1}{3}\mathrm{tr}\left(\mathbf{T}_{\mathrm{D}}^3\right),$$

and using this in $(21)_1$ $\boldsymbol{\eta} = \frac{\partial \mathcal{G}}{\partial \mathbf{T}}$ we get (see [36])

$$\boldsymbol{\eta} = \alpha_0\mathbf{I} + \alpha_1\mathbf{T}_{\mathrm{D}} + \alpha_2\mathbf{T}_{\mathrm{D}}^2, \tag{103}$$

where α_i are defined as $\alpha_0 = -\frac{2I_{\mathrm{D}_2}}{3}\frac{\partial \hat{\mathcal{G}}}{\partial I_{\mathrm{D}_3}}$, $\alpha_1 = \frac{\partial \hat{\mathcal{G}}}{\partial I_{\mathrm{D}_2}}$ and $\alpha_2 = \frac{\partial \hat{\mathcal{G}}}{\partial I_{\mathrm{D}_3}}$.

In [45] incompressible bodies were studied for $(21)_1$, using the principal values of $\boldsymbol{\tau}$ and (24) (see Section 3.3.1). As mentioned previously in the case of incompressible bodies $J = 1$ thus $\boldsymbol{\tau} = \mathbf{T}$ and the principal values τ_i mentioned in Section 3.3.1 become the principal stresses of $\mathbf{T}$, which here are denoted σ_i, then from (24) we have

$$\ln \lambda_i = \frac{\partial \mathcal{G}}{\partial \sigma_i}, \quad i = 1, 2, 3, \tag{104}$$

where we recall that λ_i are the principal stretches and that $\mathcal{G} = \mathcal{G}(\sigma_1, \sigma_2, \sigma_3)$. That function must satisfy the restriction $\mathcal{G}(\sigma_1, \sigma_2, \sigma_3) = \mathcal{G}(\sigma_2, \sigma_1, \sigma_3) = \mathcal{G}(\sigma_1, \sigma_3, \sigma_2) = \mathcal{G}(\sigma_3, \sigma_2, \sigma_1)$. Since the body is incompressible $\mathrm{tr}\boldsymbol{\eta} = 0$, which is equivalent to $\ln \lambda_1 + \ln \lambda_2 + \ln \lambda_3 = 0$ and from (104) we obtain the first order partial differential equation for $\mathcal{G}$: $\frac{\partial \mathcal{G}}{\partial \sigma_1} + \frac{\partial \mathcal{G}}{\partial \sigma_2} + \frac{\partial \mathcal{G}}{\partial \sigma_3} = 0$, whose solution is:

$$\mathcal{G} = \breve{\mathcal{G}}(\sigma_1 - \sigma_2, \sigma_1 - \sigma_3) + \breve{\mathcal{G}}(\sigma_2 - \sigma_1, \sigma_2 - \sigma_3) + \breve{\mathcal{G}}(\sigma_3 - \sigma_1, \sigma_3 - \sigma_2), \tag{105}$$

[15] The solution below is the same as the solution that leads to (102), see [30].

where the function $\breve{\mathcal{G}}$ must satisfy $\breve{\mathcal{G}}(\alpha,\beta) = \breve{\mathcal{G}}(\beta,\alpha)$. If σ_{D_i} are used to denote the principal values of the deviatoric stress $\mathbf{T}_{\mathrm{D}}$ as defined above, then it is easy to show that $\sigma_i - \sigma_j = \sigma_{\mathrm{D}_i} - \sigma_{\mathrm{D}_j}$ thus $\breve{\mathcal{G}}$ can be written solely in terms of the principal values of $\mathbf{T}_{\mathrm{D}}$. Using (105) in (104) we get:

$$\ln\lambda_1 = \frac{\partial\breve{\mathcal{G}}}{\partial\alpha}(\sigma_1-\sigma_2,\sigma_1-\sigma_3) + \frac{\partial\breve{\mathcal{G}}}{\partial\beta}(\sigma_1-\sigma_2,\sigma_1-\sigma_3) - \frac{\partial\breve{\mathcal{G}}}{\partial\alpha}(\sigma_2-\sigma_1,\sigma_2-\sigma_3) - \frac{\partial\breve{\mathcal{G}}}{\partial\alpha}(\sigma_3-\sigma_1,\sigma_3-\sigma_2), \tag{106}$$

$$\ln\lambda_2 = -\frac{\partial\breve{\mathcal{G}}}{\partial\alpha}(\sigma_1-\sigma_2,\sigma_1-\sigma_3) + \frac{\partial\breve{\mathcal{G}}}{\partial\alpha}(\sigma_2-\sigma_1,\sigma_2-\sigma_3) - \frac{\partial\breve{\mathcal{G}}}{\partial\beta}(\sigma_2-\sigma_1,\sigma_2-\sigma_3) - \frac{\partial\breve{\mathcal{G}}}{\partial\beta}(\sigma_3-\sigma_1,\sigma_3-\sigma_2), \tag{107}$$

$$\ln\lambda_3 = -\frac{\partial\breve{\mathcal{G}}}{\partial\beta}(\sigma_1-\sigma_2,\sigma_1-\sigma_3) - \frac{\partial\breve{\mathcal{G}}}{\partial\beta}(\sigma_2-\sigma_1,\sigma_2-\sigma_3) + \frac{\partial\breve{\mathcal{G}}}{\partial\alpha}(\sigma_3-\sigma_1,\sigma_3-\sigma_2) + \frac{\partial\breve{\mathcal{G}}}{\partial\beta}(\sigma_3-\sigma_1,\sigma_3-\sigma_2). \tag{108}$$

6.3 On a class of incompressible isotropic thermo-elastic bodies

In this section we show some results presented in [60] for a class of isotropic thermo-elastic body, which is a subclass of what has been shown in Section 4.3, and that corresponds to a type of incompressible solid similar to what has been described in the previous section. From the first and second laws of thermodynamics formulated in terms of the Kirchhoff stress $\boldsymbol{\tau}$ and the Hencky strain tensor $\boldsymbol{\eta}$, proposing the Gibbs potential $\mathcal{G}$ as $U = -\mathcal{G}(\boldsymbol{\tau},\theta) + \theta s + \mathrm{tr}(\boldsymbol{\tau}\boldsymbol{\eta})$ (where U is the internal energy, θ is the absolute temperature and s the specific entropy) from [60] we have

$$\boldsymbol{\eta} = \frac{\partial\mathcal{G}}{\partial\boldsymbol{\tau}}, \quad s = \frac{\partial\mathcal{G}}{\partial\theta}, \quad -\mathbf{Q}\cdot\theta\mathrm{grad}\left(\frac{1}{\theta}\right) \geqslant 0, \tag{109}$$

where we recall that $\mathbf{Q}$ is the heat flux in the current configuration. In this case the first law of thermodynamics becomes $\rho\theta\dot{s} = \mathrm{div}\mathbf{Q} + \rho r$, where

r is the internal heat production, and from (42) (rewritten in the current configuration) we obtain:

$$\rho\theta \left(\frac{\partial^2 \mathcal{G}}{\partial\theta\partial\boldsymbol{\tau}} \cdot \dot{\boldsymbol{\tau}} + \frac{\partial^2 \mathcal{G}}{\partial\theta^2}\dot{\theta} \right) = \operatorname{div}\mathbf{Q} + \rho r. \tag{110}$$

In [60] the Fourier's model has been assumed $\mathbf{Q} = \zeta\nabla\theta$, where ζ is a positive constant.

Eq. $(109)_1$ is only valid for $\mathcal{G}$ isotropic, then $\mathcal{G} = \mathcal{G}(I_1, I_2, I_3, \theta)$, from where we obtain the same constitutive equation (23). From different experimental data (see for example the introduction in [60] and the references cited therein) it is possible to see that for incompressible bodies $J = f(\theta)$, i.e., the body is incompressible for isothermal processes but the specific volume depends on the temperature[16]. Using (23) in $J = f(\theta) \Leftrightarrow \operatorname{tr}\boldsymbol{\eta} = \ln J = \ln[f(\theta)]$ we obtain the first order partial differential equation (see Section 6.2) $\ln[f(\theta)] = 3\mathcal{G}_1 + \mathcal{G}_2 I_1 + 2\mathcal{G}_3 I_2$, whose solution is:

$$\mathcal{G}(I_1, I_2, I_3, \theta) = \ln[f(\theta)]\frac{I_1}{3} + \hat{\mathcal{G}}(I_{\mathrm{D}_2}, I_{\mathrm{D}_3}, \theta), \tag{111}$$

where $I_1 = \operatorname{tr}\boldsymbol{\tau} = J\operatorname{tr}\mathbf{T}$, I_{D_2}, I_{D_3} are defined as $I_{\mathrm{D}_2} = \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau}_{\mathrm{D}}^2)$, $I_{\mathrm{D}_3} = \frac{1}{3}\operatorname{tr}(\boldsymbol{\tau}_{\mathrm{D}}^3)$ and $\boldsymbol{\tau}_{\mathrm{D}}$ is defined from $\boldsymbol{\tau} = J\mathbf{T} = -J\sigma_{\mathrm{S}}\mathbf{I} + \boldsymbol{\tau}_{\mathrm{D}}$ (with $\operatorname{tr}\boldsymbol{\tau}_{\mathrm{D}} = 0$). Using (111) above in (23) we obtain the constitutive relation (compare with (103)):

$$\boldsymbol{\eta} = \frac{\ln[f(\theta)]}{3}\mathbf{I} + \alpha_0\mathbf{I} + \alpha_1\boldsymbol{\tau}_{\mathrm{D}} + \alpha_2\boldsymbol{\tau}_{\mathrm{D}}^2, \tag{112}$$

where $\alpha_0 = -\frac{2I_{\mathrm{D}_2}}{3}\frac{\partial\hat{\mathcal{G}}}{\partial I_{\mathrm{D}_3}}$, $\alpha_1 = \frac{\partial\hat{\mathcal{G}}}{\partial I_{\mathrm{D}_2}}$ and $\alpha_2 = \frac{\partial\hat{\mathcal{G}}}{\partial I_{\mathrm{D}_3}}$.

One boundary problem considering (110) is analyzed in Section 9.4.

6.4 Inextensibility in the case of small strains

Let us consider a transversely isotropic body, which in the direction $\mathbf{a}$ is inextensible, then the constraint reads $\mathbf{a} \cdot (\boldsymbol{\varepsilon}\mathbf{a}) = 0$. It happens that such

[16] From the same experimental data we can see that in reality J also depends (slightly) on the spherical part of the Cauchy stress σ_{S} (called pressure in many other works), although only very large changes in such spherical part of the stress can have a noticeable effect on J. In [60] it is assumed that $J = f(\theta)$ neglecting the effect of σ_{S}, but as mentioned in Section 4 therein, from [166] it has been shown that assuming $J = f(\theta)$ posses some problems, such as implying thermodynamics instability among other issues.

a constraint is used as an idealization of bodies composed of a matrix filled with a much stiffer family of fibres in the direction $\mathbf{a}$, and in such a case it is expected that in compression the fibres may not present much resistance to the deformation, unlike that in tension, therefore, for such problems a better constraint is $\mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a})\leqslant 0$. Substituting the class of constitutive equation presented in (80) in $\mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a})=0$ we obtain the first order linear partial differential equation

$$\frac{\partial\mathcal{G}}{\partial I_1}+I_4\frac{\partial\mathcal{G}}{\partial I_2}+I_5\frac{\partial\mathcal{G}}{\partial I_3}+\frac{\partial\mathcal{G}}{\partial I_4}+2I_4\frac{\partial\mathcal{G}}{\partial I_5}=0, \tag{113}$$

whose solution is (see [25]) $\mathcal{G}=\bar{\mathcal{G}}(\bar{I}_1,\bar{I}_2,\bar{I}_3,\bar{I}_4)$, where $\bar{I}_1=I_4-I_1$, $\bar{I}_2=\frac{1}{2}I_1^2+I_2-I_1I_4$, $\bar{I}_3=I_1^2-2I_1I_4+I_3$ and $\bar{I}_4=-\frac{1}{3}I_1^3+I_3+I_1^2I_4-I_1I_5$, and the expression for $\boldsymbol{\varepsilon}$ is

$$\begin{aligned}\boldsymbol{\varepsilon}=\frac{\partial\bar{\mathcal{G}}}{\partial\bar{I}_1}(\mathbf{a}\otimes\mathbf{a}-\mathbf{I})+\frac{\partial\bar{\mathcal{G}}}{\partial\bar{I}_2}(-\bar{I}_1\mathbf{I}+\mathbf{T}-I_1\mathbf{a}\otimes\mathbf{a})+\frac{\partial\bar{\mathcal{G}}}{\partial\bar{I}_3}[-2\bar{I}_1\mathbf{I}-2I_1\mathbf{a}\otimes\mathbf{a}\\ +\mathbf{a}\otimes(\mathbf{T}\mathbf{a})+(\mathbf{T}\mathbf{a})\otimes\mathbf{a}]+\frac{\partial\bar{\Pi}}{\partial\bar{I}_4}\{-\bar{I}_3\mathbf{I}+\mathbf{T}^2+I_1^2\mathbf{a}\otimes\mathbf{a}-I_1[\mathbf{a}\otimes(\mathbf{T}\mathbf{a})\\ +(\mathbf{T}\mathbf{a})\otimes\mathbf{a}]\}\end{aligned} \tag{114}$$

In [25] it was shown that a stress of the form $-q\mathbf{a}\otimes\mathbf{a}$ does not produce any deformation, i.e., $\boldsymbol{\varepsilon}=\mathfrak{h}(\mathbf{T}-q\mathbf{a}\otimes\mathbf{a})=\mathfrak{h}(\mathbf{T})$.

In the case of a body with a family of fibres that do not support compression $\mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a})\leqslant 0$, and thus we have

$$\mathcal{G}(\mathbf{T})=\begin{cases}\mathcal{G}(I_1,I_2,I_3,I_4,I_5) & \text{if } \mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a})<0,\\ \bar{\mathcal{G}}(\bar{I}_1,\bar{I}_2,\bar{I}_3,\bar{I}_4) & \text{if } \mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a})=0,\end{cases}$$

where the expression for $\boldsymbol{\varepsilon}$ in the case $\mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a})<0$ is given in (80), while for the case corresponding to $\mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a})=0$ is given in (114).

As in the case of incompressibility studied in the previous section, it is possible to consider the alternative constitutive equation (81)-(83) in terms of the spectral invariants $\mathcal{G}=\mathcal{G}(\sigma_1,\sigma_2,\sigma_3,\zeta_1,\zeta_2,\zeta_3)$, $\zeta_3=1-\zeta_1-\zeta_2$. Considering that the strain tensor can be expressed as $\boldsymbol{\varepsilon}=\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon_{ij}\mathbf{t}^{(i)}\otimes\mathbf{t}^{(j)}$, taking into account (81)-(83), recalling the definition $\zeta_i=[\mathbf{a}\cdot\mathbf{t}^{(i)}]^2$, $i=1,2,3$, from the constraint $\mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a})=0$ we obtain the partial differential equation $\frac{\partial\mathcal{G}}{\partial\sigma_1}+\frac{\partial\mathcal{G}}{\partial\sigma_2}+\frac{\partial\mathcal{G}}{\partial\sigma_3}+2\left(\frac{\partial\mathcal{G}}{\partial\zeta_1}-\frac{\partial\mathcal{G}}{\partial\zeta_2}\right)\frac{\zeta_1\zeta_2}{(\sigma_1-\sigma_2)}+2\frac{\partial\mathcal{G}}{\partial\zeta_1}\frac{\zeta_1\zeta_3}{(\sigma_1-\sigma_3)}+2\frac{\partial\mathcal{G}}{\partial\zeta_2}\frac{\zeta_2\zeta_3}{(\sigma_2-\sigma_3)}=0$. The above partial differential equation has not been solved yet.

6.5 Inextensibility in the case of large strains

In this section we study briefly the development of constitutive equations for inextensible bodies in the case of large strains. We use as a starting point the constitutive equation (30) for transversely isotropic bodies. Many of the results to be presented here are very similar from the mathematical point of view to what has been shown in the previous section. The results presented in this section are taken from [47].

Let assume that the body is transversely isotropic, where $\mathbf{a}_0$ denotes the directions of the inextensible fibres in the reference configuration, then $\mathbf{a}_0 \cdot (\mathbf{E}\mathbf{a}_0) = 0$ and from (30) we obtain the same first order partial differential equation (113). We write the solution of (113) in a different but equivalent form, the solution can be written as:

$$\mathcal{G}(\mathbf{S}) = \mathcal{G}(\mathbf{S}_a) = \hat{\mathcal{G}}(\hat{I}_1, \hat{I}_2, \hat{I}_3, \hat{I}_4), \tag{115}$$

where we have defined $\mathbf{S}_a$ through the relation

$$\mathbf{S} = \mathbf{S}_a + q\mathbf{a}_0 \otimes \mathbf{a}_0 \quad \text{such that} \quad \mathbf{a}_0 \cdot (\mathbf{S}_a\mathbf{a}_0) = 0.$$

The invariants $\hat{I}_j$, $j = 1, 2, 3, 4$ are defined as

$$\hat{I}_1 = -\mathrm{tr}\,\mathbf{S}_a, \quad \hat{I}_2 = \frac{1}{2}\left[(\mathrm{tr}\,\mathbf{S}_a)^2 + \mathrm{tr}\,(\mathbf{S}_a^2)\right], \quad \hat{I}_3 = \frac{1}{2}(\mathrm{tr}\,\mathbf{S}_a)^2 + \mathbf{a}_0 \cdot (\mathbf{S}_a^2\mathbf{a}_0),$$
$$\hat{I}_4 = -\frac{1}{3}[(\mathrm{tr}\,\mathbf{S}_a)^3 + \mathrm{tr}\,(\mathbf{S}_a^3)] - (\mathrm{tr}\,\mathbf{S}_a)[\mathbf{a}_0 \cdot (\mathbf{S}_a^2\mathbf{a}_0)].$$

In [47] that function (115) with the above definitions gives the same solution as (114) (replacing $\boldsymbol{\varepsilon}$ and $\mathbf{T}$ by $\mathbf{E}$ and $\mathbf{S}$). Replacing (115) in (30) we obtain

$$\mathbf{E} = -\vartheta_0\mathbf{I} + \vartheta_1\mathbf{S}_a + \vartheta_2\mathbf{S}_a^2 + \vartheta_3\mathbf{a}_0 \otimes \mathbf{a}_0 + \vartheta_4[\mathbf{a}_0 \otimes (\mathbf{S}_a\mathbf{a}_0) + (\mathbf{S}_a\mathbf{a}_0) \otimes \mathbf{a}_0],$$

where we have defined

$$\vartheta_0 = \frac{\partial\hat{\mathcal{G}}}{\partial\hat{I}_1} + \hat{I}_1\frac{\partial\hat{\mathcal{G}}}{\partial\hat{I}_2} + 2\hat{I}_1\frac{\partial\hat{\mathcal{G}}}{\partial\hat{I}_3} + \hat{I}_3\frac{\partial\hat{\mathcal{G}}}{\partial\hat{I}_4}, \quad \vartheta_1 = \frac{\partial\hat{\mathcal{G}}}{\partial\hat{I}_2}, \quad \vartheta_2 = \frac{\partial\hat{\mathcal{G}}}{\partial\hat{I}_4},$$
$$\vartheta_3 = \frac{\partial\hat{\mathcal{G}}}{\partial\hat{I}_1} + \hat{I}_1\frac{\partial\hat{\mathcal{G}}}{\partial\hat{I}_2} + 2\hat{I}_1\frac{\partial\hat{\mathcal{G}}}{\partial\hat{I}_3} + \hat{I}_1^2\frac{\partial\hat{\mathcal{G}}}{\partial\hat{I}_4}, \quad \vartheta_4 = \frac{\partial\hat{\mathcal{G}}}{\partial\hat{I}_3} + \hat{I}_1\frac{\partial\hat{\mathcal{G}}}{\partial\hat{I}_4},$$

where the restriction $\frac{\partial\hat{\mathcal{G}}}{\partial\hat{I}_1}(\mathbf{0}) = 0$ must be satisfied.

In the same manner as in Section 6.4, we can propose a constitutive model, where the body is inextensible only when the fibres are in tension but not in compression. In such a case we propose to work with:

$$\mathcal{G}(\mathbf{S}) = \begin{cases} \mathcal{G}(\mathbf{S}) = \mathcal{G}(I_1, I_2, I_3, I_4, I_5) & \text{if } \mathcal{G}_1 + \mathcal{G}_2 I_4 + \mathcal{G}_3 I_4 \leqslant 0, \\ \hat{\mathcal{G}}(\mathbf{S}_a) = \bar{\mathcal{G}}(\hat{I}_1, \hat{I}_2, \hat{I}_3, \hat{I}_4) & \text{otherwise.} \end{cases}$$

In [47] two boundary value problems were studied, namely the inflation and extension of a cylindrical annulus (that is a special case of what it is shown in Section 9.2.3), and the circumferential shear of a cylindrical annulus. For this last problem for the cylindrical annulus $R_\mathrm{i} \leqslant R \leqslant R_\mathrm{o}$, $0 \leqslant \Theta \leqslant 2\pi$, $0 \leqslant Z \leqslant L$, we assume the presence of the stress tensor $\mathbf{T}(r) = \sigma_r(r)\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_\theta(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_z(r)\mathbf{e}_z \otimes \mathbf{e}_z$, where $r = f(R)$, $\theta = \Theta + g(R)$, $z = \lambda Z$, where $\lambda > 0$ is a constant and $\mathbf{a}_0(R) = A(R)\mathbf{E}_r + B(R)\mathbf{E}_\Theta$, $\sqrt{A^2 + B^2} = 1$. The resultant equations considering the above stress and deformation are not shown here for the sake of brevity, and can be found in Sections 5 and 6 of [47].

6.6 The rigid body in the case of small strains

A body it said to be rigid if there is no deformation irrespective of the stresses applied to it. A possible way to impose that restriction on an isotropic body is by considering the constraints $\mathbf{a}^{(i)} \cdot (\boldsymbol{\varepsilon}\mathbf{a}^{(i)}) = 0$, $i = 1, 2, 3$, where $\{\mathbf{a}^{(i)}\}$ is a set of orthonormal vectors. Using (78) in the above constraints, assuming that $\mathbf{a}^{(i)} = \mathbf{t}^{(i)}$, from $\mathbf{a}^{(i)} \cdot (\boldsymbol{\varepsilon}\mathbf{a}^{(i)}) = 0$ we obtain $\frac{\partial \mathcal{G}}{\partial \sigma_i} = 0$, $i = 1, 2, 3$, from which we conclude that $\mathcal{G}$ does not depend on the stress, i.e., $\boldsymbol{\varepsilon}$ is not affected by the stresses and moreover $\boldsymbol{\varepsilon} = \mathbf{0}$.

7 Applications

In this section we show some applications for some of the constitutive relations and equations, which have been listed in the previous section. We show such examples, first for cases considering large elastic deformations, and then for problems involving small strains (small gradient of the displacement field). Some examples of applications are: biomaterials, rubber-like solids, solids showing strain limiting behaviour, fracture mechanics, gum metals, rock, concrete, bone and electro- and magneto-sensitive materials.

7.1 Implicit constitutive relations

In the case of the implicit relations of the form (5), (7), in [80, 81] it has been shown that they can be used to model biological fibers and tissue. For example in [81], for a 1D problem, where ε and σ denote the 1D strain and stress, respectively, the first law of thermodynamics can be written as (defining that $W = \rho_{\mathrm{r}} U$) $\mathrm{d}W = -s\mathrm{d}\theta + \sigma\mathrm{d}\varepsilon$, and assuming that $U = U(\varepsilon, \sigma, \theta)$ one can show that

$$s = -\frac{\partial W}{\partial \theta}, \quad \sigma \mathrm{d}\varepsilon = \frac{\partial W}{\partial \varepsilon}\mathrm{d}\varepsilon + \frac{\partial W}{\partial \sigma}\mathrm{d}\sigma,$$

and for biological fibres the following expression for W was proposed

$$W = -C\left[\theta \ln\left(\frac{\theta}{\theta_o}\right) - (\theta - \theta_o)\right] + E[\varepsilon - \alpha(\theta - \theta_o)] - \sigma \\ +\beta\sigma[\varepsilon - \alpha(\theta - \theta_o)], \tag{116}$$

where C, θ_o, E, α and β are constants.

In [131] a similar model was proposed for applications for the modelling of biomaterials, in this case starting with an implicit relation of the form (5), rewritten in terms of the first Piola-Kirchhoff stress tensor $\mathbf{P}$ and the deformation gradient as $\mathfrak{F}(\mathbf{P}, \mathbf{F}) = \mathbf{0}$, for the particular case when $\mathfrak{F}$ is isotropic, the following expression written in terms of the principal directions of $\mathfrak{F}$ takes the form

$$\left[P_i - \frac{\Phi}{\lambda_i}\left(\lambda_i^2 - \frac{1}{J^{2q}}\right)\right]\left[P_i - \frac{\Phi}{\lambda_i}\left(\lambda_i^2 - \frac{1}{J^{2q}}\right) - \frac{\Psi}{\lambda_i}\left(\lambda_i^m - \frac{1}{\lambda_i^n}\right)\right] = 0, \tag{117}$$

where there is no sum in the repeated index, P_i $i = 1, 2, 3$ are the principal values of $\mathbf{P}$, λ_i, $i = 1, 2, 3$ are the principal stretches, $J = \det \mathbf{F} = \lambda_1\lambda_2\lambda_3$, and Φ, q, Ψ, m and n are constants.

In Sections 3.2.1 and 3.2.2 some general expressions for a class of implicit constitutive relations have been proposed, which can be used to generalize (116) and (117) for 3D problems. That has not been studied in the literature yet.

7.2 Modelling of rubber-like solids

In this section different constitutive equations are listed for the modelling of rubber-like solids (where we assume large elastic deformations), namely a constitutive equation by Muliana et al. [137], two constitutive equations proposed by Bustamante and Rajagopal [43, 45], a constitutive equation proposed by Bertoti [8], and a constitutive equation studied by Shyamkumar et al. [199].

7.2.1 A constitutive equation by Muliana et al

In [137] a constitutive equation has been proposed for rubber-like materials of the form (12):

$$\mathbf{B} = \mathbf{I} - \kappa\left(1 - e^{-\alpha K_1}\right)\mathbf{I} + \mu\frac{\left(1 - e^{-\delta K_1 + \beta K_2}\right)}{(K1 + wK_2)}\mathbf{T},$$

where α, δ, κ, β, w and μ are material constants, and $K_1 = \sqrt{(\mathrm{tr}\,\mathbf{T})^2}$, $K_2 = \sqrt{\frac{1}{2}\left[(\mathrm{tr}\,\mathbf{T})^2 - \mathrm{tr}\,(\mathbf{T}^2)\right]}$.

In [137] two boundary value problems are studied, one of them the uniaxial loading of a body, where $T_{11} = \sigma$ and $x_i = \lambda X_i$ (there is non sum in i), considering isochoric motion $\det \mathbf{F} = 1$. The other problem studied corresponds to the equibiaxial loading of a plate, where $T_{11} = T_{22} = \sigma$. The material constants are obtained by fitting the experimental data from [206], and the predictions of the constitutive equation are compared with the constitutive equation by Ogden [144]. The values for the material constants found in that paper for natural rubber are: $\mu = 63$, $\delta = 0.005[\mathrm{cm}^2/\mathrm{Kg}]$, $\kappa = 0.865$ and $\alpha = 0.116[\mathrm{cm}^2/\mathrm{Kg}]$. The constants β and ω are not found (in that paper the authors assumed that they are zero) because for the boundary value problems mentioned before (the uniaxial loading of a body) $K_2 = 0$.

7.2.2 A constitutive equation by Bustamante and Rajagopal

Considering (106)-(108) the following model for the function $\breve{\mathcal{G}}$ for incompressible isotropic solids has been proposed (see [45]):

$$\breve{\mathcal{G}}(\alpha, \beta) = \mathfrak{f}(\alpha + \beta),$$

then (106)-(108) become

$$\begin{aligned}
\ln\lambda_1 &= 2\mathfrak{f}'(2\sigma_1 - \sigma_2 - \sigma_3) - \mathfrak{f}'(2\sigma_2 - \sigma_1 - \sigma_3) - \mathfrak{f}'(2\sigma_3 - \sigma_1 - \sigma_2),\\
\ln\lambda_2 &= -\mathfrak{f}'(2\sigma_1 - \sigma_2 - \sigma_3) + 2\mathfrak{f}'(2\sigma_2 - \sigma_1 - \sigma_3) - \mathfrak{f}'(2\sigma_3 - \sigma_1 - \sigma_2),\\
\ln\lambda_3 &= -\mathfrak{f}'(2\sigma_1 - \sigma_2 - \sigma_3) - \mathfrak{f}'(2\sigma_2 - \sigma_1 - \sigma_3) + 2\mathfrak{f}'(2\sigma_3 - \sigma_1 - \sigma_2).
\end{aligned}$$

Using the experimental data for rubber from [206] the following particular expression has been proposed for $\mathfrak{f}'(x)$:

$$\mathfrak{f}'(x) = a\tanh(bx) + c\tanh(dx), \tag{118}$$

where a, b, c and d are material constants given in[17] Table 1 (see Table 1 in [45]).

$a = -0.07718$	$b = 0.04004[\text{cm}^2/\text{Kg}]$
$c = 0.5779$	$d = 0.04492[\text{cm}^2/\text{Kg}]$

Table 1: Material constants for the first constitutive equation for rubber by Bustamante and Rajagopal.

In [45] some boundary value problems are studied, namely the uniform extension/compression of a cylinder, the biaxial stretching of a thin plate and the simple shear of a slab (see Section 9.1). The predictions of (118) were compared with the constitutive equation by Ogden (see [144]), were the energy function W is given by $W = \sum_{j=1}^{M} \frac{\mu_j}{\alpha_j}\left(\lambda_1^{\alpha_j} + \lambda_2^{\alpha_j} + \lambda_3^{\alpha_j} - 3\right)$.

7.2.3 A constitutive equation by Bustamante

From Section 6.2 we work with the Gibbs potential $\hat{\mathcal{G}}$ using the invariants I_{D_2} and I_{D_3} therein and from [43] we have the constitutive model:

$$\hat{\mathcal{G}}(I_{\mathrm{D}_2}, I_{\mathrm{D}_3}) = \mathfrak{g}(I_{\mathrm{D}_2}) + \mathfrak{h}(I_{\mathrm{D}_3}),$$

where

$$\mathfrak{g}(I_{\mathrm{D}_2}) = \frac{p}{q^2}\left[q\sqrt{3I_{\mathrm{D}_2}} - \ln\left(1 + q\sqrt{3I_{\mathrm{D}_2}}\right)\right], \tag{119}$$

$$\mathfrak{h}(I_{\mathrm{D}_3}) = \frac{a}{b}\ln\left[\cosh\left(\frac{3b}{2^{1/3}}I_{\mathrm{D}_3}^{1/3}\right)\right] + \frac{c}{d}\ln\left[\cosh\left(\frac{3d}{2^{1/3}}I_{\mathrm{D}_3}^{1/3}\right)\right], \tag{120}$$

where a, b, c, d, p, q are materials parameters, which for the data for rubber from [206] are given in Table 2 (here we only show for the case of considering the data for tension of a sample). In [43] the above results were used for the analysis of a hollow sphere under inflation. Such boundary value problem is presented in Section 9.2.4 for the class of constitutive equation (103).

[17] In Table 1 we only show the case of fitting with the experimental data obtained considering the tension of a sample from [206].

a	b [cm^2/Kg]	c	d [cm^2/Kg]	p [cm^2/Kg]	q [cm^2/Kg]
-0.9445	0.4006	0.742	0.1164	0.4257	0.1745

Table 2: Values for the material constants in (119) and (120) from [43].

7.2.4 A constitutive equation by Bertoti

In [8] from (21) Bertoti proposed the following expression for incompressible solids $\mathcal{G} = \mathcal{G}(S_2, S_3)$, where $\boldsymbol{\tau} = \mathbf{T}$ and

$$S_2 = \operatorname{tr}(\mathbf{T}_{\mathrm{D}}^2), \quad S_3 = \sqrt{6}\frac{\operatorname{tr}(\mathbf{T}_{\mathrm{D}}^3)}{|\mathbf{T}_{\mathrm{D}}|},$$

where $\mathbf{T}_{\mathrm{D}}$ is the deviatoric part of the stress.

The following particular expression for $\mathcal{G}$ has been proposed for the fitting of rubber:

$$\mathcal{G}(S_2, S_3) = \sum_{n=1}^{N} \frac{1}{\alpha_n \mu_n} \left[(1 + a_n S_2 + b_n S_3)^{\alpha_n} - 1 \right],$$

where a_n, b_n, α_n, μ_n are material constants. For the case $N = 2$ the constants are shown in Table 3. The cases $N = 1$ and $N = 4$ are also treated in that

$a_1 = 0.2763$	$\alpha_1 = 0.5687$	$a_2 = 0.06559$	$\alpha_2 = 0.5651$
$\frac{a_1}{\mu_1} = \frac{a_2}{\mu_2} = \frac{1}{8\mu}$	$\mu = 0.4225$	$b_1 = a_1$	$b_2 = -a_2$

Table 3: Material constants for the constitutive equation by Bertoti. The constants a_1 and a_2 are given in 1/[MPa^2]. The constant μ is given in [MPa].

work [8], but for the sake of brevity we do not show those material constants here.

7.2.5 A constitutive equation by Shyamkumar et al.

In [199] a constitutive relation of the form (21) is proposed for an isotropic solid considering large elastic deformations, and that can show strain limiting behaviour. In [199] the following invariants are used for $\mathcal{G}$, namely $\mathcal{G} = \hat{\mathcal{G}}(p, \alpha)$, where $p = \operatorname{tr}\boldsymbol{\tau}/3$ and $\alpha = \sqrt{\boldsymbol{\tau}_{\mathrm{D}} \cdot \boldsymbol{\tau}_{\mathrm{D}}}$, where (see Section 3.3.1 and

Eq. (21)) $\boldsymbol{\tau}_{\rm D} = \boldsymbol{\tau} - \frac{(\operatorname{tr}\boldsymbol{\tau})}{3}\mathbf{I}$. Using the above expression for $\mathcal{G}$ and the above invariants they obtain

$$\ln \mathbf{V} = \frac{1}{3}\frac{\partial \hat{\mathcal{G}}}{\partial p}\mathbf{I} + \frac{\partial \hat{\mathcal{G}}}{\partial \alpha}\frac{\boldsymbol{\tau}_{\rm D}}{\alpha}. \tag{121}$$

The following expression for $\hat{\mathcal{G}}$ has been proposed:

$$\hat{\mathcal{G}}(p,\alpha) = \frac{2d}{\pi a}\left[ap\tan^{-1}(ap) - \frac{1}{2}\ln\left(1 + a^2p^2\right)\right] + \frac{c}{\sqrt{b}}\left(\sqrt{1+b\alpha^2} - 1\right), \tag{122}$$

where the material constants a, b, c and d are not shown here for the sake of brevity.

7.2.6 A constitutive relation for vulcanized unfilled rubber by Gokulnath and Saravanan

In [85] a constitutive relation is proposed for unfilled rubber by Gokulnath and Saravanan, using the Hencky strain tensor as a function of the Kirchhoff stress tensor. Assuming the solid as incompressible and using a Gibbs potential $\mathcal{G}$ that only depend on the deviatoric part of the Cauchy stress tensor[18], the relation is:

$$\boldsymbol{\eta} = -\frac{\partial \mathcal{G}}{\partial K_3^{\rm d}}K_2^{\rm d}\mathbf{I} + 2\frac{\partial \mathcal{G}}{\partial K_2^{\rm d}}\mathbf{T}_{\rm D} + 3\frac{\partial \mathcal{G}}{\partial K_3^{\rm d}}\mathbf{T}_{\rm D}^2, \tag{123}$$

where $\mathcal{G} = \mathcal{G}(K_2^{\rm d}, K_3^{\rm d})$, and $K_2^{\rm d} = \operatorname{tr}(\mathbf{T}_{\rm D}^2)$ and $K_3^{\rm d} = \operatorname{tr}(\mathbf{T}_{\rm D}^3)$.

The following very simple expression for $\mathcal{G}$ has been proposed:

$$\mathcal{G}(K_2^{\rm d}, K_3^{\rm d}) = m_1K_2^{\rm d} + m_2K_3^{\rm d}, \tag{124}$$

where m_1 and m_2 are material parameters. In [85] a detailed procedure is proposed to find from different experimental data the above material constants, giving some limits for $K_2^{\rm d}$ and $K_3^{\rm d}$ for which the constitutive model is accurate and useful. Some values for such constants are given in Table 1 of that paper. For example, using some experimental data obtained by Jones and Treloar the following parameters are obtained: $m_1 = 0.53[\text{MPa}^{-1}]$ and $m_2 = -.024[\text{MPa}]^{-2}$. It is interesting to notice that the constitutive model proposed in this work requires very few material constants for a good fitting, in comparison with other well known constitutive equations based on the use of Green elastic bodies, as the interested reader can see in the same paper.

[18] We need to recall that for incompressible bodies the Kirchhoff stress tensor becomes the Cauchy stress tensor. In [85] the Gibbs potential is denoted Φ_g.

7.3 Applications in fracture mechanics, in the modelling of metallic alloys, concrete, rock and bone

In this section we summarize some expressions for constitutive equations obtained from the implicit relations discussed in Sections 5.1 and 5.2, for the special case $|\nabla_{\mathbf{X}}\mathbf{u}| \sim O(\delta)$, $\delta \ll 1$, i.e., when we restrict ourselves to the linearized strain tensor $\boldsymbol{\varepsilon}$ and the Cauchy stress tensor $\mathbf{T}$. We discuss some applications within the context of the models given by (54), (75) and (90).

7.3.1 Modelling strain limiting behaviour

In this sub-section we discuss constitutive relations that reflect the fact that the strains in the body are limited by a value decided a priori, irrespective of the value of the stress. Such models are particularly useful in describing the response of brittle elastic bodies which respond in an elastic manner until a limiting strain is reached, and fail after that.

In [145] Ortiz et al. proposed the following expression for $\mathcal{G} = \mathcal{G}(\mathbf{T})$ (see (77)) namely $\mathcal{G}(\mathbf{T}) = -\alpha\left[I_1 - \frac{1}{\beta}\ln(1+\beta I_1)\right] + \frac{\alpha\gamma}{\iota}\sqrt{1+2\iota I_2}$, where α, β, γ, ι are material constants, and from (77) since $\boldsymbol{\varepsilon} = \mathfrak{h}(\mathbf{T}) = \frac{\partial\mathcal{G}}{\partial\mathbf{T}}$ we obtain that

$$\boldsymbol{\varepsilon} = -\alpha\left[1 - \frac{1}{(1+\beta I_1)}\right]\mathbf{I} + \frac{\alpha\gamma}{\sqrt{1+2\iota I_2}}\mathbf{T}, \tag{125}$$

where values of the constants that appear in the expression (125) are given in Table 4. In [145] using the above constitutive equation some problems involving stress concentration such as a plane plate with an elliptic hole and a stepped flat bar with shoulder fillets were studied.

$\alpha = 10^{-9}$	$\beta = 10^{-3}\frac{1}{[\mathrm{Pa}]}$	$\gamma = 10\frac{1}{[\mathrm{Pa}]}$	$\iota = 10^{-11}\frac{1}{[\mathrm{Pa}^2]}$

Table 4: Material constants for the constitutive equation for strain limiting by Ortiz et al. [145]

A modification of (125) was considered in [132], where it was assumed that $\mathcal{G}(\mathbf{T}) = -\frac{\alpha}{\beta}\ln[\cosh(\beta I_1)] + \frac{\gamma}{\iota}\sqrt{1+2\iota I_2}$, which leads to

$$\boldsymbol{\varepsilon} = -\alpha\tanh(\beta I_1)\mathbf{I} + \frac{\gamma}{\sqrt{1+2\iota I_2}}\mathbf{T}, \tag{126}$$

which was used to study the problem of a plane plate with an elliptic hole, a plane plate with an hyperbolic boundary, and the behaviour of a semi-infinite medium with a concentrated force at a point. Values for the constants that appear in (126) are given in Table 5 below. The above constitutive equation

$\alpha = 0.01$	$\beta = 9.277 \times 10^{-8} \frac{1}{[\mathrm{Pa}]}$	$\gamma = 4.02 \times 10^{-9} \frac{1}{[\mathrm{Pa}]}$	$\iota = 10^{-14} \frac{1}{[\mathrm{Pa}^2]}$

Table 5: Material constants for the constitutive equation for strain limiting by Montero et al. [132]

(126) has also been used for the study of spherical inclusions in cylinder under tension. In [133] cylinder made of rubber[19] is assumed to have inclusions made of a solid that show strain limiting behaviour considering (126). Several cases considering different distributions of spherical inclusions are analyzed using the finite element method and working with axil-symmetric models.

In [157, 158] Rajagopal proposed the following expression (127) in order to study strain limiting behaviour primarily to ascertain the qualitative response of bodies described by such models than with a view to correlating against experimentally observed facts. He used the constitutive relation:

$$\boldsymbol{\varepsilon} = \alpha \left\{ 1 - e^{\frac{-\lambda \mathrm{tr}\,\mathbf{T}}{\left[1 + \left(\mathrm{tr}\,(\mathbf{T}^2)\right)^{1/2}\right]}} \right\} \mathbf{I} + \frac{\beta}{[1 + \gamma \mathrm{tr}\,(\mathbf{T}^2)]^{1/2}} \mathbf{T}, \tag{127}$$

where λ_1, λ_2, α and n are constants. We notice that the above model reduces to the linearized elastic model when we require the model to be linear in the stress, or put differently for small strains and a linear stress, the model reduces to a linearized elastic model, but for large stresses a body described by the model behaves non-linearly and exhibits limiting strain.

In [11], Bridges and Rajagopal, based on the specific choice of a Gibbs potential develop a nonlinear elastic model which they then linearize with respect to the displacement gradient to obtain a constitutive relation that predicts a limiting value for the linearized strain. Their constitutive relation takes the form:

$$\boldsymbol{\varepsilon} = \alpha \left[-\frac{\beta \mathrm{tr}\,\mathbf{T}}{(1 + \beta \mathrm{tr}\,\mathbf{T})} \mathbf{I} + \frac{\mu}{\sqrt{1 + \gamma^2 \mathrm{tr}\,(\mathbf{T}^2)}} \mathbf{T} \right],$$

[19] For these cylinder it is assumed the behaviour can be modelled using a Neo-Hookean constitutive equation $W = \frac{\mu}{2}(\bar{I}_1 - 3) + \frac{\kappa}{2}(J-1)^2$, where μ, κ are material constants, which is proposed for nearly incompressible solids.

where α, β, γ and μ are constants. Using the above model Bridges and Rajagopal [11] study the problem of stress concentration in a body described by such a constitutive relation. Though they are primarily interested in studying the state of stress and strain in an annular region, by allowing the outer radius to become infinite they can study the problem of a hole in an infinite body. In keeping with the study of Ortiz et al. [145], they find that the strain grows much slower than the stress and in fact the strain never exceeds the limiting value that is set a priori. As with the previous model, the above model also reduces for small strains and the requirement that the model be linear in the stress to the linearized elastic model and it exhibits strain limiting behaviour.

In [12] Bulicek et al. proposed the following expression for $\mathfrak{h}(\mathbf{T})$ for a theoretical analysis of strain limiting behaviour

$$\boldsymbol{\varepsilon} = \alpha_0(\operatorname{tr}\mathbf{T}, \operatorname{tr}(\mathbf{T}^2))\mathbf{I} + \frac{1}{\mu_o(1+|\mathbf{T}|^r)^{1/r}}\mathbf{T},$$

where $\alpha_0 = \alpha_0(\operatorname{tr}\mathbf{T}, \operatorname{tr}(\mathbf{T}^2))$ is a function and μ_o is a constant. The aim of this study is purely mathematical analysis, namely the determination of existence of solutions to the partial differential equations that arise from the balance laws by assuming such a constitutive relation.

In [199] Eq. (121) for the case $|\nabla_{\mathrm{r}}\mathbf{u}| \sim O(\delta)$, $\delta \ll 1$ and from that constitutive equation we have:

$$\boldsymbol{\varepsilon} \approx \frac{1}{3}\frac{\partial \hat{\mathcal{G}}}{\partial p}\mathbf{I} + \frac{\partial \hat{\mathcal{G}}}{\partial \alpha}\frac{\mathbf{T}_{\mathrm{D}}}{\alpha}. \tag{128}$$

In [199] Eq. (122) is proposed to have strain-limiting behaviour. In [199] for a given set of material constants (see (122)) it is shown that $\left|\frac{\partial \hat{\mathcal{G}}}{\partial p}\right| \ll 1$ and $\left|\frac{\partial \hat{\mathcal{G}}}{\partial \alpha}\right| \ll 1$ thus $|\boldsymbol{\varepsilon}| \ll 1$. As a matter of fact $\lim_{|\mathbf{T}|\to\infty} |\boldsymbol{\varepsilon}| = \sqrt{3d^2+c^2}$. In [199] a non-dimensional version of (122) is obtained, and some restrictions on the material parameters are proposed such that when $|\mathbf{T}| \to 0$ the dimensionless version of the constitutive equation becomes the linearized elastic model (to be consistent with that constitutive equation). The restrictions are: $c > 0$ and $\sqrt{b}/a > 0$. In [199] a boundary value problem is studied using the finite element method, namely the tension of a plane plate wiht V-notches, where the numerical results indicate that near the tip of the notch $|\boldsymbol{\varepsilon}|$ is limited.

We finish this review mentioning the communication by Bulíček at al. [13], where the following constitutive equation is proposed for an elastic solid showing strain limiting behaviour:

$$\boldsymbol{\varepsilon} = \lambda(\operatorname{tr}\mathbf{T})(\operatorname{tr}\mathbf{T})\mathbf{I} + \mu(|\mathbf{T}_{\mathrm{D}}|)\mathbf{T}_{\mathrm{D}},$$

where $\lambda = \lambda(\operatorname{tr}\mathbf{T}) = \frac{1}{\left(1+|\operatorname{tr}\mathbf{T}|^{\gamma}\right)^{1/\gamma}}$, $\mu = \mu(|\mathbf{T}_{\mathrm{D}}|) = \frac{1}{(1+|\mathbf{T}_{\mathrm{D}}|^{\alpha})^{1/\alpha}}$, where $\alpha > 0$, $\gamma > 0$. In that work there is an analysis of the partial differential equations for boundary value problems, and it is shown the existence of weak solutions for 3D problems considering the above constitutive equation.

7.3.2 Fracture mechanics of brittle bodies

In the previous section we discussed some models that have been used to explore stress concentration in a general manner, and in this section we discuss constitutive relations that have been used specifically to study the behaviour of bodies with cracks. In [14] the following model was used to investigate the behaviour of the strains near the tip of a crack (in an anti-plane stress problem)

$$\boldsymbol{\varepsilon} = g_1(\operatorname{tr}\mathbf{T}, \operatorname{tr}(\mathbf{T}^2))\mathbf{I} + g_2(\operatorname{tr}(\mathbf{T}^2))\mathbf{T},$$

where $g_1 = g_1(\operatorname{tr}\mathbf{T}, \operatorname{tr}(\mathbf{T}^2))$ and $g_2 = g_2(\operatorname{tr}(\mathbf{T}^2))$ are scalar functions of the stresses, $g_1(0,.) = 0$ and $g_2(\operatorname{tr}(\mathbf{T}^2)) = g_2(|\mathbf{T}|^2) = \frac{1}{(1+|\mathbf{T}|^a)^{1/a}}$ where a is a constants such that $a > 0$. Similar analysis were carried out in [159] for the problem of a crack in a body under anti-plane shear, considering the simpler expression

$$\boldsymbol{\varepsilon} = \phi(\beta|\mathbf{T}|)\mathbf{T}, \tag{129}$$

where $\phi(r) = \frac{1}{1+\beta r}$, $\beta > 0$ is a constant and $\mathbf{T}$ is the non-dimensional stress.

In [116] we see some numerical results for the problem of a crack in a body under anti-plane shear, considering

$$\boldsymbol{\varepsilon} = \beta\left(1 - e^{\frac{-\lambda \operatorname{tr}\mathbf{T}}{1+|\mathbf{T}|}}\right)\mathbf{I} + \frac{1}{2\mu\left(1+\kappa|\mathbf{T}|^a\right)^{1/a}}\mathbf{T},$$

where β, μ, κ and a are constants.

In [87] a special class of (75) and (129) was proposed for the strain limiting behaviour of a solid, namely

$$\boldsymbol{\varepsilon} = \frac{\mathbf{K}(\mathbf{T})}{\left[1+\beta^a\left|\mathbf{K}^{1/2}(\mathbf{T})\right|^a\right]^{1/a}},$$

where $\mathbf{K}(\mathbf{T}) = \frac{1}{2\mu}\mathbf{T} - \frac{\lambda}{2\mu\left(\lambda+\frac{2\mu}{d}\right)d}(\mathrm{tr}\,\mathbf{T})\mathbf{I}$, where $a > 0$, $\beta \geqslant 0$, μ, λ and d are constants, where $d = 2$ and $d = 3$ for 2D and 3D boundary value problems, respectively. In [87] a boundary value problem is studied, where a cylinder under traction in the axial direction is analyzed, and the cylinder has a penny-shaped crack in the center. The problem is solved using the finite element method, and solving $(4)_1$ (in the quasi-static case and $\mathbf{b} = \mathbf{0}$) using a Airy stress potential $\mathbf{\Phi}$, wherein $T_{ij} = \epsilon_{irp}\epsilon_{jsq}\frac{\partial^2 \Phi_{pq}}{\partial x_r \partial x_s}$.

In [101] a strain limiting constitutive equation has been proposed, to study the behaviour of a body with a crack, where the faces of the crack can be in contact, which is one of the important features of such study. The special class of constitutive equation (75) is:

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu(|\mathbf{T}_{\mathrm{D}}|)}\mathbf{T}_{\mathrm{D}} + \frac{(\mathrm{tr}\,\mathbf{T})}{9\kappa(\mathrm{tr}\,\mathbf{T})}\mathbf{I}, \tag{130}$$

where $\kappa = \kappa(\mathrm{tr}\,\mathbf{T})$ is a sort of generalized bulk modulus, and $\mu = \mu(|\mathbf{T}_{\mathrm{D}}|)$ is a generalized shear modulus (here we recall that $\mathbf{T}_{\mathrm{D}}$ is the deviatoric part of the stress). In [101] an expression for κ has been proposed, which is bounded in $\mathrm{tr}\,\mathbf{T}$, namely

$$\kappa(\mathrm{tr}\,\mathbf{T}) = \kappa_o\left[1 + k + \frac{2}{\pi}\arctan(\alpha_1 \mathrm{tr}\,\mathbf{T} + \alpha_1)\right],$$

where κ_o, k, α_1 and α_2 are material constants. Something similar has been proposed for μ, namely $\mu(|\mathbf{T}_{\mathrm{D}}|) = \mu_o\left[1 + \gamma + \frac{2}{\pi}\arctan(\beta_1|\mathbf{T}_{\mathrm{D}}| + \beta_1)\right]$. In [101] the authors look for restrictions on μ, κ such that there is a unique solution for the displacement field and the stress tensor for the above problem of the body with a crack.

All the models presented in this and the previous sections have not been corroborated against actual experimental data, but rather they have been proposed on the basis of their possessing the mathematical property that independent of the magnitude of the stresses, the strains remain small. In the following sections we consider some examples for $\mathfrak{h}(\mathbf{T})$ that have been corroborated against actual experimental data, for Gum metal and other metallic alloys, concrete, rock and bone.

7.3.3 Modelling of gum metal and other metallic alloys

In [161] the following model is proposed for such gum metal:

$$\boldsymbol{\varepsilon} = \lambda_1 \mathrm{tr}\,(\mathbf{T})\mathbf{I} + 2\lambda_2 e^{n\mathrm{tr}\,(\mathbf{T})}\mathbf{T}, \quad \boldsymbol{\varepsilon} = \lambda_1 \mathrm{tr}\,(\mathbf{T})\mathbf{I} + \lambda_2\left[1 + \alpha \mathrm{tr}\left(\mathbf{T}^2\right)\right]^n\mathbf{T},$$

where λ_1, λ_2 and n are constants, which in general are different for the two models. In this review we do not provide numerical values for such constants.

In [117] Kulvait et al. proposed a constitutive equation of the form

$$\boldsymbol{\varepsilon} = -\frac{1}{9\hat{K}(|\mathbf{T}|^2)}(\mathrm{tr}\,\mathbf{T})\mathbf{I} + \frac{1}{2\hat{\mu}(|\mathbf{T}_\mathrm{D}|^2)}\mathbf{T}_\mathrm{D},$$

where[20] $\hat{K}(|\mathbf{T}|^2) = K_o\left[\frac{\tau_o^2}{(\tau_o^2+|\mathbf{T}|^2)}\right]^{\frac{s-2}{2}}$, $\hat{\mu}(|\mathbf{T}_\mathrm{D}|^2) = \mu_o\left[\frac{\tau_o^2}{\left(\tau_o^2+\frac{3}{2}|\mathbf{T}_\mathrm{D}|^2\right)}\right]^{\frac{q-2}{2}}$, and $K_o > 0$, $\tau_o > 0$, $1 < s < \infty$, $\mu_o > 0$ and $1 < q < \infty$ are constants and $\mathbf{T}_\mathrm{D} = \mathbf{T} - \frac{1}{3}(\mathrm{tr}\,\mathbf{T})\mathbf{I}$. Numerical values for the constants are given in Table 6. The constant τ_o can be considered as a characteristic value for the stress.

$\tau_o = 0.5$[GPa]	$q = 2.23$	$s = 7.65$	$K_o = 6226$[GPa]	$\mu_o = 20.2$[GPa]

Table 6: Material constants for the constitutive equation for strain limiting by Kulvait et al. [117]

In [65, 66] Devendiran et al. proposed the following two alternative expressions for a constitutive relation of the form (54) to model the behaviour of titanium alloys:

$$\boldsymbol{\varepsilon} - \hat{\alpha}_1\left\{[\mathrm{tr}\,\mathbf{T} - 2\mathrm{tr}\,(\mathbf{T}\boldsymbol{\varepsilon})]\,\mathbf{I} + 2\mathrm{tr}\,(\mathbf{T})\boldsymbol{\varepsilon}\right\} - \left\{\hat{\alpha}_2 + \hat{\alpha}_3 e^{\left[\left(1+\alpha_4\mathrm{tr}\,(\mathbf{T}^2)\right)^{n/2}\right]}\times\right.$$
$$\left.\times\left[1 + n\alpha_4\left(1+\alpha_4\mathrm{tr}\,\left(\mathbf{T}^2\right)\right)^{\frac{n}{2}-1}\left(\mathrm{tr}\boldsymbol{\varepsilon}\mathrm{tr}\,\left(\mathbf{T}^2\right) - 2\mathrm{tr}\,\left(\boldsymbol{\varepsilon}\mathbf{T}^2\right)\right)\right]\right\}\mathbf{T} = \mathbf{0}, \quad (131)$$

where $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$, α_4 and n are constants given in Table 7 for the particular case of the alloy TNTZ30. In [65] Devindiran et al. also proposed the alternative constitutive equation for the same material:

$$\boldsymbol{\varepsilon} = \beta_1\mathrm{tr}\,(\mathbf{T})\mathbf{I} + \left\{\beta_2 + \beta_3 e^{\left[\sqrt{1+\beta_4\mathrm{tr}\,(\mathbf{T}^2)}\right]^n}\right\}\mathbf{T}, \quad (132)$$

where the constants β_1, β_2, β_3, β_4 and n are given in Table 7.

In [118] another model for beta phase titanium alloys has been proposed, where

$$\mathrm{tr}\boldsymbol{\varepsilon} = \sigma_1(\mathrm{tr}\,\mathbf{T})\mathrm{tr}\,\mathbf{T}, \quad \boldsymbol{\varepsilon}_\mathrm{D} = \sigma_2(|\mathbf{T}_\mathrm{D}|)\mathbf{T}_\mathrm{D}, \quad (133)$$

[20] Where for a tensor $\mathbf{A}$ we define $|\mathbf{A}| = \sqrt{\mathrm{tr}\,(\mathbf{A}\mathbf{A}^\mathrm{T})}$.

$\hat{\alpha}_1 = -5.973 \times 10^{-6} \frac{1}{[\mathrm{MPa}]}$	$\hat{\alpha}_2 = 2.0906 \times 10^{-5} \frac{1}{[\mathrm{MPa}]}$	
$\hat{\alpha}_3 = 2.685 \times 10^{-7} \frac{1}{[\mathrm{MPa}]}$	$\alpha_4 = 1.242 \times 10^{-3} \frac{1}{[\mathrm{MPa}]^2}$	$n = 1$
$\beta_1 = -5.973 \times 10^{-6} \frac{1}{[\mathrm{MPa}]}$	$\beta_2 = 2.0906 \times 10^{-5} \frac{1}{[\mathrm{MPa}]}$	
$\beta_3 = 2.685 \times 10^{-7} \frac{1}{[\mathrm{MPa}]}$	$\beta_4 = 3.736 \times 10^{-5} \frac{1}{[\mathrm{MPa}]^2}$	

Table 7: Material constants for the constitutive equation for strain limiting by Devendiran et al. [65, 66]

where $\boldsymbol{\varepsilon}_{\mathrm{D}}$ is the deviatoric part of the strain tensor $\boldsymbol{\varepsilon}_{\mathrm{D}} = \boldsymbol{\varepsilon} - \frac{(\mathrm{tr}\,\boldsymbol{\varepsilon})}{3}\mathbf{I}$, and where for the functions σ_1 and σ_2 it is assumed $\sigma_1(0) = \sigma_2(0) = 0$:

$$\sigma_1(\mathrm{tr}\,\mathbf{T}) = \frac{1}{3\kappa}\left(\frac{\tau_\kappa^2 + |\mathrm{tr}\,\mathbf{T}|^2}{\tau_\kappa^2}\right)^{\frac{s'-2}{2}}, \quad \sigma_2(|\mathbf{T}_{\mathrm{D}}|) = \frac{1}{2\mu}\left(\frac{\tau_\mu^2 + |\mathbf{T}_{\mathrm{D}}|^2}{\tau_\mu^2}\right)^{\frac{q'-2}{2}},$$

where κ, μ, τ_κ, τ_μ, s' and q' are material constants that are not shown here, and where $1 < q' < \infty$, $1 < s' < \infty$.

7.3.4 Application to the modelling of rock

Rock is a material that can show a different behaviour in tension compared with compression, and in some cases also a nonlinear relation between ε and σ (for the uniaxial uniform tension/compression test for a sample). As well as this, rock can show small strains before breaking apart and in many cases it has a notorious porous structure (i.e., the mechanical properties can depend on density as well), thus this is an interesting candidate for applications of (54) and (75)

A nonlinear constitutive equation for rock proposed by Bustamante and Rajagopal: The paper by Bustamante and Rajagopal [32] is devoted to the application of a subclass of (75) to the modelling the elastic behaviour of rock. Two simplifications were required, to assume that for a range of external loads rock is approximately elastic, i.e., that there was no conversion of mechanical work into heat, and that rock is isotropic. From the experimental information provided, for example, in [109] it is possible to see that the first assumption is approximately satisfied, regarding the second assumption there are several types of rocks that do not show any preferred directional response with regard to their mechanical behaviour. The model (78) is of the form:

$$\mathcal{G}(\sigma_1,\sigma_2,\sigma_3) = \mathfrak{f}_1(\sigma_1) + \mathfrak{f}_1(\sigma_2) + \mathfrak{f}_1(\sigma_3) + \mathfrak{f}_2(\sigma_1)(\sigma_2+\sigma_3) + \mathfrak{f}_2(\sigma_2)(\sigma_1+\sigma_3) + \mathfrak{f}_2(\sigma_3)(\sigma_1+\sigma_2) + \mathfrak{f}_3\left(\frac{\sigma_1+\sigma_2+\sigma_3}{3}\right), \quad (134)$$

where $\mathfrak{f}_1(x) = \alpha_1 \left[d_1^{c_1 x} - c_1 \ln(d_1)x\right]$, $\mathfrak{f}_2(x) = \alpha_2 \left(d_2^{c_2 x} - 1\right)$ and $\mathfrak{f}_3(x) = 3\alpha_3 \left[d_3^{c_3 x} - c_3 \ln(d_3)x\right]$, where the constants α_i, c_i, d_i, $i = 1, 2, 3$ are given in Table 8.

$\alpha_1 = 0.011[\text{MPa}]$	$\alpha_2 = -0.0004$	$\alpha_3 = 0.001[\text{MPa}]$
$c_1 = -0.08\frac{1}{[\text{MPa}]}$	$c_2 = -0.05\frac{1}{[\text{MPa}]}$	$c_3 = -0.08\frac{1}{[\text{MPa}]}$
$d_1 = 0.1$	$d_2 = 0.2$	$d_3 = 0.3$

Table 8: Material constants for the constitutive equation for rock by Bustamante and Rajagopal. [32]

The above constitutive equation (78), (134) has been used to solve some 2D boundary value problems in [40]. In that work four boundary value problems have been studied:

- The axial-symmetric compression of a cylinder with two boundary conditions. In one case considering the shear load due to friction caused due to the interaction with the testing machine, and in the second case assuming that in one of the surfaces of the cylinder we prescribe the axial uniform displacement field, while the other surface is assumed to be glued to the testing machine. The purpose of these two boundary value problems has been to analyze how a cylinder in compression behaves in a more realistic context when interacting with a testing machine.

- A plane plate (plane strain) with a circular hole in the middle, where a far away compressive uniform horizontal load σ_H, and a vertical uniform load σ_V are applied. In that problem the results for ε_{ij} and T_{ij} are compared with the predictions of the linearized theory, in particular for a line passing thought the middle of the hole. The results comparing the predictions of the two theories are very different, in particular regarding the maximum normal load, which for the nonlinear constitutive equation for rock (78), (134) does not appear on the surface of the hole, but is produced in a point slightly inside the plate near the surface of the hole (see Figure 13 in [40]).

- In the third problem the same plane plate described above is analyzed in the case of the presence of an elliptic hole in the middle, under the same far away horizontal and vertical tractions (in compression). Similar results were obtained for the stresses and strains, in particular regarding the localization of the maximum values for the stress, see, for example, Figures 21 and 22 of [40].

- The last problem considered in [40] corresponds to a plane plate with an elliptical hole, but where a far away uniform shear load is applied on one surface of the plate, while another surface is cannot displace in the horizontal direction.

A bimodular constitutive equation for rock: In [44] another constitutive equation has been proposed for rock, whose main characteristic is the use of two sets of material moduli (Young's modulus and Possion ratios) obtained from the compression and tension tests for a sample of rock without lateral constraints or loads. The constitutive equation is of the form (75), and it is given as:

$$\boldsymbol{\varepsilon} = \frac{[1+\hat{\nu}(h)]}{\hat{E}(h)}\mathbf{T} - \frac{\hat{\nu}(h)}{\hat{E}(h)}h\mathbf{I}, \qquad \text{where} \quad h = \operatorname{tr}\mathbf{T}, \tag{135}$$

and where $\hat{E}(h) = E_{\mathrm{T}}q_{\mathrm{T}}(h) + E_{\mathrm{C}}q_{\mathrm{C}}(h)$, $\hat{\nu}(h) = \nu_{\mathrm{T}}q_{\mathrm{T}}(h) + \nu_{\mathrm{C}}q_{\mathrm{C}}(h)$, $q_{\mathrm{T}}(h) = \frac{1}{2}[1+\operatorname{erf}(Ch)]$, $q_{\mathrm{C}}(h) = \frac{1}{2}[1-\operatorname{erf}(Ch)]$, where erf is the error function, E_{T}, ν_{T} are the Young's modulus and Poisson ratio determined experimentally from a cylindrical sample of rock in tension, while E_{C}, ν_{C} are obtained from a sample of rock in compression (in both cases without lateral load), and C is a positive constant that should be very 'large'.

The above constitutive equation has been applied to two types of rock that have a relatively similar behaviour, namely granodiorite (for the compression tests), and diorite-monzonite (for the tension tests). Rock can be a notoriously heterogeneous material, thus the experimental data show some variation for different samples. In Table 9 we show some values for the material constants for the samples called CS62 and TE7 (see [44]). For these material constants it is necessary to notice that ν_{T} is not obtained directly from experiments, but from the relation (see (25) in [44]) $\nu_{\mathrm{T}}/E_{\mathrm{T}} = \nu_{\mathrm{C}}/E_{\mathbf{C}}$. The constant C that appears in the function presented in the paragraph after Eq. (135) is not obtained from experiments. It has units 1/[MPa] (or more

$E_{\mathrm{C}} = 4.8972 \times 10^4$[MPa]	$\nu_{\mathrm{C}} = 0.1737$	$E_{\mathrm{T}} = 459.5588$[MPa]	$\nu_{\mathrm{T}} = 0.0016$

Table 9: Material constants for the bimodular constitutive equation for rock. [44]

generally 1/stress), and it should be 'large' enough such that the transition between compression and tension happens rapidly.

Apart from proposing (135) in [44] some boundary value problems where studied considering homogeneous distributions for strains and stresses (see Section 8.2.2), and the propagation of S and P waves was analyzed (see Section 8.1.1) for a cylindrical sample of rock under compression.

It is important to indicate that (135) was not obtained from a potential $\mathcal{G}$ (see (77)) and from [23] that means we cannot say that this body will behaves elastically for all deformations.

A nonlinear constitutive equation for isotropic rock by Shariff and Bustamante: In [196] a different type of constitutive equation for rock has been presented, which is a special case of (78) (see Section 5.2.1). If σ_i are the principal stresses of $\mathbf{T}$, defining $h = \sigma_1 + \sigma_2 + \sigma_3$, for the deviatoric stress tensor $\mathbf{T}_{\mathrm{D}} = \mathbf{T} - \frac{(\operatorname{tr}\mathbf{T})}{3}\mathbf{I}$ the principal stresses are $\sigma_{\mathrm{D}_i} = \sigma_i - h/3$. If E and ν are the ground or referential Young's modulus and Poisson ratio for the solid, then defining $\bar{\sigma}_{\mathrm{D}_i} = \sigma_{\mathrm{D}_i}/E$ and $\bar{h} = h/E$, the following constitutive equation is proposed for isotropic rock:

$$\frac{\mathcal{G}}{E} = (1+\nu)I + \frac{(1-2\nu)}{3} g(\bar{h}), \tag{136}$$

where $I = \sum_{i=1}^{3} f(\bar{\sigma}_{\mathrm{D}_i})$, and it is assumed that $f(0) = f'(0) = g(0) = 0$, $f''(0) = g''(0) = 1$ and f', g' are strictly monotonic.

The linearized elastic constitutive equation is recovered from (136) if $f(\bar{\sigma}_{\mathrm{D}_i}) = \bar{\sigma}_{\mathrm{D}_i}^2/2$ and $g(h) = \bar{h}^2/2$.

Using (136) in (78) we obtain

$$\boldsymbol{\varepsilon} = (1+\nu)\sum_{i=1}^{3} f'(\bar{\sigma}_{\mathrm{D}_i})\left[\mathbf{t}^{(i)} \otimes \mathbf{t}^{(i)} - \frac{1}{3}\mathbf{I}\right] + \frac{(1-2\nu)}{3} g'(\bar{h})\mathbf{I}, \tag{137}$$

where $\mathbf{t}^{(i)}$ are the principal directions of $\mathbf{T}$. From (137) we get

$$\varepsilon_i = (1+\nu)\left[f'(\bar{\sigma}_{\mathrm{D}_i}) - \frac{1}{3}\sum_{j=1}^{3} f'(\bar{\sigma}_{\mathrm{D}_j})\right] + \frac{(1-2\nu)}{3} g'(\bar{h}), \quad i = 1, 2, 3.$$

The above constitutive equation satisfies the Baker-Ericksen inequality and the pressure-compression inequality (see Section 51 of [209]).

In [196] some boundary value problems are studied, such as the uniform expansion/compression of a cylinder, the compression and tension of a slab, the shear and compression of a slab, and the propagation of small amplitude waves on infinite media with initial 'large' stresses (see Section 8.2.2). In order to fit data for rock [109] it is assumed that

$$f'(x) = \frac{\tan^{-1}(ax)}{a}, \quad g'(x) = \frac{\frac{e^{bx}-1}{b} + \sinh(x)}{1+c}, \tag{138}$$

where a, b and c are material constants.

For the class of nonlinear rock studied in [196], in Table 10 we show the material constants for (138).

$E = 2600$[MPa]	$\nu = 0.1038$	$a = 0.6 \times 2600$	$b = 0.5 \times 2600$	$c = 2600$

Table 10: Material constants for the nonlinear constitutive equation for rock by Shariff and Bustamante. [196]

A constitutive equation for transversely isotropic rock: In [195] the constitutive equation (81)-(83) is used for the modelling of some types of rock that can be considered as transversely isotropic. Let us assume the existence of a Gibbs potential $\mathcal{G} = \mathcal{G}(\mathbf{T}, \mathbf{a})$, where $|\mathbf{a}| = 1$. Then if σ_i, $\mathbf{t}^{(i)}$ are the principal stresses and principal directions of $\mathbf{T}$ and $\zeta_i = \left[\mathbf{t}^{(i)} \cdot \mathbf{a}\right]^2$ (see Section 5.2.2), then the model proposed in [195] is:

$$\frac{\mathcal{G}}{E_{\mathrm{p}}^2} = b_1 K_0 + b_2 g_0(h) + b_3 K_1 + b_4 K_2^2 + b_5 K_3 g_1(h), \tag{139}$$

where

$$K_0 = \sum_{i=1}^{3} f_0(\bar{\sigma}_i^*), \quad K_\alpha = \sum_{i=1}^{3} \zeta_i f_\alpha(\bar{\sigma}_i^*), \quad \alpha = 1, 2, 3, \tag{140}$$

where E_{p} is the ground 'Young modulus' in the plane normal to $\mathbf{a}$, b_m, $m = 1, 2, 3, 4, 5$ are material constants, $\bar{\sigma}_i^* = \hat{\sigma}_i - h/3$, $\hat{\sigma}_i = \sigma_i/E_{\mathrm{p}}$, $h = \bar{h}/E_{\mathrm{p}}$ and $\bar{h} = \operatorname{tr}\mathbf{T}/3$, and g_0, g_1, f_0 and f_α are functions.

The constitutive equations (5), (139), (140) have forms that are similar to the counterpart for linearized transversely isotropic elastic bodies.

In [195] some restrictions are proposed for the functions in (139), (140), namely: $f_0(0) = f_0'(0) = g_0'(0) = f_1(0) = f_1'(0) = 0$, $f_0''(0) = g_0''(0) = f_1''(0) = 2$, $f_\beta(0) = g_1(0) = 0$, $f_\beta'(0) = g_1'(0) = 1$, $\beta = 2, 3$ (other restrictions are imposed but are not shown here). The function $\mathcal{G}$ satisfies the P-property as stated, for example, in [191, 193]. From $\boldsymbol{\varepsilon} = \frac{\partial \mathcal{G}}{\partial \mathbf{T}}$ (see (81)-(83)) we obtain

$$\boldsymbol{\varepsilon} = \sum_{i,j=1}^{3} \left(\frac{\partial \mathcal{G}}{\partial \mathbf{T}} \right)_{ij} \mathbf{t}^{(i)} \otimes \mathbf{t}^{(i)},$$

where $\left(\frac{\partial \mathcal{G}}{\partial \mathbf{T}}\right)_{ii} = \frac{\partial \mathcal{G}}{\partial \sigma_i}$, $\left(\frac{\partial \mathcal{G}}{\partial \mathbf{T}}\right)_{ij} = \frac{a_i a_j}{(\sigma_i - \sigma_j)} \left(\frac{\partial \mathcal{G}}{\partial \zeta_i} - \frac{\partial \mathcal{G}}{\partial \zeta_j} \right)$ (where $i \neq j$ and $a_i = \mathbf{t}^{(i)} \cdot \mathbf{a}$ and in the above expressions repetition of indices does not mean sum).

In [195] some boundary value problems are studied, namely the uniform axial deformation of a cylinder where $\mathbf{a} = \mathbf{e}_3$, the compression/tension and shear of a slab where $\mathbf{a} = s\mathbf{t}^{(1)} + c\mathbf{t}^{(2)}$, $s = \sin\phi$, $c = \cos\phi$ (where ϕ is the angle of $\mathbf{a}$ with respecto to $\mathbf{e}_2$).

The above constitutive equation (139) is applied for the modelling of Marcellus shale [108], wherein it is assumed that

$$f_0(x) = f_1(x) = x^2, \quad g_1(x) = f_2(x) = f_3(x) = x, \quad g_0'(x) = 2\frac{(e^{dx} - 1)}{d},$$

where d is a constant. For such a type of rock the material constants are shown in Table 11.

Tension				
$E_a = 11.5$[GPa]	$E_p = 37.06$[GPa]	$\nu_{zp} = 0.33$	$\nu_p = 0.18$	$\mu_a = 6.40$
Compression				
$E_a = 16.12$[GPa]	$E_p = 37.72$[GPa]	$\nu_{zp} = 0.35$	$\nu_p = 0.25$	$\mu_a = 5.68$

Table 11: Material constants for the nonlinear transversely isotropic constitutive equation for rock. The material constant d is equal to -0.01 in compression and -2 in tension. [195]

The different constants are obtained from such data from the following relations:

$$b_1 = c_1, \quad b_2 = \frac{c_1}{3} + c_2 + \frac{c_3}{9} + \frac{c_4}{9} + \frac{c_5}{3}, \quad b_3 = c_2, \quad b_4 = c_4 \quad b_5 = \frac{2}{3}(c_3 + c_4) + c_5,$$
$$c_1 = \frac{1 + \nu_p}{2E_p}, \quad c_2 = -\frac{\nu_p}{2E_p}, \quad c_3 = \frac{1}{4\mu_a} - \frac{(1 + \nu_p)}{2E_p}, \quad c_4 = \frac{1}{2E_a} + \frac{2\nu_a - \nu_p}{2E_p} - \frac{1}{4\mu_a},$$
$$c_5 = \frac{\nu_p - \nu_a}{E_p}.$$

An implicit constitutive relation for rock by Bustamante and Rajagopal: In this section we show an application of (61)-(63) for the modelling of dry rock. Recalling the notation $\hat{\varepsilon}_z(\sigma_z)$ and $\hat{\varepsilon}_r(\sigma_z)$ for the axial and radial components of the strain tensor, for the case of a cylinder deforming uniformly without lateral constrains, under the application of an axial uniform stress σ_z, using the experimental data from [44] it has been proposed

$$\hat{\varepsilon}_z(\sigma_z) = \hat{\varepsilon}_z^{(\mathrm{C})}(\sigma_z) q_{\mathrm{C}}(\sigma_z) + \hat{\varepsilon}_z^{(\mathrm{T})}(\sigma_z) q_{\mathrm{T}}(\sigma_z),$$
$$\hat{\varepsilon}_r(\sigma_z) = \hat{\varepsilon}_r^{(\mathrm{C})}(\sigma_z) q_{\mathrm{C}}(\sigma_z) + \hat{\varepsilon}_r^{(\mathrm{T})}(\sigma_z) q_{\mathrm{T}}(\sigma_z),$$

where $q_{\mathrm{T}}(\sigma_z) = \frac{1}{2}[1+\mathrm{erf}(C\sigma_z)]$, $q_{\mathrm{C}}(\sigma_z) = \frac{1}{2}[1-\mathrm{erf}(C\sigma_z)]$, $\hat{\varepsilon}_z^{(\mathrm{C})}(\sigma_z) = \xi_z^{(\mathrm{C})}\sigma_z + \zeta_z^{(\mathrm{C})}\sigma_z^2$, $\hat{\varepsilon}_z^{(\mathrm{T})}(\sigma_z) = \xi_z^{(\mathrm{T})}\sigma_z + \zeta_z^{(\mathrm{T})}\sigma_z^2$, $\hat{\varepsilon}_r^{(\mathrm{C})}(\sigma_z) = \xi_r^{(\mathrm{C})}\sigma_r + \zeta_r^{(\mathrm{C})}\sigma_r^2$ and[21] $\hat{\varepsilon}_r^{(\mathrm{T})}(\sigma_z) = -\nu\hat{\varepsilon}_z^{(\mathrm{T})}(\sigma_z)$, where C is a positive constant, erf is the error function, $\xi_z^{(\mathrm{C})}$, $\zeta_z^{(\mathrm{C})}$, $\xi_r^{(\mathrm{C})}$ and $\zeta_r^{(\mathrm{C})}$ are material constants obtained from a sample under compression, $\xi_z^{(\mathrm{T})}$, $\zeta_z^{(\mathrm{T})}$ are material constants for a sample of rock under tension, and ν would correspond to the Poisson ration for a sample in compression. In Table 12 we give the values for the above material constants.

$\xi_z^{(\mathrm{C})} = 1.824 \times 10^{-6}[\mathrm{MPa}]^{-1}$	$\zeta_z^{(\mathrm{C})} = -3.888 \times 10^{-7}[\mathrm{MPa}]^{-2}$
$\xi_r^{(\mathrm{C})} = 1.086 \times 10^{-7}[\mathrm{MPa}]^{-1}$	$\zeta_r^{(\mathrm{C})} = 7.631 \times 10^{-8}[\mathrm{MPa}]^{-2}$
$\xi_z^{(\mathrm{T})} = 0.001647[\mathrm{MPa}]^{-1}$	$\zeta_z^{(\mathrm{T})} = 9.699 \times 10^{-5}[\mathrm{MPa}]^{-2}$
$\nu = 0.17203$	

Table 12: Values of the material constants for the implicit constitutive relation for rock by Bustamante and Rajagopal [49].

The above model were used for the study of the inflation of cylindrical annulus in [49].

7.3.5 Modelling of Concrete

Concrete is also a material that can show a nonlinear behaviour similar to what has been described for rock, and in this section we show some constitutive equations and relations to describe the mechanical behaviour of such a material.

[21]From [44] we do not have data for $\hat{\varepsilon}_r(\sigma_z)$ in tension, therefore we assume that that components of the strain $\hat{\varepsilon}_r(\sigma_z)$ can be obtained approximately from $\hat{\varepsilon}_z(\sigma_z)$ with the Poisson ration ν. Such a ratio has been obtained from the experimental data from the same reference [44], and it is also shown in Table 12.

A constitutive equation by Grasley et al.: In this section we document an expression belonging to a sub-class of (75) which has been used successfully in modelling the behaviour of concrete. Grasley et al. [88] proposed the constitutive equation[22]

$$\boldsymbol{\varepsilon} = \beta_0 \mathbf{I} + \beta_1 \mathbf{T} + \beta_2 \mathbf{T}^2, \tag{141}$$

where $\beta_0 = \gamma_1 \mathrm{tr}\,(\mathbf{T}) + \sinh\left[\frac{\mathrm{tr}\,(\mathbf{T})^{\gamma_2}}{\gamma_3}\right]$, $\beta_1 = \gamma_4$ and $\beta_2 = 0$, where the constants γ_i, $i = 1, 2, 3, 4$ are given in Table 13.

$\gamma_1 = 2.3398 \frac{1}{[\mathrm{MPa}]}$	$\gamma_2 = 1.3105$	$\gamma_3 = 37.52[\mathrm{MPa}]^{\gamma_2}$	$\gamma_4 = 19.8398$

Table 13: Material constants for the constitutive equation for concrete proposed by Grasley et al. [88]

A constitutive equation by Murru et al.: In [139] a constitutive relation for concrete has been proposed, based on assuming that the porosity is important for the behaviour of such a solid. From Section 5.1.1 the porosity is directly connected with the density of the body and $\rho \approx \rho_{\mathrm{r}}(1 + \mathrm{tr}\,\boldsymbol{\varepsilon})$. The constitutive relation is:

$$\boldsymbol{\varepsilon} = -\frac{\nu}{E_{\mathrm{r}}(1 - \gamma\rho_{\mathrm{r}}\mathrm{tr}\,\boldsymbol{\varepsilon})}(\mathrm{tr}\,\mathbf{T})\mathbf{I} + \frac{(1+\nu)}{E_{\mathrm{r}}(1 - \gamma\rho_{\mathrm{r}}\mathrm{tr}\,\boldsymbol{\varepsilon})}\mathbf{T},$$

where E_{r}, ν are the Young's modulus and Poisson ration in the reference configuration and γ is a material constant.

In [139] for the 1D uniaxial deformation of a sample, the above constitutive relation shows strain softening and if we plot the uniaxial stress versus the strain, a local maximum (peak) is reached that can be considered as the initiation of damage, and which can be considered as a limit for the application of the above model. The material constants are fitted with experimental data, and some boundary value problems are solve using the finite element method. For the sake of brevity such material constants are not shown here.

An implicit constitutive relation for concrete by Bustamante and Rajagopal: Here we show an application of (61)-(63) for the modelling of

[22] In (141) the strains must be divided by a factor 10^6.

the mechanical behaviour of concrete. If the experimental data is used for the fitting of the implicit relation is obtained for the tension/compression of a cylinder (without laterial constraints), recalling the notation $\hat{\varepsilon}_z(\sigma_z)$, $\hat{\varepsilon}_r(\sigma_z)$ for the axial and radial strains, as functions of the applied uniform axial stress σ_z, which are obtained from experiments.

From such experimental data (see [88]) for the case of concrete the following expressions are proposed for $\hat{\varepsilon}_z(\sigma_z)$ and $\hat{\varepsilon}_r(\sigma_z)$ (see [49]):

$$\begin{aligned}\hat{\varepsilon}_z(\sigma_z) &= \xi\sigma_z + \zeta\sigma_z^5, \\ \hat{\varepsilon}_r(\sigma_z) &= \varpi_1[\exp(\vartheta_1\sigma_z) - 1] + \varpi_2[\exp(\vartheta_2\sigma_z) - 1],\end{aligned}$$

where ξ, ζ, ϖ_1, ϖ_2, ϑ_1, ϑ_2 are material constants given in Table 14 (see Table 1 in [49]).

$\xi = 2.364 \times 10^{-5}[\text{MPa}]^{-1}$	$\zeta = 1.433 \times 10^{-13}[\text{MPa}]^{-5}$
$\varpi_1 = 1.809 \times 10^{-5}$	$\zeta_1 = -0.03768[\text{MPa}]^{-1}$
$\varpi_2 = 1.904 \times 10^{-16}$	$\zeta_2 = -0.3607[\text{MPa}]^{-1}$

Table 14: Values of the material constants for the implicit constitutive relation for concrete by Bustamante and Rajagopal [49].

A constitutive equation for plain cement concrete by Gokulnath et al.: In [84] a constitutive equation has been proposed for plain cement concrete, based on the use of a Gibbs potential and $\boldsymbol{\varepsilon} = \frac{\partial \mathcal{G}}{\partial \mathbf{T}}$, assuming that for certain range of stresses and strain cement can be approximately modelled as an elastic solid. The particular expression for the above constitutive equation is:

$$\begin{aligned}\boldsymbol{\varepsilon} = &\left[2m_1 K_4 K_2^{1/2} + 2m_3 K_4 K_2^2 + 4m_5\left(K_4^2 - 1\right) K_4 K_2^2\right]\mathbf{I} \\ &+ \left\{2m_2 + K_2^{3/2}\left[3m_3 K_4^2 + 2m_4 + \left(K_4^4 - 6K_4^2 + 5\right) m_5\right]\right\}\mathbf{T}, \quad (142)\end{aligned}$$

where m_i, $i = 1, 2, 3, 4, 5$ are material parameters and $K_1 = \operatorname{tr}\mathbf{T}$, $K_2 = \operatorname{tr}(\mathbf{T}^2)$, $K_3 = \operatorname{tr}(\mathbf{T}^3)$, $K_4 = K_1/\sqrt{K_2}$ and $K_5 = K_3/K_2^{3/2}$.

7.3.6 Modelling of the mechanical behaviour of bone

Bone is a material that can behave nonlinearly in the range of small strains (see, for example, [67, 92, 114, 119, 178, 179, 183, 210, 218]). Trabecular bone is also a highly porous material, and its mechanical behaviour depends on such porosity density [113, 120, 134], therefore, the constitutive theory presented in Section 5.1.1 can be particularly interesting for the development of constitutive relations for such materials. In this section we show some works on that.

In [5] the following implicit relation has been proposed for trabecular bone:

$$[1 + \lambda(\mathrm{tr}\,\mathbf{T})]\boldsymbol{\varepsilon} - B_1[1 + \gamma(\mathrm{tr}\,\boldsymbol{\varepsilon})]\mathbf{T} - B_2[1 - \gamma(\mathrm{tr}\,\boldsymbol{\varepsilon})](\mathrm{tr}\,\mathbf{T})\mathbf{I} = \mathbf{0} \qquad (143)$$

In Figure 1 therein we can see the predictions of the model against actual experimental data. In the same paper the finite element method is used to study the behaviour of a dogbone sample under tension, assuming the sample as a homogeneous body, with an inhomogeinity in the centre of the same (that is an inclusion with different density). Circular inclusions, semi-circular inclusion, and a square inclusion are considered, as well as this, lateral inclusion and a lateral hole are analyzed. The material constants for (143) are presented in Table 15.

$\lambda = 0$	$E = 200$[MPa]	$\nu = 0.3$
$B_1 = \frac{1+\nu}{E}$	$B_2 = -\frac{\nu}{E}$	$\gamma = 35$

Table 15: Values of the material constants for the implicit constitutive relation for bone (143). See [5].

In [107] some numerical results are obtained for the same materials constants that appear in (143) for trabecullar bone.

Another variation of (143) was studied in that work [107], wherein

$$[1 + \lambda_1(\mathrm{tr}\,\mathbf{T})]\boldsymbol{\varepsilon} - B_1[1 + \lambda_2|\mathrm{tr}\,\boldsymbol{\varepsilon}|]\mathbf{T} - B_2[1 - \lambda_3|\mathrm{tr}\,\boldsymbol{\varepsilon}|](\mathrm{tr}\,\mathbf{T})\mathbf{I} = \mathbf{0},$$

where λ_i, $i = 1, 2, 3$ are material constants. In [107] the finite element method is used to study the problem of tension for a dogbone sample without cracks, and considering a small crack in the middle and in a third example a crack on the side of the sample (notch), among other problems.

7.4 Some applications for electro-elastic and magneto-elastic bodies

In this section the constitutive equations and relations discussed in Section 5.3 are applied for two types of electro-magneto-active bodies, considering actual experimental data.

7.4.1 A constitutive equation for electro-elastic bodies

A constitutive equation of the form (95) has been proposed in [220] for some types of electro-elastic solids, for the particular case (recall that in electro-elasticity we use the notation $\boldsymbol{\sigma}$ to denote the Cauchy stress tensor) $\Pi = \Pi(\boldsymbol{\sigma}, \mathbb{E})$ is an isotropic function (this is a special case of what it is shown in Section 5.3.2 and Eqs. (95), (96)). Here we need to consider (see Sections 12 and 12) $\boldsymbol{\varepsilon} = \frac{\partial \Pi}{\partial \boldsymbol{\sigma}}$, $\mathbb{P} = \frac{\partial \Pi}{\partial \mathbb{E}}$, $\operatorname{div}\boldsymbol{\sigma} + \operatorname{grad}(\mathbb{E})\mathbb{P} = \mathbf{0}$ and $\nabla \times \mathbb{E} = \mathbf{0}$, $\nabla \cdot \mathbb{D} = 0 \Leftrightarrow \nabla \cdot (\mathbb{P} + \epsilon_o \mathbb{E}) = 0$.

The particular expression for Π proposed in [220] for $BiFeO_3$ and PLZT ceramic is:

$$\Pi = \left\{ -\alpha \left[I_1 - \int_0^{I_1} \frac{1}{1+\beta(w^2)^b} \mathrm{d}w \right] + \frac{\alpha\gamma}{\iota}\sqrt{1+2\iota I_2} \right\} (g_0 + g_1 I_4) + \ln\left[\cosh\left(\frac{\sqrt{I_4} - m_2}{m_1} \right) \right] m_0 m_1 + m_3\sqrt{I_4} + \frac{\zeta_0}{2} I_4 + \frac{\epsilon_o \zeta_1}{2} I_5, \quad (144)$$

where we recall that $I_1 = \operatorname{tr}\boldsymbol{\sigma}$, $I_2 = \frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}^2)$, $I_4 = \mathbb{E} \cdot \mathbb{E}$, $I_5 = \mathbb{E} \cdot (\boldsymbol{\sigma}\mathbb{E})$, and where α, β, b, γ, ι, g_0, g_1, m_0, m_1, m_2, m_3, ζ_0 and ζ_1 are material constants. The material constants that appear in (144) are shown in Table 16.

$\alpha = 22$	$\beta = 0.01 \left[\frac{1}{\mathrm{MPa}^{2b}}\right]$	$b = 0.55$	$\gamma = 0.13 \left[\frac{1}{\mathrm{MPa}}\right]$
$\iota = 0.001 \left[\frac{1}{\mathrm{MPa}^2}\right]$	$g_0 = 0.0035$	$g_1 = -1.5 \times 10^{-6}$	$m_0 = 14.5 \left[\frac{\mu\mathrm{C}}{\mathrm{kVcm}}\right]$
$m_1 = 3 \left[\frac{\mathrm{kV}}{\mathrm{cm}}\right]$	$m_2 = 13 \left[\frac{\mathrm{kV}}{\mathrm{cm}}\right]$	$m_3 = 14.495 \left[\frac{\mu\mathrm{C}}{\mathrm{cm}^2}\right]$	$\zeta_0 = 0.14 \left[\frac{\mu\mathrm{C}}{\mathrm{kVcm}}\right]$
$\zeta_1 = 13$	$\epsilon_o = 8.85 \times 10^{-5} \left[\frac{\mu\mathrm{C}}{\mathrm{kVcm}}\right]$		

Table 16: Values of the material constants for the implicit constitutive relation for a type of electroelastic solid (144), see [220].

In [220] some boundary value problems are solved using the finite element method. The problems that are solved are: the inflation of a hollow sphere, the inflation and extension of a cylindrical annulus, the inflation, extension and telescopic shear of a cylindrical annulus, and the inflation, extension and circumferential shear of a cylindrical annulus.

The constitutive equation (144) is valid for Π isotropic, and the more realistic case of a transversely isotropic Π has been addressed partially in the recent work in [9].

7.4.2 Some constitutive relations for magneto-elastic bodies

In this review we have not shown details about magneto-elastic bodies. The reason is that most of the constitutive relations and equations are very similar to what has been presented in Sections 4.1, 4.2 and 5.3, but replacing the electric field, electric displacement and the polarization by the magnetic field, magnetic induction and magnetization, respectively (and adjusting some other minor things in other equations). As well as this, not many works have appeared in the literature with applications of such constitutive theories (unlike the case of electro-elastic solids where some few examples and applications have been studied, see [9, 220]) with the exception of the paper [204]. In that work, assuming that the magnetization $\mathbb{M}$ and the total stress $\boldsymbol{\tau}$ are the main variables, the following constitutive equations are proposed (compare with (95) and (96)):

$$\boldsymbol{\varepsilon} = \mathfrak{h}(\boldsymbol{\tau}, \mathbb{M}), \quad \mathbb{H} = \mathfrak{r}(\boldsymbol{\tau}, \mathbb{M}), \tag{145}$$

where for quasi-static problems (see (342) and (344) in the Appendix for the magneto-elastic case) $\operatorname{div}\boldsymbol{\tau} = \mathbf{0}$ and $\operatorname{div}\mathbb{B} = 0$, where (see the paragraph after (342) in the Appendix) $\mathbb{B} = \mu_o(\mathbb{H} + \mathbb{M})$.

The particular expression for $\mathfrak{h}$ proposed in [204] is:

$$\mathfrak{h}(\boldsymbol{\tau}, \mathbb{M}) = \mathscr{S} \cdot \boldsymbol{\tau} + \frac{\lambda_s}{M_s^2}\left[\frac{3}{2}\mathbb{M} \otimes \mathbb{M} - (\mathbb{M} \cdot \mathbb{M})\left(\frac{1}{2}\mathbf{I} + \frac{1}{\sigma_s}\tilde{\mathbf{T}}\right)\right], \tag{146}$$

where $\mathscr{S}$ is a fourth order tensor (the compliance of the solid) and λ_s, σ_s and M_s are the saturation values for the magnetization, stress and magnetostriction, respectively. The tensor $\tilde{\mathbf{T}}$ is defined as $\tilde{\mathbf{T}} = \frac{3}{2}\boldsymbol{\tau} - \frac{1}{2}(\operatorname{tr}\boldsymbol{\tau})\mathbf{I}$.

In the case of $(145)_2$ the authors propose:

$$\mathfrak{r}(\boldsymbol{\tau}, \mathbb{M}) = \frac{1}{kM} f^{-1}\left(\frac{M}{M_s}\right)\mathbb{M} - \frac{\lambda_s}{\mu_o M_s^2}\left[2\tilde{\mathbf{T}}\mathbb{M} - \frac{(I_\sigma^2 - 3II_\sigma)\mathbb{M}}{\sigma_s}\right], \tag{147}$$

where $M = \sqrt{\mathbb{M} \cdot \mathbb{M}}$, k is a material constant, $I_\sigma = I_1 = \operatorname{tr}\boldsymbol{\tau}$, $II_\sigma = I_2 = \frac{1}{2}[I_1^2 - \operatorname{tr}(\boldsymbol{\tau}^2)]$ and $f(x) = \coth(x) - 1/x$. The material constants for the above constitutive relations are shown in Table 17 for the plane stress problem.

In [204] some boundary value problems are solved using the finite element method for the case of plane stress problems, assuming $\mathbb{H} = -\operatorname{grad}\psi$, where ψ is a scalar magnetic potential. The main variables for the problem are ψ and the displacement field $\mathbf{u}$. The stress $\boldsymbol{\tau}$ is given in terms of the stress

$\lambda_s = 1000[\text{ppm}]$	$M_s = 6.35 \times 10^5 \left[\frac{\text{A}}{\text{m}}\right]$	$\sigma_s = 200 \times 10^6 \left[\frac{\text{N}}{\text{m}^2}\right]$
$k = 6.06 \times 10^{-5} \left[\frac{\text{m}}{\text{A}}\right]$	$\mathscr{S}_{11} = 4.3478 \times 10^{-11} \left[\frac{\text{m}^2}{\text{N}}\right]$	$\mathscr{S}_{22} = 4.3478 \times 10^{-11} \left[\frac{\text{m}^2}{\text{N}}\right]$
$\mathscr{S}_{12} = -1.3043 \times 10^{-11} \left[\frac{\text{m}^2}{\text{N}}\right]$	$\mathscr{S}_{66} = 1.1304 \times 10^{-10} \left[\frac{\text{m}^2}{\text{N}}\right]$	

Table 17: Values of the material constants for (146) and (147). The constant μ is the magnetic permeability in vaccum.

potential, and the function that is used for the finite element formulation is the function error $e_r = \int_V \sqrt{\boldsymbol{\mathfrak{F}} \cdot \boldsymbol{\mathfrak{F}}}\, \mathrm{d}V$, where $\boldsymbol{\mathfrak{F}}$ is a function that contains (146) and (147). The above function e_r is minimized in terms of ψ and $\mathbf{u}$.

8 Boundary value problems for the case of small gradient of the displacement field

In this section we consider some boundary value problems, for the subclass of the constitutive equation (75). Let us discuss about the boundary value problem for the implicit relation (5), where we work with the second Piola-Kirchhoff stress tensor $\mathbf{S}$ and the Lagrange Saint-Venant strain tensor $\mathbf{E}$ as the main variables for the problem. In such a case, in order to find the stresses and the deformation, we need to solve in parallel the equation of motion $(4)_1$, re-written in the reference configuration, and (5), i.e.:

$$\rho_{\mathrm{r}}\ddot{\mathbf{u}} = \mathrm{Div}(\mathbf{F}^{\mathrm{T}}\mathbf{S}) + \rho_{\mathrm{r}}\mathbf{b}, \quad \boldsymbol{\mathfrak{F}}(\mathbf{S},\mathbf{E}) = \mathbf{0}, \quad \mathbf{X} \in \boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B}), \tag{148}$$

with the boundary conditions $(\mathbf{F}^{\mathrm{T}}\mathbf{S})\mathbf{N} = \hat{\mathbf{p}}$ where $\hat{\mathbf{p}}$ is the external traction, for a sub-part $\partial\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})_p$ of the boundary, where $\mathbf{N}$ is the unit normal vector to that surface in the reference configuration, and $\mathbf{u} = \hat{\mathbf{u}}$ where $\hat{\mathbf{u}}$ is a specification of the displacement on some part of the boundary of the body $\mathbf{X} \in \partial\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})_x$, where $\partial\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B}) = \partial\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})_p \cup \partial\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})_x$ and $\partial\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})_p \cap \partial\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})_x = \emptyset$. We need to recall that $\mathbf{u} = \mathbf{x} - \mathbf{X} = \boldsymbol{\chi}(\mathbf{X}, t) - \mathbf{X}$ and $(1)_4$, (2), therefore, we have three equations (from the equation of motion $(148)_1$), plus six equations from the implicit relation $(148)_2$, which must be solved to find the three components of $\mathbf{x}$ and the six independent components of $\mathbf{S}$. Therefore, for this class of implicit relation, if we want to solve boundary value problems using the semi-inverse method, we need to not only propose some simplified expressions for the displacement $\mathbf{u}$, but also some simplified expression for $\mathbf{S}$, which should be compatible with the above field, and to solve the resultant simplified equations from (148). In comparison, in the classical theory of nonlinear elasticity, we just need to propose a simplified expression for $\mathbf{x}$ and from $\mathbf{T} = \mathfrak{f}(\mathbf{F})$ we obtain the components of the stress, which are replaced in the equation of motion that is now written in terms of simplified expressions for the deformation field.

Now, in the case of the subclass (75), we need to find the displacement field $\mathbf{u}$ and the Cauchy stress tensor $\mathbf{T}$ by solving

$$\rho\ddot{\mathbf{u}} = \operatorname{div}\mathbf{T} + \rho\mathbf{b}, \quad \boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) = \mathfrak{h}(\mathbf{T}), \tag{149}$$

where ρ should be found from the mass balance[23] $(4)_2$, and $\mathbf{T}\mathbf{n} = \hat{\mathbf{t}}$, for $\mathbf{x} \in \partial\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})_t$, $\mathbf{n}$ is the unit normal vector to that surface, $\mathbf{u} = \hat{\mathbf{u}}$ for $\mathbf{x} \in \partial\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})_u$, where $\hat{\mathbf{t}}$ and $\hat{\mathbf{u}}$ are the external traction, and the specification of the displacement field, respectively, and $\partial\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B}) = \partial\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})_t \cup \partial\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})_u$ and $\partial\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})_t \cap \partial\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})_u = \emptyset$. In this case we need to solve the three components of the equation of motion, plus the six components of $(149)_2$, in order to find the three components of $\mathbf{u}$ and the six independent components of $\mathbf{T}$.

Following the approach used in the classical theory of elasticity, one may be tempted to try to invert $\mathfrak{h}(\mathbf{T})$ to express the stresses in terms of the strains, and to replace such expressions in the equation of motion, to have three equations in terms of the components of the displacement field. However, as explained in [169], while it might be mathematically possible to invert such an expression, the results so obtained could involve terms of higher order in the linearized strain, which would not have been acceptable within the context of the linearized theory.

Another approach that can be used when $\boldsymbol{\varepsilon} = \mathfrak{h}(\mathbf{T})$ is to recall what is done in the linearized theory of elasticity, when the strains are expressed in terms of the stresses, for example, in the case of isotropic bodies we have $\boldsymbol{\varepsilon} = \frac{(1+\nu)}{E}\mathbf{T} - \frac{\nu}{E}\operatorname{tr}(\mathbf{T})\mathbf{I}$, where for the quasi-static case and plane stress problem (no body forces), the components of the stress are expressed in terms of an Airy stress potential, and satisfying the equations of equilibrium automatically, and by invoking the compatibility equation to obtain the well known biharmonic equation. However, it is not mandatory to consider the

[23] In the rest of this section we will assume that the mass density changes little, and that such small changes in density do not have an important impact with regard to the behaviour of the bodies, thus we do not solve $(4)_2$ and assume that ρ is approximately constant.

compatibility equations for the strains (see [164]), and secondly, even if we use them, in our case the resulting equation is a nonlinear fourth order partial differential equation for the stress potential, which so far has not been amenable to the determination of exact solutions for problems (see, for example, Section 3.3 of [18]). Therefore, for the problems presented in the following sections, in general we solve them assuming simplified expressions for the displacement field and the stress field simultaneously.

8.1 Quasi-Static Deformations

In this section we study boundary value problems considering the quasi-static case, therefore $(4)_1$ becomes

$$\operatorname{div}\mathbf{T} + \rho\mathbf{b} = \mathbf{0}, \tag{150}$$

and for all the problems considered here we assume additionally that $\mathbf{b} = \mathbf{0}$.

8.1.1 Homogeneous Distributions of Stresses

Let us consider the case of the displacement field being of the form $\mathbf{u} = \mathbf{A}_o\mathbf{x}$, where $\mathbf{A}_o$ is a constant second order tensor. Let us assume that such a displacement field is caused by the homogeneous stress distribution $\mathbf{T}_o$. Since $\mathbf{T}_o$ does not depend on $\mathbf{x}$ and $\mathbf{b} = \mathbf{0}$ the above stress field is a solution of (150). From $(2)_2$ we have $\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{A}_o + \mathbf{A}_o^{\mathrm{T}})$ and from (77) for an isotropic body we have[24]

$$\frac{1}{2}(\mathbf{A}_o + \mathbf{A}_o^{\mathrm{T}}) = \mathcal{G}_1 + \mathcal{G}_2\mathbf{T}_o + \mathcal{G}_3\mathbf{T}_o^2, \tag{151}$$

which is the equation that allows us to find the symmetric part of $\mathbf{A}_o$ in terms of $\mathbf{T}_o$, where the values of the functions $\mathcal{G}_i$, $i = 1, 2, 3$ depend on the stress tensor $\mathbf{T}_o$.

Let us examine two special cases. The first corresponds to the uniform tension/compresion of a cylinder under a uniform stress of the form (in Cylindrical coordinates) $\mathbf{T}_o = \sigma\mathbf{e}_z \otimes \mathbf{e}_z$. Assuming that such a stress tensor produces a displacement field of the form $u_r = cr$, $u_\theta = 0$, $u_z = (\lambda - 1)z$, where

[24]In this and in the following sections we consider $\boldsymbol{\varepsilon} = \mathfrak{h}(\mathbf{T})$ isotropic. In Sections 8.2.1, 8.3.1 and 8.3.5 some implicit relations of the form (54) are also studied. Several of the boundary value problems studied in this and in the following sections can also be extended for $\mathfrak{h}(\mathbf{T})$ transversely isotropic and other types of anisotropic bodies (see Sections 5.2.2 and 5.2.3), but for the sake of brevity we do not shown such results here (see [25]).

c, λ are constants, from (151) we obtain

$$c = \Pi_1, \quad \lambda - 1 = \mathcal{G}_1 + \mathcal{G}_2\sigma + \mathcal{G}_3\sigma^2,$$

i.e., we obtain directly the radial and axial deformations in terms of the stress σ.

The second problem that is interesting is the case of the simple shear of a slab (described in Cartesian coordinates). For simple shear, we assume that we apply a uniform stress of the form $\mathbf{T} = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$, and we assume that such a stress produces the displacement field $u_1 = (\lambda_1 - 1)x + \kappa y$, $u_2 = (\lambda_2 - 1)y$, $u_3 = (\lambda_3 - 1)z$, where λ_i, $i = 1, 2, 3$ and κ are constants. From (151) we obtain

$$\lambda_1 - 1 = \lambda_2 - 1 = \mathcal{G}_1 + \mathcal{G}_3\tau^2, \quad \lambda_3 - 1 = \mathcal{G}_1, \quad \frac{\kappa}{2} = \mathcal{G}_2\tau.$$

8.1.2 Non-homogeneous distributions of stresses: problems with cylindrical and spherical symmetry for unconstrained solids

Some problem involving non-homogeneous stresses and strains have been solved numerically describing the bodies in terms of cylindrical and spherical coordinates, see, for example, [19, 26, 27]. Let us study the deformation of the annulus $r_{\mathrm{i}} \leqslant r \leqslant r_{\mathrm{o}}$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant z \leqslant L$, which we assume is under the stress distribution[25] $T_{rr} = \sigma_r(r)$, $T_{\theta\theta} = \sigma_\theta(r)$, $T_{zz} = \sigma_z(r)$, $T_{r\theta} = \tau_{r\theta}(r)$, $T_{rz} = \tau_{rz}(r)$ and $T_{\theta z} = \tau_{\theta z}(r)$, which produces a displacement field of the form

$$u_r = f(r), \quad u_\theta = kr\theta + g(r) + \tau_o rz, \quad u_z = (\lambda - 1)z + h(r), \tag{152}$$

where k, τ_o and λ are constants. The above displacement field includes as special cases several deformations already studied in the literature, for example, $f(r)$ can be used to study the radial inflation of such an annulus, $g(r)$ is connected with circumferential shear, $h(r)$ with telescopic shear, τ_o is connected with torsion and k can be used to study the opening and closing of an annulus in the azimuthal direction.

[25] In this and in the following section τ_{ij} are not the components of the Kirchhoff stress tensor.

If $T_{rr} = \sigma_r(r)$, $T_{\theta\theta} = \sigma_\theta(r)$, $T_{zz} = \sigma_z(r)$, $T_{r\theta} = \tau_{r\theta}(r)$, $T_{rz} = \tau_{rz}(r)$ and $T_{\theta z} = \tau_{\theta z}(r)$, the equations of equilibrium (150) (again in absence of body forces) become

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0, \quad \frac{\mathrm{d}\tau_{r\theta}}{\mathrm{d}r} + \frac{2}{r}\tau_{r\theta} = 0, \quad \frac{\mathrm{d}\tau_{rz}}{\mathrm{d}r} + \frac{1}{r}\tau_{rz} = 0. \tag{153}$$

These last two equations can be solved exactly, and if $\tau_{r\theta_\mathrm{i}}$, τ_{rz_i} denote the stresses evaluated at $r = r_\mathrm{i}$ we have $\tau_{r\theta}(r) = \tau_{r\theta_\mathrm{i}}\left(\frac{r_\mathrm{i}}{r}\right)^2$ and $\tau_{rz}(r) = \tau_{rz_\mathrm{i}}\frac{r_\mathrm{i}}{r}$.

Using (152) in $(2)_2$ from (77) we obtain

$$f'(r) = \mathcal{G}_1 + \mathcal{G}_2\sigma_r + \mathcal{G}_3(\sigma_r^2 + \tau_{r\theta}^2 + \tau_{rz}^2), \tag{154}$$

$$k + \frac{f(r)}{r} = \mathcal{G}_1 + \mathcal{G}_2\sigma_\theta + \mathcal{G}_3(\tau_{r\theta}^2 + \sigma_\theta^2 + \tau_{\theta z}^2), \tag{155}$$

$$\lambda - 1 = \mathcal{G}_1 + \mathcal{G}_2\sigma_z + \mathcal{G}_3(\tau_{rz}^2 + \tau_{\theta z}^2 + \sigma_z^2), \tag{156}$$

$$\frac{1}{2}\left[g'(r) - \frac{g(r)}{r}\right] = \mathcal{G}_2\tau_{r\theta} + \mathcal{G}_3(\sigma_r\tau_{r\theta} + \tau_{r\theta}\sigma_\theta + \tau_{rz}\tau_{\theta z}), \tag{157}$$

$$\frac{h'(r)}{2} = \mathcal{G}_2\tau_{rz} + \mathcal{G}_3(\sigma_r\tau_{rz} + \tau_{r\theta}\tau_{\theta z} + \tau_{rz}\sigma_z), \tag{158}$$

$$\frac{\tau_o r}{2} = \mathcal{G}_2\tau_{\theta z} + \mathcal{G}_3(\tau_{r\theta}\tau_{rz} + \sigma_\theta\tau_{\theta z} + \tau_{\theta z}\sigma_z). \tag{159}$$

We have seven equations in (154)-(159) plus $(153)_1$, which can be used to find the seven unknown σ_r, σ_θ, $\tau_{\theta z}$, σ_z, f, g and h.

From $(153)_1$ we can express σ_θ in terms of σ_r as $\sigma_\theta = r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \sigma_r$; on the other hand from (155) we have

$$f(r) = r\left\{\mathcal{G}_1 + \mathcal{G}_2\left(r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \sigma_r\right) + \mathcal{G}_3\left[\tau_{r\theta}^2 + \left(r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \sigma_r\right)^2 + \tau_{\theta z}^2\right] - k\right\},$$

which after being substituted in (154) leads to

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\left\{\mathcal{G}_1 + \mathcal{G}_2\left(r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \sigma_r\right) + \mathcal{G}_3\left[\tau_{r\theta}^2 + \left(r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \sigma_r\right)^2 + \tau_{\theta z}^2\right] - k\right\}\right) = \mathcal{G}_1 + \mathcal{G}_2\sigma_r + \mathcal{G}_3(\sigma_r^2 + \tau_{r\theta}^2 + \tau_{rz}^2). \tag{160}$$

From (157) and (158) we obtain

$$h(r) = 2\int_{r_\mathrm{i}}^{r} \left[\mathcal{G}_2\tau_{rz} + \mathcal{G}_3(\sigma_r\tau_{rz} + \tau_{r\theta}\tau_{\theta z} + \tau_{rz}\sigma_z)\right] \mathrm{d}\xi + h_\mathrm{i}, \tag{161}$$

$$g(r) = 2r\int_{r_\mathrm{i}}^{r} \left\{\frac{1}{\xi}\left[\mathcal{G}_2\tau_{r\theta} + \mathcal{G}_3\left(\sigma_r\tau_{r\theta} + \tau_{r\theta}\left[\xi\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \sigma_r\right] + \tau_{rz}\tau_{\theta z}\right)\right]\right\} \mathrm{d}\xi + g_\mathrm{i}\frac{r}{r_\mathrm{i}}, \tag{162}$$

where h_i and g_i are the values of the functions $h(r)$ and $g(r)$ evaluated at $r = r_\mathrm{i}$.

Therefore, we end up with three equations to be solved, (160), (156) and (159). Eq. (160) is a nonlinear second order differential equation for σ_r and we can assume as boundary conditions that $\sigma_r(r_\mathrm{i}) = \sigma_{r_\mathrm{i}}$ and $\sigma_{rr}(r_\mathrm{o}) = \sigma_{r_\mathrm{o}}$, where σ_{r_i} and σ_{r_o} would be the external traction on the inner and outer surface of the annulus. Eqs. (156), (159) can be considered as nonlinear algebraic equations that can be used to find $\tau_{\theta z}(r)$ and $\sigma_z(r)$.

If we are studying the deformation of the spherical body $r_\mathrm{i} \leqslant r \leqslant r_\mathrm{o}$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant \phi \leqslant \pi$, the only possibility of reducing the original partial differential equations to a system of equations depending on one variable, is to assume that the spherical body deforms under the influence of the stress tensor $\mathbf{T} = \sigma_r(r)\mathbf{e}_r\otimes\mathbf{e}_r + \sigma_\theta(r)\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \sigma_\phi(r)\mathbf{e}_\phi\otimes\mathbf{e}_\phi$, which we assume produces the displacement field $u_r = f(r)$, $u_\theta = 0$, $u_\phi = 0$. From the equilibrium equations we obtain that $\sigma_\theta = \sigma_\phi$ and the equation of equilibrium reduces to $\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0$, while from (77) we obtain $f'(r) = \mathcal{G}_1 + \mathcal{G}_2\sigma_r + \mathcal{G}_3\sigma_r^2$ and $\frac{f(r)}{r} = \mathcal{G}_1 + \mathcal{G}_2\sigma_\theta + \mathcal{G}_3\sigma_\theta^2$. These equations have the same structure as that for the problem of the cylindrical annulus, therefore, we do not discuss this problem further.

An approximate analytical method of solution of boundary value problems by Sandeep et al: In [181] an approximate method has been proposed to solve some boundary value problems involving cylindrical and spherical coordinates, for the particular case of the constitutive equation (see (76)):

$$\boldsymbol{\varepsilon} = \alpha_0(\mathrm{tr}\,\mathbf{T})\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{T}^2, \tag{163}$$

where α_0, α_1 and α_2 are constants, which in that work are fitted with experimental data for two classes of titanium alloys.

In the first problem there is a study of the behaviour of a cylindrical annulus with a traction applied on the inner surface. It is assumed that for the non-zero components of the stress (in cylindrical coordinates) $T_{rr} = \sigma_r(r)$, $T_{\theta\theta} = \sigma_\theta(r)$ and $T_{zz} = \sigma_z(r)$. In such a case (quasi-static deformations and no body force) Eq. (150) becomes(153)$_1$, which can be solved assuming the existence of a stress potential $\psi = \psi(r)$ such that

$$\sigma_r = \frac{\psi}{r}, \quad \sigma_\theta = \frac{\mathrm{d}\psi}{\mathrm{d}r}. \tag{164}$$

Using (164) in (163) we have $\boldsymbol{\varepsilon}$ and replacing that in the compatibility equation $\frac{\mathrm{d}\varepsilon_{\theta\theta}}{\mathrm{d}r} = \frac{1}{r}(\varepsilon_{rr} - \varepsilon_{\theta\theta})$ the following nonlinear ordinary differential equation is found[26]:

$$(\alpha_0 + \alpha_1)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\psi}{r} + \frac{\mathrm{d}\psi}{\mathrm{d}r}\right) + \alpha_0\frac{\mathrm{d}}{\mathrm{d}r}\sqrt{\left(\frac{\alpha_0+\alpha_1}{2\alpha_2}\right)^2 - \frac{\alpha_0}{\alpha_2}\left(\frac{\psi}{2} + \frac{\mathrm{d}\psi}{\mathrm{d}r}\right)}$$
$$+\alpha_2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}\psi}{\mathrm{d}r}\right)^2 + \alpha_2\left(\frac{\psi}{r} + \frac{\mathrm{d}\psi}{\mathrm{d}r}\right)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\psi}{r}\right) = 0.$$

This equation is solved with an iteration method and the use of a variational formulation (which for the sake of brevity is not shown here), considering as a starting point the solution for the linearized elastic solid $\psi_0 = Ar + B/r$, where A, B are constants to be found, and σ_r can be equal to $\pm P$ and 0 inside or outside the annulus. The following approximate solution is found after one iteration:

$$\psi(r) \approx Ar + \frac{B}{r} - \frac{Br}{2}\left[\frac{1}{r^2} - \frac{1}{R_\mathrm{i}^2} + \left(k - \frac{1}{R_\mathrm{i}^2}\right)\ln\left(\frac{k - \frac{1}{r^2}}{k - \frac{1}{R_\mathrm{i}^2}}\right)\right], \tag{165}$$

where $k = \frac{G+2\alpha_2 A}{2\alpha_2 B}$ and $G = \alpha_1 + \alpha_0\left[1 - \frac{\alpha_0}{2\alpha_2\sqrt{\left(\frac{\alpha_0+\alpha_1}{2\alpha_2}\right)^2 - 2A\frac{\alpha_0}{\alpha_2}}}\right]$.

The above solution (165) (and the stresses calculated from it) are compared with fully numerical solutions, showing a close agreement. A similar study for the problem of the inflation of a hollow sphere has been considered, but for the sake of brevity such results are not shown here.

[26] The component of the stress $\sigma_z \neq 0$ is obtained from (163) as $\sigma_z = -\frac{(\alpha_0+\alpha_1)}{2\alpha_2} + \sqrt{\left(\frac{\alpha_0+\alpha_1}{2\alpha_2}\right)^2 + \frac{\varepsilon_{zz}}{\alpha_2}}$ assuming $\varepsilon_{zz} = 0$.

8.1.3 Non-homogeneous distributions of stresses: incompressible bodies

Eq. (160) in general cannot be solved exactly. One of the open problems would be to look for particular expressions for $\mathcal{G}$ such that some exact solutions can be found. One can ask if it is possible to repeat the classical technique developed originally by Rivlin (see, for example, Chapter D Section b of [209]), of appealing to the constraint of incompressibility, in order to find exact solutions that could be valid for any $\mathcal{G}$ for a given family of elastic bodies. We explore such a question in this section. In the classical theory of nonlinear elasticity, for incompressible isotropic bodies we have a constitutive equation of the form $\mathbf{T} = -p\mathbf{I} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^2$, and the indeterminacy of p is exploited in order to simplify the equations in the following manner. Let us consider as an example the case of the problem of inflation and extension of a cylindrical annulus, where it assumed that the deformation is of the form $r = f(R)$, $\theta = \Theta$, $z = \lambda Z$, where λ is a constant. Calculating $\mathbf{F}$ and imposing directly the constraint $\det\mathbf{F} = 1$ to the above deformation we obtain the solution $r = \sqrt{\frac{R^2-R_{\mathrm{i}}^2}{\lambda} + r_{\mathrm{i}}^2}$, where R_{i} and r_{i} are the inner radii of the annulus in the reference and current configurations. If we use the notation $\tilde{\mathbf{T}} = \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^2$, in the case of the above deformation, it is easy to see that $\tilde{\mathbf{T}}$ only depends on the radial position, and only has normal components. Substituting $\mathbf{T} = -p\mathbf{I} + \tilde{\mathbf{T}}$ in the equations of equilibrium without body forces we obtain that $p = p(r) = p(R)$ and $\frac{\mathrm{d}\tilde{T}_{rr}}{\mathrm{d}r} - \frac{\mathrm{d}p}{\mathrm{d}r} + \frac{1}{r}(\tilde{T}_{rr} - \tilde{T}_{\theta\theta}) = 0$, which can be solved easily for $p(r)$, and is subjected to the boundary conditions for $T_{rr}(r)$ known at r_{i} and r_{o}, where this last variable is the radius of the outer surface of the annulus in the current configuration, respectively. Thus, the solution is valid for any functions α_1, α_2, i.e., they are universal solutions (see [209]).

With the above discussion in mind, let us see if something similar can be carried out for the case of the incompressible isotropic body (102). In the case of (102) we have the property that $\boldsymbol{\varepsilon} = \mathfrak{h}(\mathbf{T}) = \mathfrak{h}(\mathbf{T}_{\mathrm{D}})$, where $\mathbf{T}_{\mathrm{D}} := \mathbf{T} - \frac{\mathrm{tr}\,(\mathbf{T})}{3}\mathbf{I}$. Moreover, we can add any spherical stress of the form $-\sigma_{\mathrm{S}}\mathbf{I}$ to $\mathbf{T}$ and the above result will always hold, i.e., $\mathfrak{h}(-\sigma_{\mathrm{S}}\mathbf{I} + \mathbf{T}) = \mathfrak{h}(\mathbf{T}_{\mathrm{D}})$. The constitutive equation (102) can be rewritten in an alternative way as (see [28, 38], see also (103))

$$\boldsymbol{\varepsilon} = \vartheta_0\mathbf{I} + \vartheta_1\mathbf{T}_{\mathrm{D}} + \vartheta_2\mathbf{T}_{\mathrm{D}}^2, \tag{166}$$

where we have defined $\vartheta_0 := -\frac{2}{3}I_{\mathrm{D}_2}\frac{\partial\bar{\mathcal{G}}}{\partial I_{\mathrm{D}_3}}$, $\vartheta_1 := \frac{\partial\bar{\mathcal{G}}}{\partial I_{\mathrm{D}_2}}$ and $\vartheta_2 := \frac{\partial\bar{\mathcal{G}}}{\partial I_{\mathrm{D}_3}}$, where

$I_{\mathrm{D}_2} = \frac{1}{2}\mathrm{tr}\left(\mathbf{T}_{\mathrm{D}}^2\right)$ and $I_{\mathrm{D}_3} = \frac{1}{3}\mathrm{tr}\left(\mathbf{T}_{\mathrm{D}}^3\right)$. In the following list we show three boundary value problems considering non-homogeneous deformations working with (166). The results are taken from [28, 38].

Inflation and extension of a cylindrical annulus: For the cylindrical annulus $r_{\mathrm{i}} \leqslant r \leqslant r_{\mathrm{o}}$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant z \leqslant L$ we assume the presence of the stress distribution

$$\mathbf{T} = \sigma_r(r)\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_\theta(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_z(r)\mathbf{e}_z \otimes \mathbf{e}_z, \tag{167}$$

where

$$\sigma_r(r) = -\sigma_{\mathrm{S}}(r) + \sigma_{\mathrm{D}_r}(r), \quad \sigma_\theta(r) = -\sigma_{\mathrm{S}}(r) + \sigma_{\mathrm{D}_\theta}(r), \tag{168}$$

$$\sigma_z(r) = -\sigma_{\mathrm{S}}(r) + \sigma_{\mathrm{D}_z}(r), \tag{169}$$

where the deviatoric part of the stress is assumed to be of the form $\mathbf{T}_{\mathrm{D}} = \sigma_{\mathrm{D}_r}(r)\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_{\mathrm{D}_\theta}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_{\mathrm{D}_z}(r)\mathbf{e}_z \otimes \mathbf{e}_z$.

The displacement field is assumed to be

$$\mathbf{u} = f(r)\mathbf{e}_r + (\lambda - 1)z\mathbf{e}_z, \tag{170}$$

where $\lambda > 0$ is a constant. The body is incompressible, thus $\mathrm{tr}\boldsymbol{\varepsilon} = 0$, and considering (170) we obtain

$$f(r) = \frac{(1-\lambda)}{2}r + \frac{C}{r}, \tag{171}$$

where C is a constant.

Using (167) in (150) we obtain $\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0$ and considering the boundary conditions $(168)_1$ $\sigma_r(r_{\mathrm{i}}) = -P$, $\sigma_r(r_{\mathrm{o}}) = 0$, we get

$$\sigma_{\mathrm{S}}(r) = \sigma_{\mathrm{D}_r}(r) + \int_{r_{\mathrm{o}}}^{r} \frac{1}{\xi}\left[\sigma_{\mathrm{D}_r}(\xi) - \sigma_{\mathrm{D}_\theta}(\xi)\right] \mathrm{d}\xi, \tag{172}$$

and $P = \int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}} \frac{1}{\xi}\left[\sigma_{\mathrm{D}_\theta}(\xi) - \sigma_{\mathrm{D}_r}(\xi)\right] \mathrm{d}\xi$.

Replacing (168), (169) and (170) in (166) we have (recall (171)):

$$f'(r) = \frac{1-\lambda}{2} - \frac{C}{r^2} = \vartheta_0 + \vartheta_1\sigma_{\mathrm{D}_r} + \vartheta_2\sigma_{\mathrm{D}_r}^2, \tag{173}$$

$$\frac{f(r)}{r} = \frac{1-\lambda}{2} + \frac{C}{r^2} = \vartheta_0 + \vartheta_1\sigma_{\mathrm{D}_\theta} + \vartheta_2\sigma_{\mathrm{D}_\theta}^2, \tag{174}$$

$$\lambda - 1 = \vartheta_0 + \vartheta_1\sigma_{\mathrm{D}_z} + \vartheta_2\sigma_{\mathrm{D}_z}^2. \tag{175}$$

The above is a system of three nonlinear algebraic equations. They are not independent because $\operatorname{tr}\boldsymbol{\varepsilon} = 0$. We can choose any two of them, to find, for example, σ_{D_r} and $\sigma_{\mathrm{D}_\theta}$, considering that $\operatorname{tr}\mathbf{T}_{\mathrm{D}} = 0$ thus $\sigma_{\mathrm{D}_z} = -\sigma_{\mathrm{D}_r} - \sigma_{\mathrm{D}_\theta}$. We have obtained an universal solution, because the above solution do not depend on the specific expression for $\bar{\Pi}$ given. If P is given, the constant C can be adjusted such that with the values for σ_{D_r} and $\sigma_{\mathrm{D}_\theta}$, from (173) and (174) we have that $P = \int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}} \frac{1}{\xi}\left[\sigma_{\mathrm{D}_\theta}(\xi) - \sigma_{\mathrm{D}_r}(\xi)\right] \mathrm{d}\xi$ is satisfied.

Non-homogeneous deformations of a cylindrical opened annulus: In this section we study the inflation, extension, torsion, circumferential shear, telescopic shear and closure of a cylindrical opened annulus. This problem is a generalization of the solution studied previously. Let us consider the annulus $r_{\mathrm{i}} \leqslant r \leqslant r_{\mathrm{o}}$, $0 \leqslant \theta \leqslant 2\pi - \alpha$, $0 \leqslant z \leqslant L$, where we assume we have the stress distribution:

$$\mathbf{T} = -\sigma_{\mathrm{S}}(r)\mathbf{I} + \mathbf{T}_{\mathrm{D}}(r), \tag{176}$$

where

$$\begin{aligned}\mathbf{T}_{\mathrm{D}} = {} & \sigma_{\mathrm{D}_r}(r)\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_{\mathrm{D}_\theta}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_{\mathrm{D}_z}(r)\mathbf{e}_z \otimes \mathbf{e}_z + \tau_{r\theta}(r)(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r)\\ & + \tau_{rz}(r)(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r) + \tau_{\theta z}(r)(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r).\end{aligned} \tag{177}$$

Replacing (176) (considering (177)) in (150) we obtain $\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0$, whose solution is the same as in (172) (assuming the same boundary conditions discussed there), plus the equations $\frac{\mathrm{d}\tau_{r\theta}}{\mathrm{d}r} + \frac{2\tau_{r\theta}}{r} = 0$, $\frac{\mathrm{d}\tau_{rz}}{\mathrm{d}r} + \frac{\tau_{rz}}{r} = 0$, whose solutions are:

$$\tau_{r\theta}(r) = \tau_{r\theta_{\mathrm{i}}}\left(\frac{r_{\mathrm{i}}}{r}\right)^2, \quad \tau_{rz}(r) = \tau_{rz_{\mathrm{i}}}\frac{r_{\mathrm{i}}}{r}, \tag{178}$$

where $\tau_{r\theta_{\mathrm{i}}}$ and $\tau_{rz_{\mathrm{i}}}$ are constants.

As for the displacement field we assume that is of the form:

$$\mathbf{u} = f(r)\mathbf{e}_r + [kr\theta + g(r) + \tau_o rz]\mathbf{e}_z + [(\lambda - 1)z + h(r)]\mathbf{e}_z, \tag{179}$$

where $k = \alpha/(2\pi - \alpha)$, τ_o and $\lambda > 0$ are constants. With the above displacement field we have that the cylindrical annulus becomes closed in the circumferential direction. The annulus is incompressible, then from $\operatorname{tr}\boldsymbol{\varepsilon} = 0$ and (179) we obtain

$$f(r) = (1 - \lambda - k)\frac{r}{2} + \frac{C}{r}. \tag{180}$$

Using (179) and (176) in (166) (recall (177), (180)) we obtain:

$$f'(r) = \frac{1-\lambda-k}{2} - \frac{C}{r^2} = \vartheta_0 + \vartheta_1\sigma_{\mathrm{D}_r} + \vartheta_2\left(\sigma_{\mathrm{D}_r}^2 + \tau_{r\theta}^2 + \tau_{rz}^2\right), \quad (181)$$

$$k + \frac{f(r)}{r} = \frac{1-\lambda+k}{2} + \frac{C}{r^2} = \vartheta_0 + \vartheta_1\sigma_{\mathrm{D}_\theta} + \vartheta_2\left(\tau_{r\theta}^2 + \sigma_{\mathrm{D}_\theta}^2 + \tau_{\theta z}^2\right), \quad (182)$$

$$\lambda - 1 = \vartheta_0 + \vartheta_1\sigma_{\mathrm{D}_z} + \vartheta_2\left(\tau_{rz}^2 + \tau_{\theta z}^2 + \sigma_{\mathrm{D}_z}^2\right), \quad (183)$$

$$\frac{\tau_o r}{2} = \vartheta_1\tau_{\theta z} + \vartheta_2\left(\tau_{r\theta}\tau_{rz} + \sigma_{\mathrm{D}_\theta}\tau_{\theta z} + \tau_{\theta z}\sigma_{\mathrm{D}_z}\right), \quad (184)$$

$$\frac{1}{2}\left[g'(r) - \frac{g(r)}{r}\right] = \vartheta_1\tau_{r\theta} + \vartheta_2\left(\sigma_{\mathrm{D}_r}\tau_{r\theta} + \tau_{r\theta}\sigma_{\mathrm{D}_\theta} + \tau_{rz}\tau_{\theta z}\right), \quad (185)$$

$$\frac{h'(r)}{2} = \vartheta_1\tau_{rz} + \vartheta_2\left(\sigma_{\mathrm{D}_r}\tau_{rz} + T_{r\theta}\tau_{\theta z} + \sigma_{\mathrm{D}_z}\right). \quad (186)$$

Eqs. (181)-(183) are not independent because $\mathrm{tr}\boldsymbol{\varepsilon} = 0$ (see (180)). We can choose two of them, say (181) and (182) and plus (184) we can solve for σ_{D_r}, $\sigma_{\mathrm{D}_\theta}$ and $\tau_{\theta z}$ (recall (178)). As for (185) and (186), they can be solved exactly as (compare this with (161) and (162) for unconstrained solids):

$$h(r) = 2\int_{r_\mathrm{i}}^{r} \left[\vartheta_1\tau_{rz} + \vartheta_2(\sigma_{\mathrm{D}_r}\tau_{rz} + \tau_{r\theta}\tau_{\theta z} + \tau_{rz}\sigma_{\mathrm{D}_z})\right]\,\mathrm{d}\xi + h_\mathrm{i}, \quad (187)$$

$$g(r) = 2r\int_{r_\mathrm{i}}^{r} \left\{\frac{1}{\xi}\left[\vartheta_1\tau_{r\theta} + \vartheta_2\left(\sigma_{\mathrm{D}_r}\tau_{r\theta} + \tau_{r\theta}\sigma_{\mathrm{D}_\theta} + \tau_{rz}\tau_{\theta z}\right)\right]\right\}\,\mathrm{d}\xi + g_\mathrm{i}\frac{r}{r_\mathrm{i}}, \quad (188)$$

where in the above integrals we have that $\vartheta_i = \vartheta_i(\mathbf{T}_\mathrm{D}(\xi))$ $(i = 1, 2)$, $\sigma_{\mathrm{D}_r} = \sigma_{\mathrm{D}_r}(\xi)$, $\sigma_{\mathrm{D}_\theta} = \sigma_{\mathrm{D}_\theta}(\xi)$, $\tau_{ij} = \tau_{ij}(\xi)$, $i \neq j$, and h_i, g_i are constants.

The above is a universal solution. From the algebraic equations say (181), (182) and (184) we can find $\sigma_{\mathrm{D}_r}(r)$, $\sigma_{\mathrm{D}_\theta}(r)$ and $\tau_{\theta z}(r)$, considering that $\tau_{r\theta}(r)$ and $\tau_{rz}(r)$ are known from (178). With the above solutions for the components of the deviatoric stress we can find $h(r)$ and $g(r)$ from (187) and (188) above. These solutions do not depend on the specific expression for $\bar{\mathcal{G}}$ that is used for the definitions of the ϑ_j, $j = 0, 1, 2$.

Inflation of a sphere: The last problem studied in [38] corresponds to the inflation of the hollow sphere $r_\mathrm{i} \leqslant r \leqslant r_\mathrm{o}$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant \phi \leqslant \pi$, where we assume the presence of the stress distribution

$$\mathbf{T} = -\sigma_\mathrm{S}(r)\mathbf{I} + \mathbf{T}_\mathrm{D} = \sigma_\mathrm{S}(r)\mathbf{I} + \sigma_{\mathrm{D}_r}(r)\mathbf{e}_r\otimes\mathbf{e}_r + \sigma_{\mathrm{D}_\theta}(r)\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \sigma_{\mathrm{D}_\phi}(r)\mathbf{e}_\phi\otimes\mathbf{e}_\phi. \quad (189)$$

It is assumed that the above stress produces the displacement field

$$\mathbf{u} = f(r)\mathbf{e}_r, \tag{190}$$

and from $\mathrm{tr}\boldsymbol{\varepsilon} = 0$ we obtain $f(r) = C/r^2$, where C is a constant.

Replacing (189) in (150) we obtain (assuming that $\sigma_r(r_\mathrm{i}) = -P$ and $\sigma_r(r_\mathrm{o}) = 0$):

$$\sigma_{\mathrm{D}_\phi} = \sigma_{\mathrm{D}_\theta}, \quad \sigma_\mathrm{S}(r) = \sigma_{\mathrm{D}_r}(r) + \int_{r_\mathrm{i}}^{r} \frac{2}{\xi}[\sigma_{\mathrm{D}_r}(\xi) - \sigma_{\mathrm{D}_\theta}(\xi)]\,\mathrm{d}\xi + P,$$

where $P = \int_{r_\mathrm{i}}^{r_\mathrm{o}} \frac{2}{\xi}[\sigma_{\mathrm{D}_\theta}(\xi) - \sigma_{\mathrm{D}_r}(\xi)]\,\mathrm{d}\xi$. Replacing the above expression for the deviatoric stress and the strain calculated with (190) in (166) we have

$$-2\frac{C}{r^3} = \vartheta_0 + \vartheta_1 T_{\mathrm{D}_{rr}} + \vartheta_2 T_{\mathrm{D}_{rr}}^2, \tag{191}$$

the other three components of the constitutive equation are not independent since $\mathrm{tr}\boldsymbol{\varepsilon} = 0$. On the other hand $\mathrm{tr}\,\mathbf{T}_\mathrm{D} = 0$, which taking into account (189) means $\sigma_{\mathrm{D}_\theta} = -\sigma_{\mathrm{D}_r}/2$.

Again we have a universal solution, since from (191) we can obtain σ_{D_r} for any expression for $\bar{\mathcal{G}}$.

8.1.4 A non-homogeneous tension of a plane slab

In this section we discuss a problem wherein the body is subject to a non-homogeneous distributions of stress, which we shall describe within the context of Cartesian coordinates (see Section 4.1 of [19] using the notation x, y instead x_1, x_2, respectively), for the plane slab $0 \leqslant x \leqslant L$, $0 \leqslant y \leqslant H$. We assume plane stresses and the existence of an Airy stress potential $\Upsilon(x, y)$ of the form $\Upsilon(x, y) = \varphi(x) - \tau_o xy$, where τ_o is a constant. We obtain

$$T_{11} = 0, \quad T_{22} = T_{22}(x) = \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}, \quad T_{12} = \tau_o. \tag{192}$$

From (76) for the components of the strain tensor we have (on recognizing that plane stress does not imply plane strain and on neglecting ε_{33})

$$\varepsilon_{11} = \vartheta_0 + \vartheta_2\tau_o^2, \quad \varepsilon_{22} = \vartheta_0 + \vartheta_1 T_{22} + \vartheta_2(\tau_o^2 + T_{22}^2), \quad \varepsilon_{12} = \vartheta_1\tau_o + \vartheta_2\tau_o T_{22}, \tag{193}$$

where $\varepsilon_{11} = \varepsilon_{11}(x)$, $\varepsilon_{22} = \varepsilon_{22}(x)$ and $\varepsilon_{12} = \varepsilon_{12}(x)$. The above deformation field is produced by a continuous displacement field if the compatibility equations are satisfied[27], and that means that $\frac{\mathrm{d}^2\varepsilon_{22}}{\mathrm{d}x^2} = 0$, which implies that $\varepsilon_{22} = c_1 x + c_0$, where c_0, c_1 are constants. In the light of the above results, let us assume that the components of the displacement field that is produced by the stress (192) is of the form

$$u_1(x, y) = f(x) + h(y), \quad u_2(x, y) = (c_1 x + c_0)y + g(x).$$

Calculating ε_{12} from the above displacement field it is found that to have $\varepsilon_{12} = \varepsilon_{12}(x)$, and we need $h(y) = -\frac{c_1}{2}y^2 + c_2$, where c_2 is a constant. Therefore in (193) we obtain

$$f'(x) = \vartheta_0 + \vartheta_2\tau_o^2, \quad c_1 x + c_0 = \vartheta_0 + \vartheta_1 T_{22}(x) + \vartheta_2[\tau_o^2 + T_{22}(x)^2], \tag{194}$$
$$\frac{1}{2}g'(x) = \vartheta_1\tau_o + \vartheta_2\tau_o T_{22}(x), \tag{195}$$

and from $(194)_1$, (195) we obtain $f(x) = \int_0^x (\vartheta_0 + \vartheta_2\tau_o^2)\,\mathrm{d}\xi + f_o$ and $g(x) = 2\int_0^x (\vartheta_1\tau_o + \vartheta_2\tau_o T_{22}(x))\,\mathrm{d}\xi + g_o$, where f_o and g_o are constants. Eq. $(194)_2$ can be used to find $T_{22}(x)$, given c_0 and c_1.

8.1.5 A class of non-homogeneous shear stress in a slab

In [20] for the constitutive equation for unconstrained solid (76) and the same slab described in the previous section, we assume that the stress tensor is given (using x, y, z instead x_i, $i = 1, 2, 3$, here τ_{12} is not a component of the Kirchhoff stress)

$$\mathbf{T} = -\sigma_{\mathrm{S}}(x, y, z)\sum_{i=1}^{3}\mathbf{e}_i \otimes \mathbf{e}_i + \tau_{12}(y)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1).$$

Replacing the above stress in (150) we find that $\sigma_{\mathrm{S}} = \sigma_{\mathrm{S}}(x)$ and $-\frac{\mathrm{d}\sigma_{\mathrm{S}}}{\mathrm{d}x} + \frac{\mathrm{d}\tau_{12}}{\mathrm{d}y} = 0$, which is satisfied if

$$\sigma_{\mathrm{S}}(x) = Cx + \sigma_o, \quad \tau_{12}(y) = Cy + \tau_o, \tag{196}$$

where C, σ_o and τ_o are constants.

[27] It is not mandatory to consider the compatibility equations if one works with the displacement field (see [164]), but in this section we choose to do so.

In [20] a special class of constitutive equation was considered where (see (76))

$$\boldsymbol{\varepsilon} = \alpha_1 \left[\mathbf{T} - \frac{\mathrm{tr}(\mathbf{T})}{3}\mathbf{I} \right], \tag{197}$$

where $\alpha_1 = \alpha_1(\mathbf{T}) = \mu_o \left[1 + \alpha \mathrm{tr}(\mathbf{T}_{\mathrm{D}}^2)\right]^n$, $\mathbf{T}_{\mathrm{D}} = \mathbf{T} - \frac{\mathrm{tr}(\mathbf{T})}{3}\mathbf{I}$, μ_o, α and n are material constants.

Assuming that the displacement field $\mathbf{u}$ is given by $u_1 = f(y)$, $u_2 = u_3 = 0$ from (196) and (197) we get $f'(y) = \mu_o[1 + 2\alpha(Cy + \tau_o)^2]^n(Cy + \tau_o)$ thus:

$$f(y) = \frac{\mu_o[1 + 2\alpha(Cy + \tau_o)^2]^{1+n}}{4C(1+n)\alpha},$$

which in [20] is studied for different values for the different constants that appear there.

8.1.6 The anti-plane shear

In this section we show three studies on the anti-plane shear problem, considering (76) and some of its subclasses. This is an interesting 2D problem, wherein some equations can be simplified greatly, allowing the analysis of existence of solutions, and in some cases even the obtention of exact solutions.

A first study on the anti-plane shear problem by Kulvait et al.: In [118] the anti-plane problem has been studied for the class of constitutive equation (76). In this case it is assumed that

$$\mathbf{T} = T_{13}(x_1, x_2)(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1) + T_{23}(x_1, x_2)(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2), \tag{198}$$

$$\mathbf{u} = u(x_1, x_2)\mathbf{e}_3. \tag{199}$$

The particular subclass of (76) considered in this paper is (compare, for example, with (128) and (130), see also Section 7.3.3 and (133) therein):

$$\mathrm{tr}\,\boldsymbol{\varepsilon} = \sigma_1(\mathrm{tr}\,\mathbf{T})\mathrm{tr}\,\mathbf{T}, \quad \boldsymbol{\varepsilon}_{\mathrm{D}} = \sigma_2(|\mathbf{T}_{\mathrm{D}}|)\mathbf{T}_{\mathrm{D}},$$

In the particular case of the above constitutive equation we have $\mathrm{tr}\,\mathbf{T} = 0$ and $|\mathbf{T}_{\mathrm{D}}| = \sqrt{2}|\mathbf{T}_{\mathrm{V}}|$, where $\mathbf{T}_{\mathrm{V}}$ is the vector $\mathsf{T}_{\mathrm{V}} = \begin{pmatrix} T_{13} \\ T_{23} \end{pmatrix}$. On the other hand if we define the vector $\boldsymbol{\varepsilon}_{\mathrm{V}} = \begin{pmatrix} \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix}$ from (133) we obtain $\boldsymbol{\varepsilon}_{\mathrm{V}} = \sigma_2(\sqrt{2}|\mathbf{T}_{\mathrm{V}}|)\mathbf{T}_{\mathrm{V}}$, then from this constitutive relation and (150) we obtain:

$$\varepsilon_{13} = \sigma_2(\sqrt{2}|\mathbf{T}_{\mathrm{V}}|)T_{13}, \quad \varepsilon_{23} = \sigma_2(\sqrt{2}|\mathbf{T}_{\mathrm{V}}|)T_{23}, \quad \frac{\partial T_{13}}{\partial x_1} + \frac{\partial T_{23}}{\partial x_2} = 0. \tag{200}$$

Eq. $(200)_3$ can be solved assuming the existence of a stress potential $\Phi = \Phi(x_1, x_2)$ such that $T_{13} = \frac{\partial \Phi}{\partial x_2}$, $T_{23} = -\frac{\partial \Phi}{\partial x_1}$. Using the above potential in $(200)_{1,2}$ we have two partial differential equations for u and Φ. The above system can be further reduced using the compatibility equations for ε_{13} and ε_{23}. In [118] the equations are solved numerically using the finite element method in terms of Φ.

A second analysis of the anti-plane shear problem by Rajagopal and Zappalorto: In [170] the above problem of anti-plane shear has been also studied, where an exact solution has been found for a case with concentration of stresses. The solution has been found for the following particular constitutive equation (see (76)):

$$\boldsymbol{\varepsilon} = \alpha[1 + \beta \mathrm{tr}(\mathbf{T}^2)]^n \mathbf{T}, \tag{201}$$

where α, β and $n < -1/2$ are material constants. The displacement field is of the form (199), where in [170] the authors use the notation x, y, z for x_1, x_2, x_3 and 1,2,3, respectively, then from (198) and (200) we have that T_{13} and T_{23} are T_{xz}, T_{yz}, respectively, and ε_{13}, ε_{23} become ε_{xz}, ε_{yz}, respectively. Defining the total shear strain as $\gamma = 2\sqrt{\varepsilon_{xz}^2 + \varepsilon_{yz}^2}$ and the total shear stress as $\tau = \sqrt{T_{xz}^2 + T_{yz}^2}$ from (201) we obtain

$$\gamma = 2\alpha(1 + 2\beta\tau^2)^n \tau. \tag{202}$$

From (4a) in [170] it is shown that when the stresses are 'small' we have $\gamma \approx 2\alpha\tau$, whereas then the stresses are 'large' from (4b) therein we obtain $\gamma \approx 2\alpha(2\beta)^n \tau^{2n+1}$, from where we see that as $\tau \to \infty$ implies that γ becomes a constant.

The above relation (202) is used for the analysis of a notched body under antiplane shear stress. The notch has an opening angle 2ξ, and at the tip of that notch the origin of the coordinate system is located (plane $x - y$, where the angle is symmetric with respect to x). The crack problem appears when $2\xi \to 0$. After several manipulations for this problem the stress can be calculated by solving the differential equation

$$\frac{\gamma(\tau)}{\tau\gamma'(\tau)}\frac{\partial^2 f}{\partial \tau^2} + \frac{1}{\tau}\frac{\partial f}{\partial \tau} - \frac{w^2}{\tau^2} f = 0, \tag{203}$$

where $\gamma' = \frac{\partial \gamma}{\partial \tau}$, $w = \frac{\pi}{\pi - 2\xi}$ and $f(\tau)$ satisfies:

$$x = -\frac{\partial f}{\partial \tau}\cos\phi\cos(w\phi) - \frac{wf}{\tau}\sin\phi\sin(w\phi), \tag{204}$$

$$y = -\frac{\partial f}{\partial \tau}\sin\phi\cos(w\phi) + \frac{wf}{\tau}\cos\phi\sin(w\phi), \tag{205}$$

where $\phi = \arctan(-T_{xz}/T_{yz})$.

For the case of (202) for $\gamma = \gamma(\tau)$ the solution of (203) is found to be (see Section 4 in [170])

$$f(\tau) = C_1\tau^{-w}F_1 + C_2\tau^{w}F_2,$$

where C_1, C_2 are constants, and F_1, F_2 are given in terms of the Gauss hypergeometric function and n, w, β, τ. For the case $2\xi = 0$ from (204) and (205) the following solution is found:

$$x = C_1\left[\frac{1}{\tau\gamma(\tau)}\cos^2\phi - \int_{\tau}^{\infty}\frac{\mathrm{d}\zeta}{\zeta^2\gamma(\zeta)}\right], \quad y = C_1\frac{\sin\phi\cos\phi}{\tau\gamma(\tau)}.$$

In [170] an analysis of the above solution is provided very close to the crack tip and also at a certain finite distance from it. For some intervals of values for n it is shown that near the tip of the crack $\tau \to \infty$ but γ remains bounded. At a certain finite distance the solution has the same structure as the solution found using the linearized elastic model, which for this case depend on the stress concentration factor K_{III}. Close to the tip the solution strongly depends on n, and such solutions are very different from the classical elasticity solutions. At a finite distance from the tip the solution do not depend on n.

A third study of the anti-plane shear by Yoon et al.: In [222, 223] the antiplane shear problem has been studied for some constitutive equations of the form (76), which are different from (200) and (201). In [223] the following constitutive equation has been considered:

$$\boldsymbol{\varepsilon} = \psi_0(\mathrm{tr}\,\mathbf{T}, |\mathbf{T}|)\mathbf{I} + \psi_1(|\mathbf{T}|)\mathbf{T}, \tag{206}$$

where $\psi_0(0,|\mathbf{T}|) = 0$ and $\psi_1(|\mathbf{T}|) = \frac{1}{2\mu(1+\beta^\alpha|\mathbf{T}|^\alpha)^{(1/\alpha)}}$, where α, β and μ are constants. In [222] a slightly different constitutive equation is used:

$$\boldsymbol{\varepsilon} = \psi_0(\mathrm{tr}\,\mathbf{T})\mathbf{I} + \psi_1(|\mathbf{T}|)\mathbf{T}. \tag{207}$$

In (206) and (207) we have that $|\mathbf{T}| = \sqrt{\mathbf{T}\cdot\mathbf{T}}$, where in [222] we also have $\psi_1(|\mathbf{T}|) = \frac{1}{2\mu(1+\beta|\mathbf{T}|^\alpha)^{(1/\alpha)}}$.

In [223, 222] for the stress and the displacement field it is assumed the same expressions presented in (198) and (199). As for the components of the stress T_{13} and T_{23} it is assumed that they are given in terms of a scalar stress potential Φ as presented in the paragraph after Eq. (200). Using the stress tensor (198) in (207) and considering (199) we obtain in both cases (206) and (207) that $\boldsymbol{\varepsilon} = \psi_1(|\mathbf{T}|)\mathbf{T}$, from where we can find ε_{13} and ε_{23}, which depend on x_1, x_2. These are the only non-zero components of the strain tensor, and if they satisfy the compatibility equation for the strains $\frac{\partial\varepsilon_{13}}{\partial x_2} - \frac{\partial\varepsilon_{23}}{\partial x_1} = 0$, then we can calculate a continuous displacement field from those components of the strain using $(2)_2$ (see also (199)). Using $\boldsymbol{\varepsilon} = \psi_1(|\mathbf{T}|)\mathbf{T}$ in the above compatibility equation the following second order nonlinear partial differential equation for ψ is obtained:

$$-\nabla\cdot[\psi_1(|\nabla\Phi|)\nabla\Phi] = 0, \tag{208}$$

where $|\nabla\Phi| = \sqrt{\left(\frac{\partial\Phi}{\partial x_1}\right)^2 + \left(\frac{\partial\Phi}{\partial x_2}\right)^2}$ and ψ_1 is given in the paragraph after (206).

The above Eq. (208) is solved numerically using the finite element method, and in [223] this is used to study the propagation of a crack, in particular considering the problem of a plane plate with a crack in mode III, which is in the middle of the plate or in its edges. In [222] the equation is solved to study the behaviour of a 2D plate under tension with a V-notch on the side, a plane plate with cracks with different orientations (located on the edges), and a plane plate with a crack in the middle (mode I), with different orientations.

8.2 Wave Propagation

When we turn our attention to time dependent stresses and deformations, few problems have been studied in the literature (see, for example, [22, 96, 111, 112]), and in the particular case of the subclass of constitutive equation (75), they can be classified into two types, problems involving the full study of the nonlinear equations (149) for some simple expressions for the stresses and displacement field, and incremental analysis.

8.2.1 Fully Nonlinear Problem

In this section we provide some details of some problems that have been studied such as the wave propagation in one dimensional rod, the propagation of shear waves in a semi-infinite slab, and the propagation of circumferential shear waves in a cylindrical annulus[28]. among others.

- In [22] the problem of propagation of longitudinal waves is studied for a one dimensional rod $0 \leqslant x \leqslant L$, for the constitutive equation (75), (77), assuming that the stress tensor is of the form $\mathbf{T} = \sigma(x,t)\mathbf{e}_1 \otimes \mathbf{e}_1$, which is assumed to produce a displacement field of the form $\mathbf{u} = u(x,t)\mathbf{e}_1$. For this one dimensional problem, there is only one component for the strain, namely $\varepsilon_{11} = \varepsilon = \frac{\partial u}{\partial x}$ and from (77), (125) considering the above expression for the stress we obtain $\varepsilon = \mathfrak{f}(\sigma) = -\alpha\left[1 - \frac{1}{(1+\beta\sigma)}\right] + \frac{\alpha\gamma}{\sqrt{1+\iota\sigma^2}}\sigma$. The notation $\varepsilon = \mathfrak{f}(\sigma)$ can be used to consider other expressions for the constitutive equation. From (149) we obtain the two partial differential equations

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}, \quad \frac{\partial u}{\partial x} = \mathfrak{f}(\sigma), \tag{209}$$

  which must be solved to obtain[29] $u(x,t)$ and $\sigma(x,t)$. With regard to the boundary and initial conditions, we can assume, for example, $u(L,t) = 0$, $\sigma(0,t) = \hat{\sigma}(t)$, where $\hat{\sigma}(t)$ is an external known traction, and $u(x,0) = \dot{u}(x,0) = 0$.

[28]In a recent paper by Huang et al. [96], the equations of motion corresponding to a special sub-class of (75) is considered under the prescription of Riemann data and all wave patterns that are possible are delineated. The specific wave form that arises depends on the initial data, and it is shown that rarefaction and shock waves are possible. As the paper is quite technical we shall not discuss it in detail here.

[29]Recently, in a work by Naz and Hereman [141] some closed solutions for (209) have been obtained, for the particular case that $\mathfrak{f}(\sigma) = \sigma(\beta+\sigma^2)^m$, where β and n are constants. The solutions are obtained using Lie symmetries. Some closed-form solutions (travelling wave solutions) are obtained for u and σ, which can be seen in Table 3 therein.

It is necessary to point out that in general it is not a good idea to cross-differentiate to reduce the number of equations governing the original problem (209). On equation that results from such cross-differentiation is:

$$\frac{\partial^2 \sigma}{\partial x^2} = \rho \frac{\partial^2}{\partial t^2}[\mathfrak{f}(\sigma)]. \tag{210}$$

In [22] a non-trivial exact solution (travelling wave) was found for the above equation[30], however in [39] it has been shown that such a solution (from [22]) is not a solution of the original problem (209).

In [125] some asymptotic approximate solutions have been found for (210) for the particular case $\mathfrak{f}(\sigma) = \alpha(1+\gamma\sigma^2)^n\sigma$, where α, γ and n are constants. Changing the notation y for the spatial coordinate used in that paper for x, we show here two of such solutions (which are given in implicit form). For example, from (4.22) of [125] we have:

$$\sigma(x,t) = A\sin\left\{\frac{2\pi\left[\left(1+\frac{3n}{2}\sigma^2(x,t)\right)x - t\right]}{1+\frac{3n}{2}A^2\sin^2\left[2\pi\left(\left(1+\frac{3n}{2}\sigma^2(x,t)\right)x-t\right)\right]}\right\}, \tag{211}$$

if $t \leqslant x \leqslant 1+t$, and $\sigma(x,t) = 0$ if $1+t \leqslant x < \infty$, where A is a constant. In Eq. (7.23) of the same paper we can find another such solution of the form:

$$\sigma(x,t) = A\sin\left\{\frac{2^{1-n/2}\pi\left[2^{n/2}\left(1-\frac{3n}{4}\sigma^2(x,t)\right)x - t\right]}{1-\frac{3n}{4}A^2\sin^2\left[2^{1-n/2}\left(2^{n/2}\left(1-\frac{3n}{2}\sigma^2(x,t)\right)x-t\right)\right]}\right\}, \tag{212}$$

if $2^{-n/2}t \leqslant y \leqslant 2^{-n/2}(1+t)$, and $\sigma(x,t) = 0$ if $2^{-n/2}(1+t) \leqslant x < \infty$.

- In [112] a problem that is similar in structure to that of (209) is studied. The authors study the behaviour of the semi-infinite slab $-\infty \leqslant x \leqslant \infty$, $0 \leqslant y \leqslant y_o$, $-\infty \leqslant z \leqslant \infty$ that is assumed have the shear stress

[30] The travelling wave solution of (210) is $\sigma(x,t) = \varphi(z)$,where $z = kx + \lambda t$, $\frac{k^2}{\rho\lambda^2}\varphi(z) = \mathfrak{f}(\varphi(z)) + Az + B$, where k, λ, A and B are constants. In [130] there is an analysis of the above solution, showing the existence of global solutions and singularities in the derivatives. In that paper [130] the above solution is analized for some special classes of $\mathfrak{f}$, namely: $\mathfrak{f}(\sigma) = \aleph\arctan(\vartheta\sigma)$, $\mathfrak{f}(\sigma) = \aleph(e^{\vartheta\sigma}-1)$ and $\mathfrak{f}(\sigma) = \aleph\sigma^{2n+1}$, where $\aleph$, ϑ and n are constants. Finally, in that paper a self similar solution is found for (210), which is of the form $\sigma(x,t) = \nu(\eta)$, where $\eta = \zeta x/t$ and $\eta(\nu) = \eta_o(\nu) + \phi(\nu)\left[A - \int\phi(\nu)f_2(\nu)\,\mathrm{d}\nu\right]^{-1}$, $f_2(\nu) = -\mathfrak{f}'(\nu)/C$.

distribution $\mathbf{T} = \tau(y,t)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$, which induces the deformation in the body of the form $\mathbf{u} = u(y,t)\mathbf{e}_1$. In [112] the constitutive equation $\mathfrak{h} = \beta \mathrm{tr}(\mathbf{T})\mathbf{I} + \alpha \left[1 + \frac{\gamma}{2}\mathrm{tr}(\mathbf{T}^2)\right]^n \mathbf{T}$ is used, which for the problem under consideration leads to $\mathfrak{h}(\tau) = \alpha(1+\gamma\tau^2)^n\tau$, where α, γ, n are constants. In this case (149) become

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \tau}{\partial y}, \quad \frac{\partial u}{\partial y} = \mathfrak{h}(\tau),$$

which has a structure very similar to (209), where we now need to solve for $u(y,t)$ and $\tau(y,t)$.

- Another problem to be discussed in this section is the propagation of shear waves in a cylindrical annulus defined through $r_\mathrm{i} \leqslant r \leqslant r_\mathrm{o}$, $0 \leqslant z \leqslant 2\pi$, $-\infty \leqslant z \leqslant \infty$ (see [111]), wherein it is assumed that the stress and the displacement have the forms $\mathbf{T} = \tau(r,t)(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r)$ and $\mathbf{u} = u(r,t)\mathbf{e}_\theta$, respectively. The constitutive equation considered is the same as in that [112], and (149) become the partial differential equations

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \tau}{\partial r} + \frac{2}{r}\tau, \quad \frac{\partial u}{\partial r} - \frac{u}{r} = \mathfrak{h}(\tau),$$

that must be solved for $u(r,t)$ and $\tau(r,t)$.

In [126] a similar system of partial differential equations has been analyzed for the case of 1D wave propagation in a cylindrical annulus $R_\mathrm{i} \leqslant r \leqslant R_\mathrm{o}$, where the waves propagate in the radial directions. In such a case if σ and u represent the radial normal component of the stress tensor and the radial component of the displacement field, the equations that are solved in an approximate manner are:

$$\frac{\partial \sigma}{\partial r} + \frac{\sigma}{r} = \rho \frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial u}{\partial r} = \mathfrak{h}(\sigma). \tag{213}$$

The authors of [126] considered the special case $\mathfrak{h}(\sigma) = \alpha(1+\gamma\sigma^2)^n$, and they reduce the above system of two PDE (considering also some dimensionless forms for the equations) to the single nonlinear partial differential equation

$$\frac{\partial^2 \sigma}{\partial r^2} + \frac{1}{r}\frac{\partial \sigma}{\partial r} - \frac{\sigma}{r^2} = \frac{\partial^2}{\partial t^2}\left[(1+\sigma^2)^2\right]. \tag{214}$$

One approximate solution that shows pseudo-solitary waves (in implicit form) based on the use of asymptotic methods is presented in Eq. (4.24) therein, which is reproduced here:

$$\begin{aligned}\sigma(r,t) = &\Bigg\{\Bigg[\left(1+\frac{3n}{2}\sigma^2(r,t)\ln r\right)r-t\Bigg]\Bigg[1+\frac{3n}{2}A^2\sin^2\Bigg[2\pi\Bigg(\Bigg(1\\ &+\frac{3n}{2}\sigma^2(r,t)\ln r\Bigg)r-t\Bigg)\Bigg]\ln\Bigg[\left(1+\frac{3n}{2}\sigma^2(r,t)\ln r\right)r-t\Bigg]\Bigg]^{-1}\Bigg\}^{1/2}\times\\ &\times A\sin\Bigg\{\Bigg[2\pi\Bigg[\left(1+\frac{3n}{2}\sigma^2(r,t)\ln r\right)r-t\Bigg]\Bigg]\Bigg[1\\ &+\frac{3n}{2}A^2\sin^2\Bigg[2\pi\Bigg(\left(1+\frac{3n}{2}\sigma^2(r,t)\ln r\right)r-t\Bigg)\Bigg]\ln\Bigg[1\\ &+\frac{3n}{2}\sigma^2(r,t)\ln r\Bigg)r-t\Bigg]\Bigg]^{-1}\Bigg\},\end{aligned} \tag{215}$$

if $R_\mathrm{i}+t\leqslant r\leqslant R^\star+t$, and $\sigma(r,t)=0$ if $R^\star+t\leqslant r\leqslant R_\mathrm{o}+t$. Another such an approximate solution is shown in Eq. (7.11) therein, which for the sake of brevity is not presented in this review.

A study on instability and non-existence for the 1D bar problem: In [150] a study on instability and non-existence has been presented for the 1D problem (209), with the initial conditions $u(x,0)=u_0(x)$ and $\dot{u}(x,0)=v_0(x)$. In that work it is stated that the deformation is unstable if $\sigma\mathfrak{h}(\sigma)\leqslant 0$. The above inequality (see (75)) is equivalent to $\sigma\varepsilon\leqslant 0$, which means the uniaxial stress and strain have opposite signs.

Defining the following functions $\mathcal{F}(t)=\int_0^L \rho u^2\,\mathrm{d}x$, it is possible to show that $\dot{\mathcal{F}}(t)=2\int_0^L \rho u\dot{u}\,\mathrm{d}x$ and $\ddot{\mathcal{F}}(t)=2\int_0^L \rho u\ddot{u}\,\mathrm{d}x+2\int_0^L \rho|\dot{u}|^2\,\mathrm{d}x$. On the other hand from $(209)_1$ integrating by parts we obtain

$$\int_0^L \rho u\ddot{u}\,\mathrm{d}x = -\int_0^L \sigma\mathfrak{h}(\sigma)\,\mathrm{d}x,$$

then

$$\ddot{\mathcal{F}}(t) = -2\int_0^L \sigma\mathfrak{h}(\sigma)\,\mathrm{d}x + 2\int_0^L \rho|\dot{u}|^2\,\mathrm{d}x \geqslant 2\int_0^L \rho|\dot{u}|^2\,\mathrm{d}x.$$

After some manipulations in [150] it is shown that the system is unstable if $\dot{\mathcal{F}}(0)=2\int_0^L \rho u_0(x)v_0(x)\,\mathrm{d}x>0$, which can be satisfied by selecting special expressions for u_0 and v_0.

About non-existence in [150] it is shown that the solution does not exists if $\mathfrak{h}(\sigma)$ satisfies

$$\sigma\mathfrak{h}(\sigma) - (2-\omega)W \geqslant 0,$$

where $0 < \omega < 1$ and $W'(\sigma) = \mathfrak{h}(\sigma)$ with $W(0) = 0$. By defining[31] $\mathcal{G}(t) = \int_0^L \rho u^2 \, \mathrm{d}x + \beta(t+t_o)^2$, where $\beta > 0$, $t_o > 0$, in [150] it is shown that the solution blows up in a finite time if $\dot{\mathcal{G}}(0) > 0$. Here $\dot{\mathcal{G}}(0) = 2\int_0^L \rho u(x,0)\dot{u}(x,0)\,\mathrm{d}x + 2\beta t_o > 0$.

An exact solution for a class of implicit relation for a body whose mechanical properties depend on density: In [176] an exact solution has been found for a subclass of the constitutive relation (55), wherein $\boldsymbol{\varepsilon} + A_1\mathbf{T} + A_2(\mathrm{tr}\,\boldsymbol{\varepsilon})\mathbf{T} + A_3(\mathrm{tr}\,\mathbf{T})\mathbf{I} + A_4(\mathrm{tr}\,\boldsymbol{\varepsilon})\mathbf{I} + A_5(\mathrm{tr}\,\mathbf{T})\boldsymbol{\varepsilon} + A_6(\mathrm{tr}\,\mathbf{T})(\mathrm{tr}\,\boldsymbol{\varepsilon})\mathbf{I} + A_7(\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) + A_8\mathrm{tr}\,(\boldsymbol{\varepsilon}\mathbf{T})\mathbf{I} = \mathbf{0}$, and where A_i, $i = 1, 2, 3, \ldots 8$ are constants. The study is focused in the particular subclass of the above implicit relation (58) $\boldsymbol{\varepsilon} = B_1[1 + \lambda_2(\mathrm{tr}\,\boldsymbol{\varepsilon})]\mathbf{T} + B_2[1 + \lambda_3(\mathrm{tr}\,\boldsymbol{\varepsilon})](\mathrm{tr}\,\mathbf{T})\mathbf{I}$, where B_1, B_2, λ_2 and λ_3 are constants. The solution is found for the deformation of a 1D bar, wherein the notations σ, ε, u and x are used for the 1D stress, strain, displacement field and position along the bar (see (209)). In such a case defining $B_1 = (1+\nu)/E$ and $B_2 = -\nu/E$, where E, ν are the ground Young's modulus and Poisson ratio of the solid, and $\alpha = B_1\lambda_2 + B_2\lambda_3$, the 1D version of (58) and of the equation of motion $(149)_1$ (see also $(209)_1$ and without body forces) are:

$$\varepsilon = \frac{\sigma}{E} + \alpha\varepsilon\sigma, \quad \rho\ddot{u} = \frac{\partial\sigma}{\partial x}, \tag{216}$$

where $\varepsilon = \frac{\partial u}{\partial x}$. Expressing ε in terms of σ from $(216)_1$, taking the derivative of that equation twice in time, and of $(216)_2$ once in x, combining the two equation it is found that (see [39] for a criticism on such procedure of reducing the equations):

$$\ddot{\sigma} = \frac{E}{\rho_\mathrm{r}}\frac{\partial^2\sigma}{\partial x^2}(1-\alpha\sigma)^2 - \frac{2\alpha\dot{\sigma}^2}{(1-\alpha\sigma)}, \tag{217}$$

where (see Section 5.1.1) $\rho_\mathrm{r} \approx \rho(1 + \mathrm{tr}\,\boldsymbol{\varepsilon})$. A travelling wave solution is found assuming that (see the discussion in the paragraph after (210) and the footnote therein) $\sigma(x,t) = \sigma(\xi)$, where $\xi = x - ct$, where c is a constant.

[31] The reader must not confuse the function $\mathcal{G}$ here with the Gibbs potential defined and used in Sections 3.3.1 and 5.2.1.

Replacing this in (221) the following nonlinear ordinary differential equation is obtained:

$$c^2\sigma'' = \frac{E}{\rho_{\mathrm{r}}}\sigma''(1-\alpha\sigma)^2 - \frac{2\alpha c^2(\sigma')^2}{(1-\alpha\sigma)}, \tag{218}$$

where the notation $\sigma' = \frac{\mathrm{d}\sigma}{\mathrm{d}\xi}$ is used. The exact solution found for (218) is (in implicit form):

$$\sigma(\xi) - \frac{c}{\sqrt{E/\rho_{\mathrm{r}}}\alpha}\tanh^{-1}\left[\frac{\sqrt{E/\rho_{\mathrm{r}}}}{c}(\alpha\sigma(\xi)-1)\right] = C_1\xi + \frac{C_2+1}{\alpha},$$

where C_1, C_2 are constants.

8.2.2 Infinitesimal wave analysis as a consequence of time dependent stress superposed on a time independent stress

In this section some results concerning the extension of the classical incremental analysis for the class of constitutive equations (75) are reviewed (see [4]). It is assumed the presence of a time independent stress, to which a small time dependent stress is superimposed (incremental stress). There are many possible applications for such an analysis, in particular regarding the determination of mechanical properties, a case in point being non-destructive testing. Let $\mathbf{T} = \mathbf{T}(\mathbf{x})$ and $\mathbf{u} = \mathbf{u}(\mathbf{x})$ denote the initial time independent stress and displacement field, which are solutions of the boundary value problem (75), (150) in the case of quasi-static deformations (and no body forces), i.e., $\operatorname{div}\mathbf{T} = \mathbf{0}$ and $\frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) = \mathfrak{h}(\mathbf{T})$, where $\mathbf{Tn} = \hat{\mathbf{t}}(\mathbf{x})$ on $\mathbf{x} \in \partial\boldsymbol{\kappa}_r(\mathscr{B})_t$ and $\mathbf{u} = \hat{\mathbf{u}}(\mathbf{x})$ on $\mathbf{x} \in \partial\boldsymbol{\kappa}_r(\mathscr{B})_u$, where $\boldsymbol{\kappa}_r(\mathscr{B}) = \boldsymbol{\kappa}_r(\mathscr{B})_t \cup \boldsymbol{\kappa}_r(\mathscr{B})_u$ and $\boldsymbol{\kappa}_r(\mathscr{B})_t \cap \boldsymbol{\kappa}_r(\mathscr{B})_u = \emptyset$. Let us add to the above stress the incremental time dependent stress $\Delta\mathbf{T}(\mathbf{x},t)$, where it is assumed that $\frac{|\Delta\mathbf{T}|}{|\mathbf{T}|} \sim O(\delta)$, $\delta \ll 1$, which it is supposed to produce an increment in the displacement field $\Delta\mathbf{u}(\mathbf{x},t)$. Now the key ingredients in this analysis are the assumptions that the stress $\mathbf{T}(\mathbf{x}) + \Delta\mathbf{T}(\mathbf{x},t)$ and the displacement field $\mathbf{u}(\mathbf{x}) + \Delta\mathbf{u}(\mathbf{x},t)$ are also solutions of the equation of motion[32] (149), therefore, substituting such

[32] One ought to mention that in general for this problem it is also necessary to determine the density, which from $(4)_2$ is given as $\rho = \frac{\rho_{\mathrm{r}}}{\det\mathbf{F}}$. In the case of small gradient of the displacement we have $\det\mathbf{F} \approx 1+\operatorname{tr}\nabla\mathbf{u} = 1+\operatorname{div}\mathbf{u} = 1+\operatorname{tr}\boldsymbol{\varepsilon}$, therefore, $\rho \approx \rho_{\mathrm{r}}(1+\operatorname{tr}\boldsymbol{\varepsilon})^{-1} \approx \rho_{\mathrm{r}}(1-\operatorname{tr}\boldsymbol{\varepsilon})$. There may be some problems for which even the small changes in the density due to the factor $\operatorname{tr}\boldsymbol{\varepsilon}$, we can have an important influence in the mechanical behaviour of the material (see Sections 5.1.1 and 7.3.6). In this section it is assumed that that is

fields in (149), recalling that $\mathbf{T} = \mathbf{T}(\mathbf{x})$ and $\mathbf{u} = \mathbf{u}(\mathbf{x})$ are solutions of the equilibrium equations, it is obtained

$$\rho \Delta \ddot{\mathbf{u}} = \operatorname{div} \Delta \mathbf{T}. \tag{219}$$

Regarding (75), using $\mathbf{T}(\mathbf{x}) + \Delta\mathbf{T}(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}) + \Delta\mathbf{u}(\mathbf{x}, t)$ in that equation, recalling that $\frac{|\Delta\mathbf{T}|}{|\mathbf{T}|} \sim O(\delta)$, $\delta \ll 1$, the approximation $\boldsymbol{\varepsilon} + \Delta\boldsymbol{\varepsilon} \approx \mathfrak{h}(\mathbf{T}) + \frac{\partial \mathfrak{h}}{\partial \mathbf{T}} \cdot \Delta\mathbf{T}$ is valid, where $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}})$, and where $\Delta\boldsymbol{\varepsilon} = \frac{1}{2}[\nabla(\Delta\mathbf{u}) + \nabla(\Delta\mathbf{u})^{\mathrm{T}}]$ and the fourth order tensor $\frac{\partial \mathfrak{h}}{\partial \mathbf{T}}$ is evaluated at $\mathbf{T}$, i.e., it does not depend on $\Delta\mathbf{T}$. Let $\boldsymbol{\varepsilon} = \mathfrak{h}(\mathbf{T})$, on defining[33] $\boldsymbol{\mathscr{C}} = \boldsymbol{\mathscr{C}}(\mathbf{T}) = \frac{\partial \mathfrak{h}}{\partial \mathbf{T}}$ the following incremental constitutive equation is obtained (see [4])

$$\Delta\boldsymbol{\varepsilon} = \boldsymbol{\mathscr{C}} \cdot \Delta\mathbf{T}. \tag{220}$$

With regard to the boundary conditions, for the original quasi-static problem it is assumed the application of a time-independent traction $\hat{\mathbf{t}}(\mathbf{x})$, and that the increment in stress $\Delta\mathbf{T}(\mathbf{x}, t)$ is caused by the addition of a time-dependent increment in the traction $\Delta\hat{\mathbf{t}}(\mathbf{x}, t)$, therefore, for $\mathbf{T}(\mathbf{x}) + \Delta\mathbf{T}(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}) + \Delta\mathbf{u}(\mathbf{x}, t)$ the boundary conditions become $[\mathbf{T}(\mathbf{x}) + \Delta\mathbf{T}(\mathbf{x}, t)]\mathbf{n} = \hat{\mathbf{t}}(\mathbf{x}) + \Delta\hat{\mathbf{t}}(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}) + \Delta\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{u}}(\mathbf{x}) + \Delta\hat{\mathbf{u}}(\mathbf{x}, t)$, and it is obtained

$$\Delta\mathbf{T}(\mathbf{x}, t)\mathbf{n} = \Delta\hat{\mathbf{t}}(\mathbf{x}, t) \quad \mathbf{x} \in \partial\boldsymbol{\kappa}_r(\mathscr{B})_t, \quad \Delta\mathbf{u}(\mathbf{x}, t) = \Delta\hat{\mathbf{u}}(\mathbf{x}, t) \quad \mathbf{x} \in \partial\boldsymbol{\kappa}_r(\mathscr{B})_u.$$

If it assumed that $\Delta\mathbf{T}$ and $\Delta\mathbf{u}$ do not depend on time, the left side of (219) is zero, and the counterpart for (75) of the incremental equations are obtained, which are used, for example, in the nonlinear theory of elasticity, to study among other things, stability of the solutions for boundary value problems, especially when it is assumed that $\Delta\hat{\mathbf{t}} = \mathbf{0}$. If $\Delta\mathbf{T}$ and $\Delta\mathbf{u}$ depend on time, but $\Delta\hat{\mathbf{t}} = \mathbf{0}$ and $\Delta\hat{\mathbf{u}} = \mathbf{0}$ and Eq. (219) can be used to explore uniqueness/non-uniqueness for the solution of a boundary value problem, and also stability, by looking for the existence of such small wave solutions from (219).

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not the case, and that the mechanical behaviour of the material is not influenced by small changes in the deformation, therefore, we adopt the approximation $\rho \approx \rho_{\mathrm{r}}$ is adopted and it is assumed that the density is constant in time.

[33]If it is assumed that $\mathfrak{h}(\mathbf{T}) = \frac{\partial \mathcal{G}}{\partial \mathbf{T}}$, then $\boldsymbol{\mathscr{C}} = \frac{\partial^2 \mathcal{G}}{\partial \mathbf{T} \partial \mathbf{T}}$ and this fourth order tensor has the symmetries $\mathscr{C}_{ijkl} = \mathscr{C}_{jikl} = \mathscr{C}_{ijlk} = \mathscr{C}_{klij}$. We assume that $\mathfrak{h}(\mathbf{T}) = \frac{\partial \mathcal{G}}{\partial \mathbf{T}}$ for the rest of this section.

Let us study the problem of propagation of infinitesimal waves in an infinite medium, assuming that $\mathbf{T}$ and $\boldsymbol{\varepsilon}$ are constant tensors. There are two ways to carry out the analysis, in the first case it can be assumed that $\mathscr{C}(\mathbf{T})$ has an inverse, and in the second case it is assumed that that is not necessarily the case. If $\mathscr{C}(\mathbf{T})$ has an inverse, from (220) $\Delta\mathbf{T} = \mathscr{C}^{-1}\cdot\Delta\boldsymbol{\varepsilon}$ is obtained and (219) becomes

$$\rho\Delta\ddot{\mathbf{u}} = \operatorname{div}(\mathscr{C}^{-1}\cdot\Delta\boldsymbol{\varepsilon}), \quad \text{where} \quad \Delta\boldsymbol{\varepsilon} = \frac{1}{2}[\nabla(\Delta\mathbf{u}) + \nabla(\Delta\mathbf{u})^{\mathrm{T}}]. \tag{221}$$

A solution of (221) of the form $\Delta\mathbf{u}(\mathbf{x},t) = \Delta\mathbf{u}_o g(\zeta)$ is sought, where $\zeta = k\mathbf{p}\cdot\mathbf{x} - \lambda t$, k, λ being constants and $\mathbf{p}$ being in the direction of propagation of the wave, $\Delta\mathbf{u}_o$ is a constant vector and g is a scalar function that is smooth enough for our purposes. If the above expression is substituted in (221) after some manipulations

$$\left(\mathbf{Q} - \frac{\rho\lambda^2}{k^2}\mathbf{I}\right)\Delta\mathbf{u}_o = \mathbf{0}, \tag{222}$$

is obtained, where it is been defined[34] $Q_{ij} = \overset{-1}{\mathscr{C}}_{iklj}\, p_k p_l$ and $\nu = \frac{\lambda}{k}$ is identified as the wave speed. The above equation (222) has nontrivial solutions if $\det\left(\mathbf{Q} - \frac{\rho\lambda^2}{k^2}\mathbf{I}\right) = 0$, which is the equation that must be used to find ν. It is noticed that $\mathbf{Q}$ is a function of the initial time-independent stress since $\mathscr{C} = \mathscr{C}(\mathbf{T})$.

Next, the case when $\mathscr{C}(\mathbf{T})$ does not have inverse is studied, looking for a solution for $\Delta\mathbf{T}(\mathbf{x},t)$ and $\Delta\mathbf{u}(\mathbf{x},t)$ of the form $\Delta\mathbf{T}(\mathbf{x},t) = \Delta\mathbf{T}_o f(\zeta)$ and $\Delta\mathbf{u}(\mathbf{x},t) = \Delta\mathbf{u}_o g(\zeta)$, where $\zeta = k\mathbf{p}\cdot\mathbf{x} - \lambda t$, where $\Delta\mathbf{T}_o$ is a constant second order tensor and f is a scalar function smooth enough for our purposes. Replacing these expressions in (219) and (220) are obtained

$$\rho\lambda^2\Delta\mathbf{u}_o g''(\zeta) = \Delta\mathbf{T}_o\mathbf{p}kf'(\zeta), \tag{223}$$

$$\frac{1}{2}(\Delta\mathbf{u}_o\otimes\mathbf{p} + \mathbf{p}\otimes\Delta\mathbf{u}_o)kg'(\zeta) = \mathscr{C}\cdot\Delta\mathbf{T}_o f(\zeta). \tag{224}$$

Further progress can be made with the above solutions in the following way. Let us assume that the functions f and g are the same and are given as $f(\zeta) = g(\zeta) = e^{\hat{I}\zeta}$, where $\hat{I}$ is the imaginary number. Substituting this in (223), (224) the equation $[M][V] = [0]$ is obtained, where the matrix

[34]Here $\overset{-1}{\mathscr{C}}_{iklj}$ corresponds to the components $iklj$ of the fourth order tensor $\mathscr{C}^{-1}$.

$[M] = [M]_{9\times 9}$ and the vector $[V] = [V]_{9\times 1}$ are defined below and $[0]_{9\times 1}$ is a column vector with all components equal to 0. The non-zero components of $[M]$ are: $M_{11} = M_{22} = M_{33} = \rho\lambda^2$, $M_{14} = -M_{41} = k\hat{I}p_1$, $M_{25} = -M_{52} = k\hat{I}p_2$, $M_{36} = -M_{63} = k\hat{I}p_3$, $M_{17} = -M_{71} = \frac{k\hat{I}p_2}{2}$, $M_{18} = -M_{81} = \frac{k\hat{I}p_3}{2}$, $M_{27} = -M_{72} = \frac{k\hat{I}p_1}{2}$, $M_{29} = -M_{92} = \frac{k\hat{I}p_3}{2}$, $M_{38} = -M_{83} = \frac{k\hat{I}p_1}{2}$, $M_{39} = -M_{93} = \frac{k\hat{I}p_2}{2}$, $M_{44} = \mathscr{C}_{1111}$, $M_{45} = M_{54} = \mathscr{C}_{1122}$, $M_{46} = M_{64} = \mathscr{C}_{1133}$, $M_{47} = M_{74} = \mathscr{C}_{1112}$, $M_{48} = M_{84} = \mathscr{C}_{1113}$, $M_{49} = M_{94} = \mathscr{C}_{1123}$, $M_{55} = \mathscr{C}_{2222}$, $M_{56} = M_{65} = \mathscr{C}_{2233}$, $M_{57} = M_{75} = \mathscr{C}_{2212}$, $M_{58} = M_{85} = \mathscr{C}_{2213}$, $M_{59} = M_{95} = \mathscr{C}_{2223}$, $M_{66} = \mathscr{C}_{3333}$, $M_{67} = M_{76} = \mathscr{C}_{3312}$, $M_{68} = M_{86} = \mathscr{C}_{3313}$, $M_{69} = M_{96} = \mathscr{C}_{3323}$, $M_{77} = \mathscr{C}_{1212}$, $M_{78} = M_{87} = \mathscr{C}_{1213}$, $M_{79} = M_{97} = \mathscr{C}_{1223}$, $M_{88} = \mathscr{C}_{1313}$, $M_{89} = M_{98} = \mathscr{C}_{1323}$, $M_{99} = \mathscr{C}_{2323}$. As well as this, $[V] = (\Delta u_{o_1}, \Delta u_{o_2}, \Delta u_{o_3}, \Delta T_{o_{11}}, \Delta T_{o_{22}}, \Delta T_{o_{33}}, 2\Delta T_{o_{12}}, 2\Delta T_{o_{13}}, 2\Delta T_{o_{23}})^{\mathrm{T}}$. The system of equations $[M][V] = [0]$ has non-trivial solutions if $\det[M] = 0$. In this case this last equation would give a relation between λ, k and $\mathbf{p}$ for such a wave to exist.

In [97] Eqs. (221) are solved for some cases formulated in cylindrical coordinates, wherein $\mathbf{T} = \mathbf{T}(r)$ and $\Delta\mathbf{T} = \Delta\mathbf{T}(r,t)$ and $\Delta\mathbf{u} = \Delta\mathbf{u}(r,t)$. The problems are: the propagation of small amplitude wave on a cylindrical annulus under inflation, and a cylindrical annulus showing circumferential shear and inflation. In such a case $\mathbf{T} = \mathbf{T}(r)$ and $\mathbf{u}$ are obtained solving numerically (149), and using such solutions in (221) (in particular for the evaluation of the fourth order tenso $\boldsymbol{\mathscr{C}} = \boldsymbol{\mathscr{C}}(\mathbf{T})$). Those equations (which are linear partial differential equations with coefficients that depends on r) are solved numerically using the finite element method.

8.3 Modelling of 2D beams and trust problems (small strains)

In this section different models are presented for beams and trusts, considering the constitutive equations and relations (54) and (75).

8.3.1 A theory for beams of rectangular cross section by Erbay et al

In [71] the problem of modelling 2D beams of rectangular section is studied for some subclasses of the implicit constitutive relations studied in Section 5.1.1 (see in particular (59) and (68) therein), in the particular case of small gradient of the displacement field, where we have the density as one of the main variable (see Section 5.1.1), and we recall that for this case that mass density is related with the terms $\mathrm{tr}\boldsymbol{\varepsilon}$. The particular subclass of constitutive relation considered is (compare with (59), (68)):

$$\boldsymbol{\varepsilon} = \frac{(1+\nu)}{E_o(1+\beta\mathrm{tr}\boldsymbol{\varepsilon})}\mathbf{T} - \frac{\nu}{E_o(1+\beta\mathrm{tr}\boldsymbol{\varepsilon})}(\mathrm{tr}\,\mathbf{T})\mathbf{I}, \tag{225}$$

where E_o, ν are the 'ground' Young's modulus and Poisson ratio, and β is a material constant. From (225) it is possible to define a sort of density-dependent Young's modulus E as: $E = E(\rho) \approx E(\mathrm{tr}\boldsymbol{\varepsilon}) = E(\varepsilon_{\mathrm{V}}) = E_o(1 + \beta\mathrm{tr}\boldsymbol{\varepsilon})$, where $\varepsilon_{\mathrm{V}} = \mathrm{tr}\boldsymbol{\varepsilon}$.

The beam is defined as $0 \leqslant x_1 \leqslant L$, $-h/2 \leqslant x_2 \leqslant h/2$, $-b/2 \leqslant x_3 \leqslant b/2$, where L is the length of the beam, and h, b are the height and width of its cross section. In [71] following the standard theory for linearized elastic beams, it is assumed that the only non-zero components of the stress tensor are $T_{11} = T_{11}(x_1, x_3)$, $T_{33} = T_{33}(x_1, x_3)$ and $T_{13} = T_{13}(x_1, x_3)$. The displacement field is assumed to be $\mathbf{u} = u_1(x_1, x_3)\mathbf{e}_1 + u_3(x_1, x_3)\mathbf{e}_3$. In [71] Eq. (225) is inverted, where that inversion must be understood as a mathematical device, since from the physical point of view it is not rational to do that. In such a case, using the above definition for E from (150) (when $\mathbf{b} = \mathbf{0}$) and (225), the following six equations are obtained:

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{13}}{\partial x_3} = 0, \quad \frac{\partial T_{13}}{\partial x_1} + \frac{\partial T_{33}}{\partial x_3} = 0, \tag{226}$$

$$T_{11} = \frac{E(\varepsilon_{\mathrm{V}})}{(1-\nu^2)}\frac{\partial u_1}{\partial x_1} + \frac{\nu}{(1-\nu)}T_{33}, \quad T_{22} = \frac{E(\varepsilon_{\mathrm{V}})\nu}{(1-\nu^2)}\frac{\partial u_1}{\partial x_1} + \frac{\nu}{(1-\nu)}T_{33}, \tag{227}$$

$$E(\varepsilon_{\mathrm{V}})\frac{\partial u_3}{\partial x_3} = T_{33} - \nu(T_{11} + T_{22}), \quad E(\varepsilon_{\mathrm{V}})\frac{\partial u_1}{\partial x_3} = -E\frac{\partial u_3}{\partial x_1} + 2(1+\nu)T_{13}, \tag{228}$$

where we recall that ε_{V} is defined as $\varepsilon_{\mathrm{V}} = \mathrm{tr}\boldsymbol{\varepsilon}$. An asymptotic analysis is performed for the above equations, assuming $h \ll L$. In [71] the final system of equations (after that asymptotic analysis) is not solved (see also [72]).

8.3.2 A model for beams by Gu et al. with applications for gum metal solids

In [91] a model for beams of rectangular cross section has been proposed by Gu et al., with applications to gum metal, considering the constitutive equation

$$\boldsymbol{\varepsilon} = \lambda_1(\mathrm{tr}\,\mathbf{T})\mathbf{I} + 2\lambda_2 \exp(\eta \mathrm{tr}\,\mathbf{T})\mathbf{T}, \tag{229}$$

where λ_1, λ_2 and η are material constants.

Here it is assumed that the plane where the stresses are important is the plane $x_1 - x_2$, where the authors use the notation x, y instead x_1 and x_2, respectively. It is assumed that $\mathbf{T} = T_{11}(x,y)\mathbf{e}_1 \otimes \mathbf{e}_1 + T_{22}(x,y)\mathbf{e}_2 \otimes \mathbf{e}_2 + T_{12}(x,y)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$, then the counterparts of (226) in this case are:

$$\frac{\partial T_{11}}{\partial x} + \frac{\partial T_{12}}{\partial y} = 0, \quad \frac{\partial T_{12}}{\partial x} + \frac{\partial T_{22}}{\partial y} = 0. \tag{230}$$

For the non-zero components of the displacement field and the stress tensor (where in general $\mathbf{u} = \mathbf{u}(x,y)$ and $\mathbf{T} = \mathbf{T}(x,y)$), it is assumed the series approximations (see [91])

$$\begin{aligned}
\mathbf{u}(x,y) &\approx \mathbf{u}^{(0)}(x) + y\mathbf{u}^{(1)}(x) + \frac{y^2}{2}\mathbf{u}^{(2)}(x) + \frac{y^3}{6}\mathbf{u}^{(3)}(x), \\
\mathbf{T}(x,y) &\approx \sum_{k=0}^{4} \frac{y^k}{k!}\mathbf{T}^{(k)}(x).
\end{aligned}$$

The above is replaced in (229) and (230), from where a system of equations for $T_{ij}^{(k)}(x)$ and $u_i^{(k)}(x)$ is obtained. The boundary conditions are $\mathbf{Tn} = \mathbf{q}$ at $y = 0$ and $y = 2h$, where $2h$ is the height of the cross section of the beam. The neutral axis is assumed to be constant in x. The system of equations mentioned above, which for the sake of brevity are not shown here, are solved numerically with the collocation method.

8.3.3 A theory for 2D beams of arbitrary cross

Here some elements for the beam model presented in [53] are shown. Following the theory for beams by [205], it is assumed that the only important component of the stress tensor is the normal stress (in the direction x) that is denoted σ, which is caused due to bending in the direction x along the beam. The longitudinal strain in that direction is denoted ε, then from (75) we have $\varepsilon = \mathfrak{h}(\sigma)$. The deflection of the beam is denoted w, the neutral axis for the beam and the bending moment are denoted $\bar{y}$ and M respectively. The symbol y denotes the vertical position across the section of the beam from the neutral axis. In [53] it is assumed that $\sigma = \sigma(x,y)$, $\bar{y} = \bar{y}(x)$, $w = w(x)$ and

$M = M(x)$. The important difference with the classical theory for beams [205] is that here it is assumed that the neutral axis can depend on the axial position along the beam x. That is also an important difference with the other four models presented in Sections 8.3.1, 8.3.2, 8.3.4 and 8.3.5.

The longitudinal strain is assumed to be of the same form as in the classical theory for beams, i.e., $\varepsilon = -y\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$. If $\mathcal{A}$ represents the cross section for the beam, the balance of forces in the x direction implies that $\int_{\mathcal{A}} \sigma(x,y)\,\mathrm{d}A(y) = 0$, the constitutive equation (75) becomes $-y\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \mathfrak{h}(\sigma)$, and the balance of moment of forces means $M(x) = -\int_{\mathcal{A}} y\sigma(x,y)\,\mathrm{d}A(y)$. The above three equations can be used to find σ, $\bar{y}$ and $\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$ (integrating twice this last variable w can be obtained) if $M = M(x)$ is known. In the special case $\mathfrak{h}(\sigma) = \sigma/E$ is a linearized constitutive equation, where E is the Young's modulus, from $\varepsilon = \mathfrak{h}(\sigma)$ we can obtain σ as $\sigma = -Ey\frac{\mathrm{d}w}{\mathrm{d}x^2}$, and replacing that in $\int_{\mathcal{A}} \sigma(x,y)\,\mathrm{d}A(y) = 0$, $M(x) = -\int_{\mathcal{A}} y\sigma(x,y)\,\mathrm{d}A(y)$ we get the well known equations for 2D beams for linearized elastic solids.

If the auxiliary coordinate $\tilde{y}$ is defined from $y = \tilde{y} - \bar{y}$, we have that$\sigma = \sigma(x,y) = \sigma(x, \tilde{y} - \bar{y}(x)) = \sigma(x,\tilde{y})$. Using $\tilde{y}$ the above three equations $\int_{\mathcal{A}} \sigma(x,y)\,\mathrm{d}A(y) = 0$, $-y\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \mathfrak{h}(\sigma)$ and $M(x) = -\int_{\mathcal{A}} y\sigma(x,y)\,\mathrm{d}A(y)$ mentioned in the previous paragraph become:

$$\int_{\mathcal{A}} \sigma(x,\tilde{y})\,\mathrm{d}A(\tilde{y}) = 0, \quad -[\tilde{y} - \bar{y}(x)]\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \mathfrak{h}(\sigma(x,\tilde{y})), \tag{231}$$

$$M(x) = -\int_{\mathcal{A}} [\tilde{y} - \bar{y}(x)]\sigma(x,\tilde{y})\,\mathrm{d}A(\tilde{y}). \tag{232}$$

In [53] the above three equations have been solved with an 'incremental' method after obtaining dimensionless version of them. If $M = M(x)$ is known (that is the case for statically determined beams) and depends on some external load, then such load is divided in N increments such that $M_{n+1} \approx M_n + \Delta M_n$, $n = 0, 1, 2, ..., N$. It is assumed that $|\Delta M_n|$ are small enough such that the following approximations are valid: $\sigma_n \approx \sigma_n + \Delta\sigma_n$, $\bar{y}_{n+1} \approx \bar{y}_n + \Delta\bar{y}_n$ and $\breve{w}_{n+1} \approx \breve{w}_n + \Delta\breve{w}_n$ (where we have defined $\breve{w} = \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$), and where $|\Delta\sigma_n|/|\sigma_n|$, $|\Delta\bar{y}_n|/|\bar{y}_n|$ and $|\Delta\breve{w}_n|/|\breve{w}_n|$ are assumed to be very small. For any increment n it is assumed that $\int_{\mathcal{A}} \sigma_n(x,\tilde{y})\,\mathrm{d}A(\tilde{y}) = 0$, $-[\tilde{y} - \bar{y}_n(x)]\breve{w}_n(x) = \mathfrak{h}(\sigma_n(x,\tilde{y}))$ and $M_n(x) = -\int_{\mathcal{A}}[\tilde{y} - \bar{y}_n(x)]\sigma_n(x,\tilde{y})\,\mathrm{d}A(\tilde{y})$ are satisfied. Using $\sigma_n \approx \sigma_n + \Delta\sigma_n$, $\bar{y}_{n+1} \approx \bar{y}_n + \Delta\bar{y}_n$ and $\breve{w}_{n+1} \approx \breve{w}_n + \Delta\breve{w}_n$ in the above equations a system of two linear equations are obtained for the increments $\Delta\bar{y}_n$, $\Delta\breve{w}_n$:

$$\alpha_n \Delta\bar{y}_n + \beta_n \Delta\breve{w}_n = 0, \quad \gamma_n \Delta\bar{y}_n + \zeta_n \Delta\breve{w}_n = \Delta M_n, \tag{233}$$

where $\Delta\sigma_n = \frac{h^2}{L^2 \mathcal{C}_n}[\Delta\bar{y}_n \breve{w}_n - (\tilde{y} - \bar{y}_n)\Delta\breve{w}_n]$, $\alpha_n = \breve{w}_n \int_{\mathcal{A}} \frac{1}{\mathcal{C}_n} \mathrm{d}A(\tilde{y})$, $\beta_n = \int_{\mathcal{A}} \frac{(\tilde{y}-\bar{y}_n)}{\mathcal{C}_n} \mathrm{d}A(\tilde{y})$, $\gamma_n = -\frac{\breve{w}_n h^2 \beta_n}{L^2}$, $\zeta_n = \frac{h^2}{L^2} \int_{\mathcal{A}} \frac{(\tilde{y}-\bar{y}_n)}{\mathcal{C}_n} \mathrm{d}A(\tilde{y})$, $\mathcal{C}_n = \frac{\mathrm{d}\mathfrak{h}}{\mathrm{d}\sigma}(\sigma_n)$ and L, h are the total length of the beam and total height of the cross section of the beam, respectively.

The above method has been used in [53] to study numerically two problems, namely the cantilever beam with a point load and the three-point flexural test for a beam, where $\mathfrak{f}(\sigma)$ has been taken for rock and also for strain limiting behaviour (see Sections 7.3.1 and 7.3.4). There are important differences in the behaviour of the beam when compared with the predictions of the classical theory for beams, in particular regarding the distributions and magnitudes for σ.

In [55] the shear stress τ_{xy} (that also appears in the analysis of beams) is analyzed for the constitutive equations $\varepsilon = \mathfrak{f}(\sigma)$. Using as a starting point (231), (232) and the procedure presented, for example, in [110], the following equation is obtained

$$\tau_{xy}(x, \tilde{y}) = \frac{1}{t(\tilde{y})} \int_{\tilde{y}}^{h} \frac{\partial \sigma}{\partial x}(x, \xi) \, \mathrm{d}A(\xi),$$

where $\sigma = \sigma(x, \tilde{y})$ is the same normal stress calculated from (231), (232), $t = t(\tilde{y})$ is the width of the cross section of the beam at a position $\tilde{y}$, and ξ is an auxiliary coordinate such that $y \leqslant \xi \leqslant h$. Using the same numerical method mentioned before for the normal stress σ, the stress τ_{xy} has been calculated for the same examples for the beams listed previously and for the constitutive equations for rock and strain limiting behaviour (see Sections 7.3.1 and 7.3.4), comparing the predictions with the results obtained using the linearized theory of elasticity, where (see, for example, [110]) $\tau_{xy}(x, y) = \frac{V(x)}{I_z t(y)} \int_y^{h-\bar{y}} \xi \, \mathrm{d}A(\xi)$, where $V(x)$ is the inner shear force for the beam and I_z is the second moment of area for the section.

8.3.4 A model for beams of rectangular section by Janecka et al.

In [106] a model for Euler beams of rectangular section has been proposed for the class of constitutive equation (75), which has some similarities with the theory presented in the previous section.

Let us consider a beam of length L (in the direction of the axis x), with a cross section of height $2h$ (axis y) and width $2b$. For the stress tensor it is assumed the same distribution mentioned in Sections 8.3.1 and 8.3.2. Integrating such equations (see (230)) in y and z we obtain the classical relations $\frac{\mathrm{d}V}{\mathrm{d}x} - q = 0$ and $\frac{\mathrm{d}M}{\mathrm{d}x} - V = 0$, where V is the inner shear force and M is the bending moment. As for the displacement field and the linearized strain tensor it is assumed that

$$\mathbf{u}(x,y) = -y\frac{\mathrm{d}w}{\mathrm{d}x}\mathbf{e}_1 + w(x)\mathbf{e}_2 \quad \Rightarrow \quad \boldsymbol{\varepsilon} = -y\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\mathbf{e}_1 \otimes \mathbf{e}_1,$$

where $w = w(x)$ is the deflexion for the beam.

In the theory of Janecka et al. [106] the authors look for $T_{11}(x,y)$ (which is the same as $\sigma(x,y)$ in the previous section) and $w(x)$ solving:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\int_{y=-h}^{h} yT_{11}\,\mathrm{d}y\right) = q, \quad -y\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \mathfrak{h}(T_{11}), \tag{234}$$

where $\mathfrak{h}$ corresponds to the component $\mathfrak{h}_{11}$ from (75). In [106] it is assumed that the neutral axis is constant and does not depend on the load (unlike the assumption the model presented in the previous section).

Regarding $\mathfrak{h}$ in [106] different constitutive equations have been considered, namely for Gum metal alloys and Concrete, where (see Sections 7.3.3 and 7.3.5) $\boldsymbol{\mathfrak{h}}(\mathbf{T}) = \lambda_1(\mathrm{tr}\,\mathbf{T})\mathbf{I} + 2\lambda_2 e^{\eta \mathrm{tr}\,\mathbf{T}}\mathbf{T}$, $\boldsymbol{\mathfrak{h}}(\mathbf{T}) = \lambda_1(\mathrm{tr}\,\mathbf{T})\mathbf{I} + \lambda_2(1+\alpha|\mathbf{T}|^2)^n\mathbf{T}$ and $\boldsymbol{\mathfrak{h}}(\mathbf{T}) = \gamma_1(\mathrm{tr}\,\mathbf{T})\mathbf{I} + \sinh[(\mathrm{tr}\,\mathbf{T})^{\gamma_2}/\gamma_3]\mathbf{I} + \gamma_4\mathbf{T}$, where λ_1, λ_2, η, α, n, γ_1, γ_2, γ_3 and γ_4 are material constants.

Eqs. (234) are solve numerically with the collocation method, and it has been found that the predictions of the same are very different in comparison with the predictions of the linearized theory for beams.

8.3.5 A study of beams of rectangular section made of solids whose mechanical properties depend on the mass density

In [148] a model for beams of rectangular section has been proposed, with focus on solids whose mechanical properties depend on the mass density (see Section 5.1.1). The class of implicit constitutive relation considered here is (see (54)), $\boldsymbol{\mathfrak{G}}(\rho, \boldsymbol{\varepsilon}, \mathbf{T}, \mathbf{X}) = \mathbf{0}$, in particular (see, also (58)):

$$\boldsymbol{\varepsilon} = (\phi_{11} + \phi_{12}\mathrm{tr}\,\boldsymbol{\varepsilon})\mathbf{T} + (\phi_{21} + \phi_{22}\mathrm{tr}\,\boldsymbol{\varepsilon})(\mathrm{tr}\,\mathbf{T})\mathbf{I}. \tag{235}$$

In [148] a beam of length $2L$ and rectangular cross section of height h and width b is considered. The notation (x, y, z) is used to replace (x_1, x_2, x_3), where the axis z is aligned in the direction of $2L$, x is in the direction of h, and y in the direction of b. The origin of the coordinate system is located in the center of the beam. Similarly as the previous sections the neutral axis is assumed to be located at $x = 0$ and is constant in z, i.e. it does not depend on the magnitude of the load, as well as this, it is assumed that $\mathbf{T} = \sigma_z(x, z)\mathbf{e}_3 \otimes \mathbf{e}_3$.

Demanding that for a part of the beam, which is cut by an imaginary line, is in static equilibrium, the following relations are obtained (see and compare with Section 8.3.3):

$$\int_{-h/2}^{h/2} \sigma_z \, \mathrm{d}x = 0, \quad M = -\int_{-h/2}^{h/2} x\sigma_z b \, \mathrm{d}x. \tag{236}$$

In that paper some analytical solutions for $\mathbf{T}$, $\mathbf{u}$ and $\boldsymbol{\varepsilon}$ are obtained for the beam. Let us show briefly one of them. One solution is obtained based on assuming that the linearized strain is of the form[35] $\boldsymbol{\varepsilon} = (\alpha x+\xi)(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + (\beta x + \chi)\mathbf{e}_3 \otimes \mathbf{e}_3$, where α, β, ξ and χ are constants to be determined. As a results $\mathrm{tr}\,\boldsymbol{\varepsilon} = (2\alpha + \beta)x$, then from (235) considering the expression for the stress tensor mentioned in the paragraph after (235), this implies that $\sigma_z = \frac{\beta x+\chi}{A+B(2\alpha+\beta)x+B(2\xi+\chi)}$, where $A = \phi_{11} + \phi_{21}$, $B = \phi_{12} + \phi_{22}$. The above stress σ_z can be substituted in (236). One the other hand from (235) considering the expression for the strain and the stress $\varepsilon_{11} = (\phi_{21}+\phi_{22}\mathrm{tr}\,\boldsymbol{\varepsilon})\sigma_z$, we get the restriction $\frac{\phi_{21}+\phi_{22}(2\xi+\chi)+\phi_{22}(2\alpha+\beta)x}{\phi_{11}+\phi_{21}+(\phi_{12}+\phi_{22})(2\xi+\chi)+(\phi_{12}+\phi_{22})(2\alpha+\beta)x} = \tilde{L}$, where $\tilde{L}$ is a constant in x. The above restriction is satisfied if $\frac{\phi_{22}}{\phi_{12}+\phi_{22}} = \tilde{L}$ and $\frac{\phi_{21}}{\phi_{11}+\phi_{21}} = \tilde{L}$, and in [148] it is shown that the above means that $\alpha = \tilde{L}\beta$, $\xi = \tilde{L}\chi$. Thus from (236) the authors get:

$$\frac{\tilde{L}bh}{\phi_{22}(2\tilde{L}+1)}\tilde{b} = -M, \quad \frac{1}{\tilde{a}} - \frac{h}{\ln\left(\frac{1+\frac{h\tilde{a}}{2}}{1-\frac{h\tilde{a}}{2}}\right)} = \tilde{b}, \tag{237}$$

where $\tilde{a}$ and $\tilde{b}$ are defined as $\tilde{a} = \frac{B(2\alpha+\beta)}{A+B(2\xi+\chi)}$ and $\tilde{b} = \frac{\chi}{\beta}$. The above two equations (237) can be used to find β and χ (here it is assumed that M is constant

[35] In [148] it is shown that this strain tensor can be calculated from a continuous displacement field $\mathbf{u}$. Here χ is just a constant and it must not be confused with the deformation mentioned in Section 2.

in x). From $(2)_2$ and the expression for $\boldsymbol{\varepsilon}$ discussed in the paragraph after (236) the displacement field is given by $\mathbf{u} = \left\{\frac{\alpha}{2}\left[\left(x-\frac{\xi}{\alpha}\right)^2 - y^2\right] - \frac{\beta z^2}{2} - \frac{\xi^2}{2\alpha}\right\}\mathbf{e}_1 + \alpha\left(x-\frac{\xi}{\alpha}\right) y\mathbf{e}_2 + \beta\left(x-\frac{\xi}{\alpha}\right) z\mathbf{e}_3$.

In [148] one special subclass of (235) is considered, wherein (see Section 5.1.1) $\phi_{11} = \frac{1+\nu_{\mathrm{r}}}{E_{\mathrm{r}}}$, $\phi_{12} = n\frac{(1+\nu_{\mathrm{r}})}{E_{\mathrm{r}}}$, $\phi_{21} = -\frac{\nu_{\mathrm{r}}}{E_{\mathrm{r}}}$ and $\phi_{22} = -n\frac{\nu_{\mathrm{r}}}{E_{\mathrm{r}}}$, where E_{r}, ν_{r} are the Young modulus and Poisson ration in the reference configuration and n is a material constant.

8.3.6 Modelling of trust elements and a numerical technique to solve some boundary value problems

The problem of modelling trust elements considering a subclass of (54) is treated in [190]. For the case there is no body force and considering (54) we need to solve $\operatorname{div}\mathbf{T} = \mathbf{0}$, $\mathfrak{G}(\mathbf{T},\boldsymbol{\varepsilon}) = \mathbf{0}$.

Eq. (150) (with $\mathbf{b} = \mathbf{0}$) is solved using the 3D stress potentials $\Phi_i = \Phi_i(x,y,z)$ and ψ_i where (using the notation x, y and z for x_i, $i = 1,2,3$):

$$T_{11} = \frac{\partial^2\Phi_1}{\partial y^2} + \frac{\partial^2\Phi_2}{\partial z^2} + \psi_1(y,z), \quad T_{22} = \frac{\partial^2\Phi_1}{\partial x^2} + \frac{\partial^2\Phi_3}{\partial z^2} + \psi_2(x,z),$$
$$T_{33} = \frac{\partial^2\Phi_2}{\partial x^2} + \frac{\partial^2\Phi_3}{\partial y^2} + \psi_3(x,y), \quad T_{12} = -\frac{\partial^2\Phi_1}{\partial x\partial y}, \quad T_{13} = -\frac{\partial^2\Phi_1}{\partial x\partial z},$$
$$T_{23} = -\frac{\partial^2\Phi_1}{\partial y\partial z}.$$

In [190] for simplicity it is assumed that $\Phi_1 = \Phi(x,y)$ and $\Phi_2 = \Phi_3 = \psi_1 = \psi_2 = \psi_3 = 0$, then two are the main variables for the problem: the displacement field $\mathbf{u}$ and the stress potential Φ. The finite element method is proposed to solve boundary value problems, where for the displacement field and the stress potential the following interpolations are used:

$$\mathbf{u} = \sum_{i=1}^{N} d^i\boldsymbol{\xi}^i(\mathbf{x}) + \boldsymbol{\Xi}(\mathbf{x}), \quad \phi = \sum_{i=1}^{N}\phi^i\varphi_b^i(\mathbf{x}),$$

where d^i, ϕ^i are constants to be determined and $\boldsymbol{\xi}^i$, φ_b^i are basis functions. The solution of the problem is found by minimizing the function

$$\delta = \int_V \sqrt{\mathfrak{f}(\mathbf{T},\boldsymbol{\varepsilon})\cdot\mathfrak{f}(\mathbf{T},\boldsymbol{\varepsilon})}\,\mathrm{d}v. \tag{238}$$

If $\mathbf{u}$ and Φ are solutions of the boundary value problem $\delta = 0$ must hold exactly. The constants d^i and ϕ^i are sought such that δ in (238) is minimized. One constitutive equation used in this paper is of the form $\mathfrak{G}(\mathbf{T}, \boldsymbol{\varepsilon}) = \mathbf{0}$, where $\boldsymbol{\varepsilon} = K_1\beta_1\mathbf{I} + \left(\beta_2 K_2^{\beta_3} + \beta_2\right)\mathbf{T}$, where $K_1 = \operatorname{tr}\mathbf{T}$, $K_2 = \operatorname{tr}(\mathbf{T}^2)$ and β_j, $j = 1, 2, 3, 4$ are constants.

The above numerical technique is used for the analysis of axial elements, considering plane trusses that are statically determined, and also the case they are statically undetermined. 3D space truss elements are also studied. The numerical results for the displacement field are compared with the predictions of the linearized constitutive equation, and the predictions are considerably different.

In a very recent work [10] the idea of using such stress potentials as basis for a finite element analysis has been proposed, for a very similar problem, considering plane stresses.

8.4 A study on hyperbolicity for boundary value problems

Hyperbolicity is a mathematical property that is very convenient for the solution and analysis of partial differential equations. In [189] a study on restrictions on $\mathfrak{h}$ (see(75)) is provided such that (149) are hyperbolic. That paper starts with the analysis of 1D problems, wherein (149) becomes (using the notation u, ε, σ and $\mathfrak{h}$ for the 1D displacement field, strain, stress and constitutive function, assuming as well for simplicity that $\rho = 1$ and that there is no body force):

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}, \quad \varepsilon = \frac{\partial u}{\partial x} = \mathfrak{h}(\sigma). \tag{239}$$

Taking the derivative in time of $(239)_2$ we have $\frac{\partial \varepsilon}{\partial t} = \frac{\partial^2 u}{\partial x \partial t} = \frac{\partial \mathfrak{h}}{\partial \sigma}\frac{\partial \sigma}{\partial t}$, then using the notation $v = \frac{\partial u}{\partial t}$ Eqs. (239) become the first order partial differential equations:

$$\frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x} = 0, \quad \frac{\partial \mathfrak{f}}{\partial \sigma}\frac{\partial \sigma}{\partial t} - \frac{\partial v}{\partial x} = 0. \tag{240}$$

Defining the vectors and matrices $\mathbf{q} = \begin{pmatrix} v \\ \sigma \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & \partial\mathfrak{f}/\partial\sigma \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ the above system (240) can be re-written as:

$$\mathbf{A}\frac{\partial \mathbf{q}}{\partial t} + \mathbf{B}\frac{\partial \mathbf{q}}{\partial x} = \mathbf{0}.$$

In [189] it is shown that these equations are hyperbolic when $\frac{\partial \mathfrak{f}}{\partial \sigma} > 0$.

In that paper there is an analysis of the fully 3D problem. For simplicity assuming $\rho = 1$ and no body force, using the notation (see Section 8.2.2) $\mathscr{C}_{ijkl} = \frac{\partial \mathfrak{h}_{ij}}{\partial T_{kl}}$ and defining the vectors and matrices[36] $\mathbf{v} = (\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}, \frac{\partial u_2}{\partial t}, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)^{\mathrm{T}}$, $\mathbf{A}^0 = \begin{pmatrix} \mathbf{I}_3 & \mathbf{0}_{3\times 6} \\ \mathbf{0}_{6\times 3} & \boldsymbol{\mathscr{C}}^{-1} \end{pmatrix}$, $\mathbf{A}^i = \begin{pmatrix} \mathbf{0}_{3\times 3} & \bar{\mathbf{A}}_i^1 \\ (\bar{\mathbf{A}}_i^1)^{\mathrm{T}} & \mathbf{0}_{6\times 6} \end{pmatrix}$ (where $\bar{\mathbf{A}}_i^1$, $i = 1, 2, 3$ are matrices filled with 0, -1, which for the sake of brevity are not shown here) then (149) can be rewritten as

$$\mathbf{A}^0(\mathbf{x}, t)\frac{\partial \mathbf{v}}{\partial t} + \sum_{i=1}^{3} \mathbf{A}^i(\mathbf{x}, t)\frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \mathbf{0}.$$

For the 3D problem hyperbolicity can be obtained if the eigenvalues λ obtained from the following problem are real and distinct:

$$\det\left(\sum_{i=1}^{3} n_i \mathbf{A}^i - \lambda \mathbf{A}^0\right) = 0.$$

9 Boundary value problems. Large elastic deformations

For the case of large elastic deformations, for the class of constitutive equation[37] (103), some exact solutions have been found, following the same procedure shown in Section 8.1.3, for the case of small strains (small gradient

[36] The notation σ_1, σ_2, σ_3, σ_4, σ_5 and σ_6 is used for T_{11}, T_{22}, T_{33}, T_{12}, T_{13} and T_{23}, respectively. On the other hand $\mathbf{I}_3$ is the 3×3 identity matrix, and $\mathbf{0}_{a\times b}$ is a matrix with a rows and b columns filled with zeroes.

[37] Some boundary value problems are also been treated for some subclasses of (5) and (8). In the case of implicit constitutive relations of the form (9), in a very recent work [219] we can see a study of controllable solutions for the case of compressible bodies.

of the displacement field). Let us assume that the Gibbs potential presented in Section 3.3.1 is given as in Section 6.2. We study two classes of problems, wherein the strains and stresses are distributed homogeneously and inhomogeneously. These solutions are taken from [36], and for the sake of brevity we only show some details of the same.

9.1 Solutions for homogeneous distributions for the stresses and strains

In this section we show some details for three boundary value problems with homogeneous distributions for the stresses and strains, namely the uniform tension/compression of a cylinder, the biaxial tension of a thin slab, and the uniform shear of a block.

9.1.1 The uniform tension/compression of a cylinder

Let us consider the cylinder defined in the reference configuration as $0 \leqslant R \leqslant R_{\mathrm{o}}$, $0 \leqslant \Theta \leqslant 2\pi$, $0 \leqslant Z \leqslant L$ and we assume that this cylinder has the stress distribution

$$\mathbf{T} = \sigma_{z_o}\mathbf{e}_z \otimes \mathbf{e}_z = -\sigma_{\mathrm{S}}\mathbf{I} + \mathbf{T}_{\mathrm{D}},$$

where σ_{z_o} is a constant and

$$\mathbf{T}_{\mathrm{D}} = \sigma_{\mathrm{D}_r}\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_{\mathrm{D}_\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_{\mathrm{D}_z}\mathbf{e}_z \otimes \mathbf{e}_z. \tag{241}$$

Since $\operatorname{tr}\mathbf{T}_{\mathrm{D}} = 0$ we have $\sigma_{\mathrm{D}_r} + \sigma_{\mathrm{D}_\theta} + \sigma_{\mathrm{D}_z} = 0$, as well as this, $0 = -\sigma_{\mathrm{S}} + \sigma_{\mathrm{D}_r}$, $0 = -\sigma_{\mathrm{S}} + \sigma_{\mathrm{D}_\theta}$ and $\sigma_{z_o} = -\sigma_{\mathrm{S}} + \sigma_{\mathrm{D}_z}$, then $\sigma_{\mathrm{S}} = \sigma_{\mathrm{D}_r} = \sigma_{\mathrm{D}_\theta}$ and $\sigma_{z_o} = -\sigma_{\mathrm{D}_r} + \sigma_{\mathrm{D}_z}$.

The deformation is assumed to be $r = CR$, $\theta = \Theta$ and $z = \lambda Z$, where C, λ are positive constants. The cylinder is incompressible then $C^2\lambda = 1$, thus $C = 1/\sqrt{\lambda}$. Using the above deformation to determine $\boldsymbol{\eta}$ and replacing this and (241) in (103) we obtain[38]

$$-\frac{1}{2}\ln\lambda = \alpha_0 + \alpha_1\sigma_{\mathrm{D}_r} + \alpha_2\sigma_{\mathrm{D}_r}^2, \tag{242}$$

$$-\frac{1}{2}\ln\lambda = \alpha_0 + \alpha_1\sigma_{\mathrm{D}_\theta} + \alpha_2\sigma_{\mathrm{D}_\theta}^2, \tag{243}$$

$$\ln\lambda = \alpha_0 + \alpha_1\sigma_{\mathrm{D}_z} + \alpha_2\sigma_{\mathrm{D}_z}^2. \tag{244}$$

[38]Notice the typo in Eq. (34) of [36].

From (242) and (243) we see again that $\sigma_{\mathrm{D}_r} = \sigma_{\mathrm{D}_\theta}$. The three equations (242)-(244) are not independent since $\mathrm{tr}\,\boldsymbol{\eta} = 0$, and we can choose, for example, (242) and (244). Since $\sigma_{\mathrm{D}_r} + \sigma_{\mathrm{D}_\theta} + \sigma_{\mathrm{D}_z} = 0$ and $\sigma_{\mathrm{D}_r} = \sigma_{\mathrm{D}_\theta}$ we obtain $\sigma_{\mathrm{D}_r} = \sigma_{\mathrm{D}_\theta} = -\sigma_{\mathrm{D}_z}/2$ and only (244) remains independent. With the above results, from (244) if σ_{D_z} is given we can find λ.

9.1.2 Biaxial tension of a thin slab

Let us consider the slab defined in the reference configuration $-L_i/2 \leqslant X_i \leqslant L_i/2$, $i = 1, 2, 3$ and $L_3 \ll L_1$, $L_3 \ll L_2$. The stress is assumed to be:

$$\mathbf{T} = \sigma_{1_o}\mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_{2_o}\mathbf{e}_2 \otimes \mathbf{e}_2 = -\sigma_{\mathrm{S}}\mathbf{I} + \mathbf{T}_{\mathrm{D}}, \tag{245}$$

where σ_{1_o}, σ_{2_o} are constants and

$$\mathbf{T}_{\mathrm{D}} = \sum_{i=1}^{3} \sigma_{\mathrm{D}_i}\mathbf{e}_i \otimes \mathbf{e}_i. \tag{246}$$

From (245) we have $\sigma_{\mathrm{S}} = \sigma_{\mathrm{D}_3}$ and since $\mathrm{tr}\,\mathbf{T}_{\mathrm{D}} = 0$ we get $\sigma_{\mathrm{D}_1} + \sigma_{\mathrm{D}_2} + \sigma_{\mathrm{D}_3} = 0$, i.e., $\sigma_{\mathrm{S}} = \sigma_{\mathrm{D}_3} = -\sigma_{\mathrm{D}_1} - \sigma_{\mathrm{D}_2}$ and $\sigma_{1_o} = 2\sigma_{\mathrm{D}_1} + \sigma_{\mathrm{D}_2}$, $\sigma_{2_o} = \sigma_{\mathrm{D}_1} + 2\sigma_{\mathrm{D}_2}$.

The above deviatoric stress tensor (246) is assumed to produce the deformation $x_i = \lambda_i X_i$ (here there is no sum in i), where $\lambda_i > 0$ are constants, then $\boldsymbol{\eta} = \sum_{i=1}^{3} \ln\lambda_i \mathbf{e}_i \otimes \mathbf{e}_i$. The solid is incompressible thus $\lambda_1\lambda_2\lambda_3 = 1$, which is equivalent to $\ln\lambda_1 + \ln\lambda_2 + \ln\lambda_3 = 0$. Using (246) and the above Hencky strain tensor in (103) we obtain

$$\ln\lambda_i = \alpha_0 + \alpha_1\sigma_{\mathrm{D}_i} + \alpha_2\sigma_{\mathrm{D}_i}^2, \quad i = 1, 2, 3.$$

The above three equations are not independent since $\ln\lambda_1 + \ln\lambda_2 + \ln\lambda_3 = 0$. We can choose the first two equations $i = 1, 2$, and for given σ_{D_1}, σ_{D_2} from these equations we can find λ_1 and λ_2, recalling that $\lambda_3 = 1/(\lambda_1\lambda_2)$.

9.1.3 Uniform simple shear of a slab

For the same slab described in the previous section $-L_i/2 \leqslant X_i \leqslant L_i/2$, $i = 1, 2, 3$, we study the simple shear problem. From [36] we can see that there are two types of such shear problems. In one of them we can assume the deformation $x_1 = X_1 + \gamma X_2$, $x_2 = X_2$, $x_3 = X_3$, where γ is a positive constant. If $\boldsymbol{\eta}$ is calculated with the above deformation field, from (103) we can see that in general $\mathbf{T}$ must not only have a shear component in the plane 1-2, but it must include normal components in the three directions 1, 2 and 3. This is very similar to the classical problem studied in [207, 209]. A second option, which is studied here, is to assume the presence of the stress tensor:

$$\mathbf{T} = \tau_o(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \tag{247}$$

In this case $\mathrm{tr}\,\mathbf{T} = 0$ as a result $\mathbf{T}_\mathrm{D} = \mathbf{T}$.

For the deformation let us assume that it is given as: $x_1 = \lambda_A X_1 + \gamma_A X_2$, $x_2 = \lambda_B X_2 + \gamma_B X_1$ and $x_3 = \lambda_C X_3$, where λ_A, λ_B, λ_C, γ_A and γ_B are positive constants. The solid is incompressible, thus $J = (\lambda_A\lambda_B - \gamma_A\gamma_B)\lambda_C = 1$, from where we can get $\lambda_C = 1/(\lambda_A\lambda_B - \gamma_A\gamma_B)$ assuming that $\lambda_A\lambda_B \neq \gamma_A\gamma_B$. The Hencky strain tensor $\boldsymbol{\eta}$ can be calculated as $\boldsymbol{\eta} = \sum_{i=1}^{3} \ln\lambda_i \mathbf{v}^{(i)}\otimes\mathbf{v}^{(i)}$, where λ_i, $\mathbf{v}^{(i)}$ are the principal stretches and the principal directions of $\mathbf{B}$, respectively. In this problem (see [36]) we have $\lambda_1 = \sqrt{\frac{\lambda_a-\lambda_b}{2}}$, $\lambda_2 = \sqrt{\frac{\lambda_a+\lambda_b}{2}}$, $\lambda_3 = \lambda_C = 1/(\lambda_A\lambda_B - \gamma_A\gamma_B)$, where $\lambda_a = \lambda_A^2 + \lambda_B^2 + \gamma_A^2 + \gamma_B^2$ and $\lambda_b = \sqrt{[(\lambda_A-\lambda_B)^2 + (\gamma_A+\gamma_B)^2][(\lambda_A+\lambda_B)^2 + (\gamma_A-\gamma_B)^2]}$ (the expressions for the components for $\mathbf{v}^{(i)}$, $i = 1, 2$ can be found in [36], and for the sake brevity are not shown here, while $\mathbf{v}^{(3)} = \mathbf{e}_3$).

Using (247) and the above expression for $\boldsymbol{\eta}$ in (103) we obtain:

$$\ln\lambda_1\left(v_1^{(1)}\right)^2 + \ln\lambda_2\left(v_1^{(2)}\right)^2 = \alpha_0 + \alpha_1\tau_o, \tag{248}$$

$$\ln\lambda_1\left(v_2^{(1)}\right)^2 + \ln\lambda_2\left(v_2^{(2)}\right)^2 = \alpha_0 + \alpha_1\tau_o, \tag{249}$$

$$-\ln(\lambda_A\lambda_B - \gamma_A\gamma_B) = \ln\lambda_3 = \alpha_0, \tag{250}$$

$$\ln\lambda_1 v_1^{(1)} v_2^{(1)} + \ln\lambda_2 v_1^{(2)} v_2^{(2)} = \alpha_1\tau_o, \tag{251}$$

where $v_j^{(i)}$ are the components j of the eigenvectors $\mathbf{v}^{(i)}$. The above equations (248)-(251) are not independent, since $\mathrm{tr}\,\boldsymbol{\eta} = 0$. We can choose, for example, (248), (249) and (251). Such equations can be used to find λ_A, λ_B, γ_A and γ_B (if τ_o is given). We can see that we have 4 unknowns and 3 equations, thus this problem can present non-unique solutions.

9.2 Solutions for inhomogeneous distributions for the stresses and strains

In the case of inhomogeneous distributions for the stresses and strains, in [36] there is a study of the classical universal solutions for Green elastic solids, inquiring if those solutions are also solutions for $(4)_1$ and (103) (when $\ddot{\mathbf{x}}$ and $\mathbf{b} = \mathbf{0}$). In [36] it is shown that that is indeed the case, and some details of those solutions are shown in this section. In this section we study briefly the 5 universal solutions listed by Ericksen (see, for example, [77, 209]).

9.2.1 Bending, stretching and shearing of a rectangular block

Let us consider the slab in the reference configuration defined as (changing the notation X_i, $i = 1, 2, 3$ for X, Y and Z, respectively) $X_1 \leqslant X \leqslant X_2$, $-Y_o \leqslant Y \leqslant Y_o$, $-Z_o \leqslant Z \leqslant Z_o$. It is assumed that the Cauchy stress is given as:

$$\mathbf{T} = -\sigma_{\mathrm{S}}\mathbf{I} + \mathbf{T}_{\mathrm{D}}(r), \tag{252}$$

where

$$\mathbf{T}_{\mathrm{D}}(r) = \sigma_{\mathrm{D}_r}(r)\mathbf{e}_r\otimes\mathbf{e}_r + \sigma_{\mathrm{D}_\theta}(r)\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \sigma_{\mathrm{D}_z}(r)\mathbf{e}_z\otimes\mathbf{e}_z + \tau_{\theta z}(r)(\mathbf{e}_\theta\otimes\mathbf{e}_z + \mathbf{e}_z\otimes\mathbf{e}_\theta). \tag{253}$$

From $\operatorname{tr}\mathbf{T}_{\mathrm{D}} = 0$ we have $\sigma_{\mathrm{D}_z} = -\sigma_{\mathrm{D}_r} - \sigma_{\mathrm{D}_\theta}$.

It is assumed that the above slab deforms as $r = \sqrt{2AX}$, $\theta = BY$ and $z = Z/(AB) - BCY$, where $A \neq 0$, $B \neq 0$ and C are constants. It is possible to see that $J = 1$. The Hencky strain tensor is calculated as (see Section 9.1.3) $\boldsymbol{\eta} = \sum_{i=1}^{3} \ln \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}$, where λ_i, $\mathbf{v}^{(i)}$ are the principal stretches and the principal directions of $\mathbf{B}$. For this problem we have $\lambda_1 = A/r$, $\lambda_2 = \sqrt{\lambda_a - \lambda_b}/(\sqrt{2}AB)$, $\lambda_3 = \sqrt{\lambda_a + \lambda_b}/(\sqrt{2}AB)$, where

$$\lambda_a = 1 + A^2B^2(C^2 + r^2), \quad \lambda_b = \sqrt{1 + 2A^2B^4(C^2 - r^2) + A^4B^8(C^2 + r^2)^2}.$$

As for $\mathbf{v}^{(i)}$ we have $\mathbf{v}^{(1)} = \mathbf{e}_r$, and for the non-zero components of $\mathbf{v}^{(k)}$, $k = 2, 3$ their components given by are $v_2^{(k)} = \frac{1}{\sqrt{1+\frac{1}{C^2}\left(r-\frac{\lambda_k^2}{rB^2}\right)^2}}$, $v_3^{(k)} = \frac{\frac{1}{C}\left(r-\frac{\lambda_k^2}{rB^2}\right)}{\sqrt{1+\frac{1}{C^2}\left(r-\frac{\lambda_k^2}{rB^2}\right)^2}}$, $k = 2, 3$.

Using (252), (253) in $(4)_1$ we obtain $\frac{\mathrm{d}T_{rr}}{\mathrm{d}r} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0$, whose solution is the same as (172), i.e., $\sigma_{\mathrm{S}}(r) = \sigma_{\mathrm{D}_r}(r) + \int_{r_{\mathrm{o}}}^{r} \frac{1}{\xi}[\sigma_{\mathrm{D}_r}(\xi) - \sigma_{\mathrm{D}_\theta}(\xi)]\,\mathrm{d}\xi$ (if we assume $T_{rr}(r_{\mathrm{i}}) = -P$ and $T_{rr}(r_{\mathrm{o}}) = 0$) and $P = \int_{r_{\mathrm{i}}}^{r_{\mathrm{i}}} \frac{1}{\xi}[\sigma_{\mathrm{D}_r}(\xi) - \sigma_{\mathrm{D}_\theta}(\xi)]\,\mathrm{d}\xi$, where $r_{\mathrm{i}} = \sqrt{2AX_1}$ and $r_{\mathrm{o}} = \sqrt{2AX_2}$.

Replacing (253) and the above expression for $\boldsymbol{\eta}$ in (103) we obtain:

$$\ln\left(\frac{A}{r}\right) = \alpha_0 + \alpha_1\sigma_{\mathrm{D}_r} + \alpha_2\sigma_{\mathrm{D}_r}^2, \tag{254}$$

$$\ln\lambda_2\left(v_2^{(2)}\right)^2 + \ln\lambda_2\left(v_2^{(3)}\right)^2 = \alpha_0 + \alpha_1\sigma_{\mathrm{D}_\theta} + \alpha_2\left(\sigma_{\mathrm{D}_\theta}^2 + \tau_{\theta z}^2\right), \tag{255}$$

$$\ln\lambda_2\left(v_3^{(2)}\right)^2 + \ln\lambda_2\left(v_3^{(3)}\right)^2 = \alpha_0 + \alpha_1\sigma_{\mathrm{D}_z} + \alpha_2\left(\tau_{\theta z}^2 + \sigma_{\mathrm{D}_z}^2\right), \tag{256}$$

$$\ln\lambda_2 v_2^{(2)}v_3^{(2)} + \ln\lambda_3 v_2^{(3)}v_3^{(3)} = \alpha_1\tau_{\theta z} + \alpha_2\tau_{\theta z}(\sigma_{\mathrm{D}_\theta} + \sigma_{\mathrm{D}_z}). \tag{257}$$

The above four equations are not independent since $\mathrm{tr}\,\boldsymbol{\eta} = 0$. We can choose, for example, (254), (255) and (257) to find $\sigma_{\mathrm{D}_r}(r)$, $\sigma_{\mathrm{D}_\theta}(r)$ and $\tau_{\theta z}(r)$ (recall that $\sigma_{\mathrm{D}_z} = -\sigma_{\mathrm{D}_r} - \sigma_{\mathrm{D}_\theta}$) in terms of the deformation.

9.2.2 Straightening, stretching and shearing of a sector of a hollow cylindrical annulus sector

Let us consider the hollow cylindrical annulus sector defined in the reference configuration as $R_1 \leqslant R \leqslant R_2$, $-\Theta_o \leqslant \Theta \leqslant \Theta_o$, $-Z_o \leqslant Z \leqslant Z_o$. The stress tensor is assumed to be of the form (we use the notation x, y and z for x_i, $i = 1, 2, 3$ in the current configuration):

$$\mathbf{T} = -\sigma_{\mathrm{S}}(x)\mathbf{I} + \mathbf{T}_{\mathrm{D}}(x), \tag{258}$$

where

$$\mathbf{T}_{\mathrm{D}}(x) = \sum_{i=1}^{3}\sigma_{\mathrm{D}_i}(x)\mathbf{e}_i\otimes\mathbf{e}_i + \tau_{23}(x)(\mathbf{e}_2\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_2). \tag{259}$$

The above stress tensor (258), (259) satisfy(4)$_1$ (assuming $\ddot{\mathbf{x}} = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$) if $\sigma_{\mathrm{S}}(x) = \sigma_{\mathrm{D}_1}(x)$. As well as this from $\mathrm{tr}\,\mathbf{T}_{\mathrm{D}} = 0$ we have $\sum_{i=1}^{3}\sigma_{\mathrm{D}_i} = 0$.

From Section 92 of [209] (see also [77, 36]) we assume that the deformation is given as $x = AB^2R^2/2$, $y = \Theta/(AB)$, $z = Z/B + C\Theta/(AB)$, where A, B, C are constants and $AB \neq 0$. It is easy to see that for the above deformation $J = 1$. As for the principal stretches we obtain $\lambda_1 = AB^2R$, $\lambda_2 = \frac{1}{ABR}\sqrt{\frac{\lambda_a - \lambda_b}{2}}$, $\lambda_3 = \frac{1}{ABR}\sqrt{\frac{\lambda_a+\lambda_b}{2}}$, where $\lambda_a = 1 + C^2 + A^2R^2$ and $\lambda_b = \sqrt{(1+C^2)^2 + 2A^2(C^2-1)R^2 + A^4R^4}$. The eigenvectors of $\mathbf{B}$ are $\mathbf{v}^{(1)} = \mathbf{e}_1$ and for the non-zero components of $\mathbf{v}^{(k)}$, $k = 2, 3$ we have $v_2^{(k)} = \frac{1}{\sqrt{1+\frac{1}{C^2}\left(\lambda_k^2A^2B^2R^2-1\right)^2}}$, $v_3^{(k)} = \frac{1}{\sqrt{\frac{C^2}{(\lambda_k^2A^2B^2R^2-1)^2}+1}}$, $k = 2, 3$.

Using (259) and the above expressions for the eigenvectors and principal stretches in (103) we get the four equations:

$$\ln \lambda_1 = \alpha_0 + \alpha_1 \sigma_{\mathrm{D}_1} + \alpha_2 \sigma_{\mathrm{D}_1}^2, \qquad (260)$$

$$\ln \lambda_2 \left(v_2^{(2)}\right)^2 + \ln \lambda_3 \left(v_2^{(3)}\right)^2 = \alpha_0 + \alpha_1 \sigma_{\mathrm{D}_2} + \alpha_2 (\sigma_{\mathrm{D}_2}^2 + \tau_{23}^2), \qquad (261)$$

$$\ln \lambda_2 \left(v_3^{(2)}\right)^2 + \ln \lambda_3 \left(v_3^{(3)}\right)^2 = \alpha_0 + \alpha_1 \sigma_{\mathrm{D}_3} + \alpha_2 (\tau_{23}^2 + \sigma_{\mathrm{D}_3}^2), \qquad (262)$$

$$\ln \lambda_2 v_2^{(2)} v_3^{(2)} + \ln \lambda_3 v_2^{(3)} v_3^{(3)} = \alpha_1 \tau_{23} + \alpha_2 \tau_{23} (\sigma_{\mathrm{D}_2} + \sigma_{\mathrm{D}_3}). \qquad (263)$$

Again the above four equations are not independent. We can solve, for example, (260), (262) and (263) to find σ_{D_1}, σ_{D_2} and τ_{23} in terms of the deformation, where also $\sigma_{\mathrm{D}_3} = -\sigma_{\mathrm{D}_1} - \sigma_{\mathrm{D}_2}$.

9.2.3 Inflation, bending, torsion, extension and shearing of an annular wedge

In this problem we are interested in the deformation of the annular wedge defined in the reference configuration as $R_1 \leqslant R \leqslant R_2$, $0 \leqslant \Theta \leqslant \Theta_o$, $0 \leqslant Z \leqslant L$. The stress tensor is assumed to be of the same form as in Section 9.2.1, and for brevity such expressions are not repeated here.

As for the deformation we assume (see [77] and Section 92 in [209]) that the wedge deforms as $r = \sqrt{AR^2 + B}$, $\theta = C\Theta + DZ$, $z = E\Theta + FZ$, where A, B, C, D, E and F are constants. The solid is incompressible and $J = 1$ is satisfied if $A(CF - DE) = 1$. Regarding $\boldsymbol{\eta}$ the expressions for λ_i, $i = 1, 2, 3$ are (compare with Section 9.2.1) $\lambda_1 = AR/r$, $\lambda_2 = \sqrt{\lambda_a - \lambda_b}/(\sqrt{2}R)$, $\lambda_3 = \sqrt{\lambda_a + \lambda_b}/(\sqrt{2}R)$, where $\lambda_a = E^2 + C^2 r^2 + R^2(F^2 + D^2 r^2)$ and $\lambda_b = \sqrt{[E^2 + C^2 r^2 + (F^2 + D^2 r^2)R^2]^2 - 4(DE - CF)^2 r^2 R^2}$. For the sake of brevity we do not show the expressions for the components of $\mathbf{v}^{(i)}$, $i = 1, 2, 3$ and the interested reader can see [36].

Replacing the stress tensor and the Hencky strain tensor in (103) we have $\ln\left(\frac{AR}{r}\right) = \alpha_0 + \alpha_1 \sigma_{\mathrm{D}_r} + \alpha_2 \sigma_{\mathrm{D}_r}^2$ in the place of (254), and the rest of the equations are the same as (255)-(257), where here the expressions for λ_i and $\mathbf{v}^{(i)}$ are different. We can choose three of those four equations (255)-(257) to find, for example, $\sigma_{\mathrm{D}_r}(r)$, $\sigma_{\mathrm{D}_\theta}(r)$ and $\tau_{\theta z}(r)$ in terms of the deformation, considering that $\sigma_{\mathrm{D}_z} = -\sigma_{\mathrm{D}_r} - \sigma_{\mathrm{D}_\theta}$.

9.2.4 Inflation or eversion of a sector of a spherical shell

Here we are interested in studying the deformation of a hollow sphere defined in the reference configuration as $R_\mathrm{i} \leqslant R \leqslant R_\mathrm{o}$, $0 \leqslant \Theta \leqslant 2\pi$, $0 \leqslant \Phi \leqslant \Phi_\mathrm{o} \leqslant \pi$. The stress tensor that appear in such a sphere is assumed to be (in spherical coordinates in the current configuration):

$$\mathbf{T} = -\sigma_\mathrm{S}(r)\mathbf{I} + \mathbf{T}_\mathrm{D}(r), \tag{264}$$

where

$$\mathbf{T}_\mathrm{D}(r) = \sigma_{\mathrm{D}_r}(r)\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_{\mathrm{D}_\theta}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_{\mathrm{D}_\phi}(r)\mathbf{e}_\phi \otimes \mathbf{e}_\phi, \tag{265}$$

where $\operatorname{tr}\mathbf{T}_\mathrm{D} = 0 \Leftrightarrow \sigma_{\mathrm{D}_r} + \sigma_{\mathrm{D}_\theta} + \sigma_{\mathrm{D}_z} = 0$. Using the above stress tensor in $(4)_1$ we obtain $\sigma_{\mathrm{D}_\phi} = \sigma_{\mathrm{D}_\theta}$ and $\sigma_\mathrm{S}(r) = \sigma_{\mathrm{D}_r}(r) + 2\int_{r_\mathrm{i}}^{r} \frac{1}{\xi}\left[\sigma_{\mathrm{D}_r}(\xi) - \sigma_{\mathrm{D}_\theta}(\xi)\right]\mathrm{d}\xi + C$, where C is a constant, which can be found from the boundary conditions at $R = R_\mathrm{i}$ or $R = R_\mathrm{o}$, which for brevity is not discussed here.

As for the deformation from Section 92 of [209] it is assumed that in the current configuration we have $r = \sqrt[3]{A \pm R^3}$, $\theta = \pm\Theta$, $\phi = \Phi$, where the $+$ means we are interested in the inflation of the sphere, and the $-$ means we are interested in the eversion of the same. For the above deformation $J = 1$. Using the Hencky strain tensor calculated with the above deformation and the stress tensor (264), (265) in (103) we obtain:

$$\ln\left(\frac{R^2}{r^2}\right) = \alpha_0 + \alpha_1\sigma_{\mathrm{D}_r} + \alpha_2\sigma_{\mathrm{D}_r}^2, \tag{266}$$

$$\ln\left(\frac{r}{R}\right) = \alpha_0 + \alpha_1\sigma_{\mathrm{D}_\theta} + \alpha_2\sigma_{\mathrm{D}_\theta}^2. \tag{267}$$

The above two equations are not independent, on the other hand from $\operatorname{tr}\mathbf{T}_\mathrm{D} = 0$ we have $\sigma_{\mathrm{D}_\theta} = -\sigma_{\mathrm{D}_r}/2$. We can solve, for example, (266) to find σ_{D_r} as a function of the deformation.

9.2.5 Azimuthal shear of a cuboid

In this problem, for the annular wedge (called cuboid in the original works [200, 201]) $R_1 \leqslant R \leqslant R_2$, $0 \leqslant \Theta \leqslant \Theta_o$, $0 \leqslant Z \leqslant L$, we assume the deformation $r = AR$, $\theta = B\ln R + C\Theta$, $z = Z/(A^2C)$, where A, B and C are constants. It is easy to show that $J = 1$. This deformation is interesting in the sense that although it is inhomogeneous, the principal stretches and principal directions of $\mathbf{B}$ are constants. We have $\lambda_1 = A\sqrt{\frac{\lambda_a-\lambda_b}{2}}$, $\lambda_2 = A\sqrt{\frac{\lambda_a+\lambda_b}{2}}$ and $\lambda_3 = 1/(A^2C)$, where $\lambda_a = 1 + B^2 + C^2$ and $\lambda_b = \sqrt{(1+B^2)^2 + 2(B^2-1)C^2 + C^4}$. On the other hand $\mathbf{v}^{(3)} = \mathbf{e}_z$ and for the nonzero components of $\mathbf{v}^{(q)}$, $q = 1,2$ we have $v_1^{(q)} = \frac{1}{\sqrt{1+\frac{1}{A^4B^2}(\lambda_q^2-A^2)^2}}$ and $v_2^{(q)} = \frac{1}{\sqrt{1+\frac{A^4B^2}{(\lambda_q^2-A^2)^2}}}$, $q = 1,2$.

Considering the above results regarding the components of $\boldsymbol{\eta}$, for the stress we assume the following expression:

$$\mathbf{T} = -\sigma_{\mathrm{S}}(r,\theta)\mathbf{I} + \mathbf{T}_{\mathrm{D}}, \tag{268}$$

where the deviatoric stress tensor has constant components and it is given as[39]

$$\mathbf{T}_{\mathrm{D}} = \sigma_{\mathrm{D}_r}\mathbf{e}_r\otimes\mathbf{e}_r + \sigma_{\mathrm{D}_\theta}\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \sigma_{\mathrm{D}_z}\mathbf{e}_z\otimes\mathbf{e}_z + \tau_{r\theta}(\mathbf{e}_r\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_r), \tag{269}$$

where σ_{D_r}, $\sigma_{\mathrm{D}_\theta}$, σ_{D_z} and $\tau_{r\theta}$ are all constants, and $\sigma_{\mathrm{D}_r} + \sigma_{\mathrm{D}_\theta} + \sigma_{\mathrm{D}_z} = 0$. Replacing (268), (269) in (103) we obtain the partial differential equations $-\frac{\partial\sigma_{\mathrm{S}}}{\partial r} + \frac{1}{r}(\sigma_{\mathrm{D}_r} - \sigma_{\mathrm{D}_\theta}) = 0$ and $-\frac{1}{r}\frac{\partial\sigma_{\mathrm{S}}}{\partial\theta} + 2\frac{\tau_{r\theta}}{r} = 0$ whose solution is $\sigma_{\mathrm{S}}(r,\theta) = (\sigma_{\mathrm{D}_r} - \sigma_{\mathrm{D}_\theta})\ln(r/r_{\mathrm{i}}) + 2\tau_{r\theta}\theta + C_o$, where C_o is a constant.

Replacing (269) and $\boldsymbol{\eta}$ calculated with the above λ_i and $\mathbf{v}^{(i)}$, $i = 1,2,3$ in (103) we obtain

$$\ln\lambda_1\left(v_1^{(1)}\right)^2 + \ln\lambda_2\left(v_1^{(2)}\right)^2 = \alpha_0 + \alpha_1\sigma_{\mathrm{D}_r} + \alpha_2\left(\sigma_{\mathrm{D}_r}^2 + \tau_{r\theta}^2\right), \tag{270}$$

$$\ln\lambda_1\left(v_2^{(1)}\right)^2 + \ln\lambda_2\left(v_2^{(2)}\right)^2 = \alpha_0 + \alpha_1\sigma_{\mathrm{D}_\theta} + \alpha_2\left(\tau_{r\theta}^2 + \sigma_{\mathrm{D}_\theta}^2\right), \tag{271}$$

$$\ln\lambda_3 = \alpha_0 + \alpha_1\sigma_{\mathrm{D}_z} + \alpha_2\sigma_{\mathrm{D}_z}^2, \tag{272}$$

$$\ln\lambda_1 v_1^{(1)}v_2^{(1)} + \ln\lambda_2 v_1^{(2)}v_2^{(2)} = \alpha_1\tau_{r\theta} + \alpha_2\tau_{r\theta}(\sigma_{\mathrm{D}_r} + \sigma_{\mathrm{D}_\theta}). \tag{273}$$

The above four equations are not independent, we can choose, for example, (270), (271) and (273) to find σ_{D_r}, $\sigma_{\mathrm{D}_\theta}$ and $\tau_{r\theta}$ in terms of the deformation, considering that $\sigma_{\mathrm{D}_r} + \sigma_{\mathrm{D}_\theta} + \sigma_{\mathrm{D}_z} = 0$.

[39] In Section 6.5 of [36] it is assumed that the nonzero components of $\mathbf{T}_{\mathrm{D}}$ can depend on the radial position r, but from the constitutive equations therein, it is possible to see that such components must be constants.

9.2.6 Circumferential shear of a cylindrical annulus

In this section we show the problem of the circumferential shear of a cylindrical annulus, which for the class of constitutive equation for incompressible solids (104) has been studied in [58].

Let us consider the annulus defined in the reference configuration as $R_{\mathrm{i}} \leqslant R \leqslant R_{\mathrm{o}}$, $0 \leqslant \Theta \leqslant 2\pi$, $0 \leqslant Z \leqslant L$, which deforms as:

$$r = f(R), \quad \theta = \Theta + g(R), \quad z = Z, \tag{274}$$

with the boundary conditions $f(R_{\mathrm{i}}) = R_{\mathrm{i}}$, $f(R_{\mathrm{o}}) = R_{\mathrm{o}}$, $g(R_{\mathrm{i}}) = 0$, $g(R_{\mathrm{o}}) = \psi$.

It is assumed that the stress compatible with the above deformation is given as $\mathbf{T} = \mathbf{T}(r) = \mathbf{T}(R) = -\sigma_{\mathrm{S}}(R)\mathbf{I} + \mathbf{T}_{\mathrm{D}}(R)$, where

$$\mathbf{T}_{\mathrm{D}}(R) = \sigma_{\mathrm{D}_r}(R)\mathbf{e}_r\otimes\mathbf{e}_r + \sigma_{\mathrm{D}_\theta}(R)\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \sigma_{\mathrm{D}_z}(R)\mathbf{e}_z\otimes\mathbf{e}_z + \tau_{r\theta}(R)(\mathbf{e}_r\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_r), \tag{275}$$

where $\mathrm{tr}\,\mathbf{T}_{\mathrm{D}} = 0$. From the constraint of incompressibility we obtain $f(R) = R$. Using $\mathbf{T}$ above in $(4)_1$ (assuming $\ddot{\mathbf{x}} = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$) we obtain the solutions (compare, for example, with the results shown in 9.2.1):

$$\sigma_{\mathrm{S}}(R) = \sigma_{\mathrm{D}_r}(R) + \int_{R_{\mathrm{i}}}^{R} \frac{1}{\xi}\left[\sigma_{\mathrm{D}_r}(\xi) - \sigma_{\mathrm{D}_\theta}(\xi)\right] \mathrm{d}\xi + \sigma_o, \quad \tau_{r\theta}(R) = \frac{\tau_o}{R^2}, \tag{276}$$

where σ_o and τ_o are constants.

The body considered in [58] is isotropic, then the eigenvectors of $\boldsymbol{\eta}$ (calculated from (274)) and $\mathbf{T}_{\mathrm{D}}$ should be the same, which if $\tau_{r\theta}(R) \neq 0$ is satisfied if $g'(R) = [\sigma_{\mathrm{D}_\theta}(R) - \sigma_{\mathrm{D}_r}(R)]/\tau_{r\theta}(R)$.

From (275) it is possible to calculate the principal stresses σ_{D_i} of $\mathbf{T}_{\mathrm{D}}$, but they are long expressions that for the sake of brevity are not shown here, and which are of the form $\sigma_{\mathrm{D}_i} = \sigma_{\mathrm{D}_i}(\sigma_{\mathrm{D}_r}(R), \sigma_{\mathrm{D}_\theta}(R))$, $i = 1, 2, 3$, where we recall $(276)_2$ and that from $\mathrm{tr}\,\mathbf{T}_{\mathrm{D}} = 0$ we have $\sigma_{\mathrm{D}_z} = -\sigma_{\mathrm{D}_r} - \sigma_{\mathrm{D}_\theta}$. On the other hand from (274) the principal stretches are $\lambda_1 = \frac{1}{\sqrt{2}}\sqrt{2 + R^2 g'(R)^2 - Rg'(R)\sqrt{4 + R^2 g'(R)^2}}$, $\lambda_2 = \frac{1}{\sqrt{2}}\sqrt{2 + R^2 g'(R)^2 + Rg'(R)\sqrt{4 + R^2 g'(R)^2}}$ and $\lambda_3 = 1$. Using the above expressions for the principal stresses and principal stretches[40] in (106)-(108) (recalling the above expression for $g'(R)$) we obtain two algebraic equations (we recall that one of the equations (106)-(108) is not independent), which can be solved to find σ_{D_r} and $\sigma_{\mathrm{D}_\theta}$.

[40] For the sake of brevity we do not show expressions for the principal directions of $\mathbf{B}$ here.

9.3 Propagation of small amplitude waves in a body with an initial 'large' time-independent stresses. Incremental formulation

In this section we present a theory of propagation of small amplitude waves that propagate in a body with an initial time-independent relatively large stresses, expanding the results shown in Section 8.2.2 and [4], for the case of large elastic deformations. These results are taken from [54].

Let us consider a nonlinear incompressible isotropic elastic body using (21), (103), where the Hencky strain tensor is a function of the Cauchy stress tensor, i.e. $\boldsymbol{\eta} = \mathfrak{f}(\mathbf{T}) = \alpha_0\mathbf{I}+\alpha_1\mathbf{T}_{\mathrm{D}}+\alpha_2\mathbf{T}_{\mathrm{D}}^2$, where $\alpha_0 = -\frac{2I_{\mathrm{D}_2}}{3}\frac{\partial\hat{\mathcal{G}}}{\partial I_{\mathrm{D}_3}}$, $\alpha_1 = \frac{\partial\hat{\mathcal{G}}}{\partial I_{\mathrm{D}_2}}$, $\alpha_2 = \frac{\partial\hat{\mathcal{G}}}{\partial I_{\mathrm{D}_3}}$ and $\mathbf{T}_{\mathrm{D}} = \mathbf{T} - \frac{\operatorname{tr}(\mathbf{T})}{3}\mathbf{I}$. We assume the existence of the following three configurations: a reference configuration $\boldsymbol{\kappa}_{\mathrm{r}}(\mathcal{B})$, a configuration $\boldsymbol{\kappa}_0(\mathcal{B})$ where we have large elastic deformations by applying a time-independent load (where there is a stress tensor $\mathbf{T}^0 = \mathbf{T}^0(\mathbf{x}^0)$), and a configuration $\boldsymbol{\kappa}_t(\mathcal{B})$ that is obtained by applying a 'small' time-dependent incremental stress $\Delta\mathbf{T} = \Delta\mathbf{T}(\mathbf{x}, t)$ to the above body in that configuration $\boldsymbol{\kappa}_0(\mathcal{B})$.

The position of a particle in $\boldsymbol{\kappa}_{\mathrm{r}}(\mathcal{B})$ is denoted $\mathbf{X}$, the position of the same particle in the configuration $\boldsymbol{\kappa}_0(\mathcal{B})$ is denoted $\mathbf{x}^0$, and the position in the configuration $\boldsymbol{\kappa}_t(\mathcal{B})$ is denoted $\mathbf{x}$. We have $\mathbf{x}^0 = \boldsymbol{\chi}^0(\mathbf{X})$ and $\mathbf{x} = \boldsymbol{\chi}(\mathbf{x}^0, t)$ and in [54] the incremental displacement field $\mathbf{u}$ is defined through $\boldsymbol{\chi}(\mathbf{x}^0, t) = \mathbf{x}^0 + \mathbf{u}(\mathbf{x}^0, t)$.

We assume that $|\Delta\mathbf{T}|/|\mathbf{T}^0|$ is small enough such that $\left|\frac{\partial\mathbf{u}}{\partial\mathbf{x}^0}\right| \sim O(\delta)$, $\delta \ll 1$, then if $\mathbf{F}^0$ denotes the deformation gradient from $\boldsymbol{\kappa}_{\mathrm{r}}(\mathcal{B})$ to $\boldsymbol{\kappa}_0(\mathcal{B})$, $\mathbf{B}^0 = \mathbf{F}^0\left(\mathbf{F}^0\right)^{\mathrm{T}}$, and if $\mathbf{F}$ denotes the deformation gradient from $\boldsymbol{\kappa}_{\mathrm{r}}(\mathcal{B})$ to $\boldsymbol{\kappa}_t(\mathcal{B})$, from [54] we have the approximation

$$\boldsymbol{\eta} = \frac{1}{2}\ln\mathbf{B} = \frac{1}{2}\ln\left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right) \approx \frac{1}{2}\ln\mathbf{B}^0 + \mathbf{H},$$

where in Cartesian coordinates $H_{ij} = \Psi_{ijkl}\frac{\partial u_k}{\partial x_l^0}$ and Ψ_{ijkl} depends on $\mathbf{B}^0$. For the sake of brevity such Ψ_{ijkl} are not listed here. Using the above approximation in $\boldsymbol{\eta} = \mathfrak{f}(\mathbf{T})$ we have $\frac{1}{2}\ln\mathbf{B} = \mathfrak{f}(\mathbf{T})$ and $\frac{1}{2}\ln\mathbf{B}^0 = \mathfrak{f}(\mathbf{T}^0)$ from where we get $\frac{1}{2}\ln\mathbf{B}^0 + \mathbf{H} \approx \mathfrak{f}(\mathbf{T}^0) + \mathscr{C}^0\cdot\Delta\mathbf{T}$, where $\mathscr{C}^0 = \frac{\partial\mathfrak{f}}{\partial\mathbf{T}}(\mathbf{T}^0)$. From the above results we obtain the incremental constitutive relation:

$$\mathbf{H} = \mathscr{C}^0\cdot\Delta\mathbf{T}.$$

The body is incompressible, which means $\operatorname{tr}\boldsymbol{\eta} = 0$ and $\operatorname{tr}\boldsymbol{\eta}^0 = 0$, from where in [54] it is shown that that is equivalent to $\frac{\partial u_i}{\partial x^0} = 0$. In that paper it is proved that that restriction is satisfied automatically.

Obtaining the incremental equation of motion (see [4] and [54]) we can find $\mathbf{u} = \mathbf{u}(\mathbf{x}^0, t)$ and $\Delta\mathbf{T} = \Delta\mathbf{T}(\mathbf{x}^0, t)$ solving the two linear partial differential equations (in index notation and Cartesian coordinates):

$$\Psi_{ijkl}\frac{\partial u_k}{\partial x_l^0} = \mathscr{C}^0_{ijkl}\Delta T_{kl}, \quad \rho\ddot{u}_i = \frac{\partial}{\partial x_j^0}\left(\Delta T_{ji} - \frac{\partial u_j}{\partial x_m^0}T_{mi}\right),$$

where Ψ_{ijkl} depends on $\mathbf{B}^0$, and where $\mathbf{B}^0$, $\mathscr{C}^0_{ijkl}$ are functions of $\mathbf{T}^0$.

For initial time independent $\mathbf{T}^0$, $\boldsymbol{\eta}^0 = \frac{1}{2}\ln\mathbf{B}^0$, for homogeneous distributions for the stresses and strains, we can solve $(4)_1$ ($\mathbf{b} = \mathbf{0}$) for infinite media assuming

$$\mathbf{u} = \mathbf{U}\exp\left[\hat{I}w(s\mathbf{n}\cdot\mathbf{x}^0 - t)\right], \quad \Delta\mathbf{T} = \boldsymbol{\Lambda}\exp\left[\hat{I}w(s\mathbf{n}\cdot\mathbf{x}^0 - t)\right],$$

where $\mathbf{U}$, $\boldsymbol{\Lambda}$ are a vector and a symmetric second order tensor with constant components, $\hat{I}$ is the imaginary unit, $\nu = 1/s$ is the speed of the small amplitude waves, w is the frequency of such small amplitude waves and $\mathbf{n}$ is a unit vector field (direction of polarization).

Using $\mathscr{C}^0_{ijkl}$ calculated with the constitutive equation for rubber from [43], results for ν have been obtained for different cases for $\mathbf{T}^0$ homogeneous. The predictions for ν differ significantly from the predictions for speed of small amplitude waves, using the incremental theory for Green elastic solids by Ogden [143], where the deformation field is decomposed into an initial large deformation, to which an incremental time-dependent deformation is superimposed. The boundary value problems studied in that paper are: the uniform extension of a cylinder (see Section 9.1.1), the biaxial extension of a slab (see Section 9.1.2), and the triaxial extension of a slab.

9.4 A boundary value problem for a class of thermo-elastic body

In this section we show one boundary value problem for the class of isotropic incompressible thermo-elastic solid presented in Section 6.3 (see Eq. (112) therein). This problem has been taken from [60]. The problem corresponds to the inflation and extension of the cylindrical annulus[41] defined in the reference configuration[42] $R_{\rm i} \leqslant R \leqslant R_{\rm o}$, $0 \leqslant \Phi \leqslant 2\pi$, $0 \leqslant Z \leqslant L$. It is assumed that the deformation of the annulus is given by:

$$r = r(R), \quad \phi = \Phi, \quad z = \lambda_z Z, \tag{277}$$

where λ_z is a positive constant. About the stress we assume $\mathbf{T} = -\sigma_{\rm S}(r)\mathbf{I} + \mathbf{T}_{\rm D}$, where[43] $\operatorname{tr}\mathbf{T}_{\rm D} = 0$ and $\mathbf{T}_{\rm D} = \sigma_{{\rm D}_r}(r)\mathbf{e}_r\otimes\mathbf{e}_r + \sigma_{{\rm D}_\phi}(r)\mathbf{e}_\phi\otimes\mathbf{e}_\phi + \sigma_{{\rm D}_z}(r)\mathbf{e}_z\otimes\mathbf{e}_z$.

This problem is quasi-static thus $\dot{\boldsymbol{\tau}} = \mathbf{0}$. We assume as well that there is no internal source of heat and $\dot{\theta} = 0$, and from the first law of thermodynamic (110) we get $\operatorname{div}\mathbf{Q} = 0$, which for the Fourier's constitutive equation for heat transfer becomes $\nabla^2\theta = 0$ (see Sections 5.4 and 6.3, considering that $\mathbf{Q} = \zeta\nabla\theta$). In this problem it is assumed that $\theta = \theta(r)$ and the equation $\nabla^2\theta = 0$ is satisfied if

$$\theta(r) = C_0 + C_1 \ln r, \tag{278}$$

where C_0, C_1 are constants that depend on the boundary conditions for the thermal part of the problem.

The most important difference in this case in comparison with what has been presented in Sections 9.1.1-9.2.5 is that $J = J(\theta)$. From (277), (278) and Section 6.3 we have $J = \frac{{\rm d}r}{{\rm d}R}\frac{r}{R}\lambda_z = f(\theta(r))$ whose solution is:

$$R(r) = \sqrt{2\lambda_z \int_{r_{\rm i}}^{r} \frac{\xi}{f(\theta(\xi))}\,{\rm d}\xi + R_{\rm i}^2}.$$

We can try to invert analytically the above equation to find $r = r(R)$ if we have an explicit expression for $f = f(\theta)$ (considering also (278)).

From (277) we can calculate $\boldsymbol{\eta}$, and from the above expressions for J and $\mathbf{T}_{\rm D}$ we can calculate $\boldsymbol{\tau}_{\rm D} = J\mathbf{T}_{\rm D}$. Replacing them in (112) we obtain the

[41] In [60] several other boundary value problems have been analyzed, considering homogeneous deformations and non-homogeneous distributions of strains and stresses (which are similar to what has been presented in [36]). These problems are very similar to what has been shown in the previous Sections 9.1.1-9.2.5, and for the sake of brevity we do not show such results here.

[42] We change the notation Θ and θ that are commonly used for the angular position in the reference and current configurations, for Φ and ϕ, respectively, in order to avoid problems with the notation for the absolute temperature used in Section 6.3.

[43] Here we define $\boldsymbol{\tau}_{\rm D} = J\mathbf{T}_{\rm D}$.

relations[44]:

$$\ln\left(\frac{\mathrm{d}r}{\mathrm{d}R}\right) = \frac{\ln[f(\theta(r))]}{3} + \alpha_0 + \alpha_1 f(\theta(r))\sigma_{\mathrm{D}_r}(r) + \alpha_2[f(\theta(r))]^2[\sigma_{\mathrm{D}_r}(r)]^2, \quad (279)$$

$$\ln\left(\frac{r}{R}\right) = \frac{\ln[f(\theta(r))]}{3} + \alpha_0 + \alpha_1 f(\theta(r))\sigma_{\mathrm{D}_\phi}(r) + \alpha_2[f(\theta(r))]^2[\sigma_{\mathrm{D}_\phi}(r)]^2 \quad (280)$$

From $J = \frac{\mathrm{d}r}{\mathrm{d}R}\frac{r}{R}\lambda_z = f(\theta(r))$ considering (278) we can obtain $\frac{\mathrm{d}r}{\mathrm{d}R} = \frac{R(r)f(\theta(r))}{r\lambda_z}$ thus the left sides of (279) and (280) are known functions of r, considering that from (278) we also have $\theta = \theta(r)$. Therefore, from (279) and (280) we have two nonlinear algebraic equations that can be used to find $\sigma_{\mathrm{D}_r}(r)$ and $\sigma_{\mathrm{D}_\phi}(r)$. The final expressions for the stresses depend on the boundary conditions for the mechanical part of the problem, which for brevity are not discussed here.

Similar boundary value problems are studied as in [36] following the same method presented above.

9.5 Other problems

In this section we present results for other problems involving large elastic deformations, such as in the modelling of strings, rods and trusts elements, and we review some works on ellipticity, monotonicity and invertibility, considering some of these new classes of implicit constitutive relations.

9.5.1 A model for strings

In [180] there is a study on nonlinear elastic strings, in particular for semi-infinite strings. If s denotes the position along the string in the reference configuration, then $0 \leqslant s < \infty$. The vector $\mathbf{r} = \mathbf{r}(s,t)$ is the position of the string in the current configuration, from where it is defined $\nu = |\mathbf{r}_s|$ (that is the stretching strain for the string). The subscript s means the partial derivative in s. If the contact force in the string and the body forces are denoted $\mathbf{n}$ and $\mathbf{f}$, respectively, the equation of motion for the string (compare, for example, with $(4)_1$) is:

$$(\rho A)(s)\mathbf{r}_{tt} = \mathbf{n}_s + \mathbf{f},$$

where $(\rho A)(s)$ is the mass density per unit of reference length and the subscript t means derivative in time.

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[44] Recall that no all relations in (112) are independent since $\mathrm{tr}\,\mathbf{T}_\mathrm{D} = 0$ and $J = f(\theta)$.

The main constitutive assumption in [180] is (which is an implicit relation):

$$\mathbf{n} = N\frac{\mathbf{r}_s}{|\mathbf{r}_s|}, \quad \nu = \hat{\nu}(N, s),$$

where $N = |\mathbf{n}|$ is the norm of the contact force, and the function $\hat{\nu}$ is defined as

$$\hat{\nu}(N) = \begin{cases} \nu_1 & N \geqslant N_1 \\ \frac{N}{E} & 0 < N < N_1 \end{cases},$$

where ν_1 is a constant and $E = N_1/\nu_1$ is the elastic modulus for the string in the reference configuration. The above constitutive model gives a stretch-limiting behaviour for that elastic string.

One boundary value problem has been studied in that paper [180], wherein $\mathbf{f} = \mathbf{0}$, considering the boundary conditions $\mathbf{r}(0, t) = \mathbf{0}$ and $\mathbf{n}(\infty, t) = (\zeta t + \tau)\mathbf{e}_1$, where ζ and τ are constants. In [180] it is assumed that there exists a[45] $\sigma = \sigma(t)$ such that $N_1 < N < \infty$ if $0 \leqslant s < \sigma(t)$, while for $\sigma(t) < s < \infty$ it is assumed that $0 < N < N_1$ and $\sigma' \geqslant 0$ (where $(\ )'$ is the time derivative). Here $\sigma = \sigma(t)$ denotes a shock front. In [180] there is a study of existence and uniqueness of such shock solutions, and the dissipation of energy due to shock.

9.5.2 A model for rods

One work [173] has been published considering a model for rods using some constitutive relations similar to (5). Regarding the kinematic for the rod, in that paper [173] the position of the centerline of the rod in the current configuration is denoted $\mathbf{r} = \mathbf{r}(s)$, where $0 \leqslant s \leqslant L$ is a coordinate along the rod in the reference configuration, which is assumed to be straight and of length L. Three mutually unit orthogonal vectors $\mathbf{d}_i = \mathbf{d}_i(s)$ are defined, one of them $\mathbf{d}_3$ is tangent to $\mathbf{r}$, while $\mathbf{d}_1$, $\mathbf{d}_2$ are tangent to the cross section of the rod, thus $\mathbf{d}_3 = \mathbf{d}_1 \times \mathbf{d}_2$. In this section $\mathbf{u} = \mathbf{u}(s)$ and $\mathbf{v} = \mathbf{v}(s)$ are used to denote some types of strains. In particular $\mathbf{u}$ must not be confused with the displacement field (see $(2)_2$). The vector $\mathbf{u}$ is calculated as $\mathbf{u} = \frac{1}{2}\mathbf{d}_k \times \mathbf{d}_k'$, where $(\ )'$ represents the derivative in s. The components u_1 and u_2 of the above vector in the directions $\mathbf{d}_1$ and $\mathbf{d}_2$ are the flexural strains, while the component u_3 in the direction $\mathbf{d}_3$ is the torsional strain (twist of the rod).

[45] Here $\sigma(t)$ is just a function of time, and it should not be confused with the notation for the 1D stress used in previous sections.

The components of $\mathbf{v}$ are calculated indirectly from $\mathbf{r}'(s) = v_k(s)\mathbf{d}_k(s)$, where v_1 and v_2 are the components of the shear strain and v_3 is the dilatation strain (tension/compression). The restriction $v_3(s) > 0$ is imposed.

The balance laws for the rod are (quasi-static case):

$$\mathbf{n}'(s) + \mathbf{f}(s) = \mathbf{0}, \quad \mathbf{m}'(s) + \mathbf{r}'(s) \times \mathbf{n}(s) + \mathbf{l}(s) = \mathbf{0},$$

where $\mathbf{n}$ is the contact force, $\mathbf{f}$ is the external body force, $\mathbf{m}$ is the contact couple and $\mathbf{l}$ is the external body couple. The contact force and contact couple are expressed as $\mathbf{n} = n_k(s)\mathbf{d}_k(s)$ and $\mathbf{m} = m_k(s)\mathbf{d}_k(s)$, respectively, where n_1, n_2 are the shear forces, n_3 is the tension in the rod, m_1, m_3 are the bending couples and m_3 is the twisting couple.

In [173] a strain limiting constitutive relation has been proposed for $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{m}$, $\mathbf{n}$. The constitutive relation is proposed taking as basis the relation $\mathbf{e} = \left(a^{-p} + |b\mathbf{T}|^p\right)^{-1/p}\mathbf{T}$, where $a, b, p > 0$ are constants, and $\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})$ is the Almansi-Hamel strain tensor. The constitutive relation is:

$$\begin{aligned}
u_\mu &= \left[\gamma^p + Q^*(\mathbf{m},\mathbf{n})^{p/2}\right]^{-1/p} \frac{1}{\alpha^2} m_\mu, \\
u_3 &= \left[\gamma^p + Q^*(\mathbf{m},\mathbf{n})^{p/2}\right]^{-1/p} \frac{1}{(\beta^2\eta^2 - \iota^2)} (\eta^2 m_3 - \iota n_3), \\
v_\mu &= \left[\gamma^p + Q^*(\mathbf{m},\mathbf{n})^{p/2}\right]^{-1/p} \frac{1}{\zeta^2} n_\mu, \\
v_3 - 1 &= \left[\gamma^p + Q^*(\mathbf{m},\mathbf{n})^{p/2}\right]^{-1/p} \frac{1}{(\beta^2\eta^2 - \iota^2)} (-\iota m_3 + \beta^2 n_3),
\end{aligned}$$

where $\mu = 1, 2$ and $Q^*(\mathbf{m},\mathbf{n}) = \frac{1}{\alpha^2}(m_1^2 + m_2^2) + \frac{1}{\zeta^2}(v_1^2 + v_2^2) + \frac{\eta^2}{(\beta^2\eta^2-\iota^2)} m_3^2 + \frac{\beta^2}{(\beta^2\eta^2-\iota^2)} n_3^2 - \frac{2\iota}{(\beta^2\eta^2-\iota^2)} m_3 n_3$, and where α, β, η, ι and ζ are material constants.

In [173] an analysis of some boundary value problems is provided, in particular for the case of equilibrium states where $\mathbf{f} = \mathbf{0}$ and $\mathbf{l} = \mathbf{0}$. For the sake of brevity that analysis is not presented here.

9.5.3 Modelling of axial elements (trusts)

In [214] a constitutive relation is proposed for the modelling of axial elements (trusts) for the case of large elastic deformations (see Section 8.3.6). A numerical method is proposed for the solution of the equations, which is applied for some examples involving structures.

The model consists of nodes, for each n^{th} node a balance of the forces 'meeting' at that node is considered. The force in each axial element if denoted F_m, and the reaction forces in the support are denoted $\mathsf{F}^{\text{react}}$. The displacement of each node is u_n and the axial stretch of the jth member connecting the nodes n and m is denoted λ_j.

The sum of all forces for all nodes equal zero (quasi-static case) and it becomes:

$$[\mathsf{C}]\{\mathsf{F}_m\} + \left\{ \begin{array}{c} \mathsf{F}_n^{\text{known}} \\ \mathsf{F}_n^{\text{react}} \end{array} \right\} = \{\mathsf{0}\}, \tag{281}$$

where $\mathsf{F}_n^{\text{known}}$ represents the known external applied forces.

The implicit constitutive relation (74) $\boldsymbol{\mathfrak{G}}(\rho, \mathbf{T}, \mathbf{B}) = \mathbf{0}$ is replaced by the following constitutive relation for trust elements:

$$\mathfrak{G}_j\left(\frac{\mathsf{F}_j}{A_j}, \lambda_j\right) = 0, \tag{282}$$

where A_j is the cross section area of the jth member before deformation, and $\mathfrak{G}_j$ is the constitutive relation for the jth member.

Numerically, boundary value problems are solved minimizing the following function

$$\delta = \sqrt{\sum_{j=1}^{m_t} \left[\mathfrak{G}_j\left(\frac{\mathsf{F}_j}{A_j}, \lambda_j\right)\right]^2}, \tag{283}$$

where $\delta = 0$ means the implicit constitutive relation (282) is satisfied exactly. In (283) m_t represents the total number of elements or members in a structure.

In [214] the following particular expression for (282) is considered that is valid for rubber[46]:

$$\frac{\sigma}{2m}\left[1 - \frac{0.9\sigma}{(2.4 + \sigma)}\right] - \ln \lambda_j = 0, \tag{284}$$

where $\sigma = \frac{2}{3}\lambda_j \frac{\mathsf{F}_j}{A_j}$ and m is a material parameter.

The above constitutive relation (284) along with (281) and (283) are used to study some problems, such as a triangular structure made of 3 elements (statically determined) under a vertical load, and a problem with a structure made of 6 elements that is statically indetermined.

[46]The authors also consider a linearized constitutive relation and a model based on the constitutive equation by Mooney for rubber.

9.5.4 Some works on ellipticity, monotonicity and invertibility

We finish this section mentioning two works by Mai et al [127, 128], where the strong ellipticity condition, monotonicity and invertibility are studied for (29) and for some boundary value problems. In [127] the following particular expression for (29) has been considered

$$\mathbf{E} = \phi_0(\mathrm{tr}\,\mathbf{S})\mathbf{I} + \phi_1(|\mathbf{S}|)\mathbf{S}, \tag{285}$$

where ϕ_0 and ϕ_1 are functions. In that work there is study of strong ellipticity for the boundary value problem and invertibility for (285).

In [128] the subclass of (29) that is studied is

$$\mathbf{E} = \phi_1(|\mathbf{S}|)\mathbf{S} + \phi(|\mathbf{S}|^2)\mathbf{S}^2,$$

where ϕ_1 and ϕ_1 are functions. In that work the authors study monotonicity and invertibility for some boundary value problems as well.

10 Implicit constitutive relations for inelastic deformations and other types of implicit constitutive relations

In all the previous sections we have only considered elastic deformations, i.e., problems without dissipation of mechanical work into heat. However, the modelling of inelastic deformations is also important from the practical point of view, and in different works that are reviewed in this section, we show that implicit constitutive relations can also be useful and interesting for such problems.

10.1 Some implicit constitutive relations for the modelling of inelastic deformations

In this section we review different implicit constitutive relations for the modelling of inelastic bodies, which are not visco-elastic.

10.1.1 An implicit constitutive relation by Rajagopal & Srinivasa

In [165] an implicit constitutive relation for 1D inelastic deformations has been proposed. If ε, σ and θ represent the 1D strain (small strains), the 1D stress and the absolute temperature, the implicit relation for elastic bodies (compare with (5)) is:

$$\mathfrak{H}(\varepsilon, \sigma, \theta) = 0.$$

Taking the time derivative of the above we obtain

$$A\dot{\varepsilon} + B\dot{\sigma} + \alpha\dot{\theta} = 0,$$

where $A = \frac{\partial \mathfrak{H}}{\partial \varepsilon}$, $B = \frac{\partial \mathfrak{H}}{\partial \sigma}$ and $\alpha = \frac{\partial \mathfrak{H}}{\partial \theta}$. Assuming as a starting point that the above relation is valid for elastic deformations, in [165] the following implicit relation for inelastic deformations has been proposed:

$$\mathbf{A}_o \cdot \dot{\mathsf{X}} = \xi(\mathsf{X}, \dot{\mathsf{X}}), \tag{286}$$

where X is a vector with components $\mathsf{X} = \begin{pmatrix} \varepsilon \\ \sigma \\ \theta \end{pmatrix}$, $\mathbf{A}_o$ is a second order tensor defined as $\mathbf{A}_o = \frac{\partial \mathfrak{H}}{\partial \mathsf{X}}$, and ξ is the rate of dissipation, which in [165] is assumed to be given as

$$\xi = \xi(\mathsf{X}, \dot{\mathsf{X}}) = \omega(\mathsf{X}) H(\boldsymbol{\mu} \cdot \dot{\mathsf{X}}) \boldsymbol{\mu} \cdot \dot{\mathsf{X}},$$

where $\boldsymbol{\mu}$ is a vector field, and H is the Heaviside step function. Considering the above for (286) we have

$$\mathbf{A}_o \cdot \dot{\mathsf{X}} = \begin{cases} \omega(\mathsf{X}) \boldsymbol{\mu} \cdot \dot{\mathsf{X}} & \text{if} \quad \boldsymbol{\mu} \cdot \dot{\mathsf{X}} > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{287}$$

The above constitutive relation has been applied for the modelling of perfectly plastic deformations, wherein (287) becomes $\dot{\sigma} - E\dot{\varepsilon} = -H(\sigma\dot{\varepsilon})H(|\sigma| - \sigma_Y)\dot{\varepsilon}$, where E and σ_Y are the ground Young's modulus and yield stress for the solid. For loss of cohesion (see Section 4.3 in [165]) the authors propose $\varepsilon\dot{\sigma} - \sigma\dot{\varepsilon} = H(\sigma + k\varepsilon - \sigma_Y)[(k\dot{\varepsilon} - \sigma)\varepsilon]H(\sigma\dot{\varepsilon})$.

The above theory, which is valid for 1D deformations and stresses, is extended for 3D problems and finite deformations in [168], where the following generalization of (286) is proposed:

$$\mathfrak{F}(\mathbf{T}, \dot{\mathbf{T}}, \mathbf{F}, \dot{\mathbf{F}}) = \mathbf{0}.$$

The above relation must satisfy the principle of frame-indifferent, and that is the case if

$$\mathfrak{F}(\mathbf{R}^{\mathrm{T}}\mathbf{T}\mathbf{R}, \dot{\overline{\mathbf{R}^{\mathrm{T}}\mathbf{T}\mathbf{R}}}, \mathbf{U}, \dot{\mathbf{U}}) = \mathbf{0}. \tag{288}$$

In [168] instead considering the polar decomposition, the authors use the $\mathbf{QR}$ decomposition, where $\mathbf{F} = \tilde{\mathbf{R}}\tilde{\mathbf{F}}$, $\tilde{\mathbf{R}}$ is the rotation and $\tilde{\mathbf{F}}$ is an upper triangular matrix $\tilde{\mathsf{F}} = \begin{pmatrix} \tilde{F}_{11} & \tilde{F}_{12} & \tilde{F}_{13} \\ 0 & \tilde{F}_{22} & \tilde{F}_{23} \\ 0 & 0 & \tilde{F}_{33} \end{pmatrix}$. The components of $\tilde{\mathbf{F}}$ can be directly found from the components of $\mathbf{C}$. The tensor $\tilde{\mathbf{F}}$ is frame invariant (since it depends on $\mathbf{C}$) and for the case of small gradient of the displacement field we have that $\boldsymbol{\varepsilon} \approx \tilde{\mathbf{F}} + \mathbf{I}$. Then (288) is replaced by

$$\mathfrak{F}(\tilde{\mathbf{T}}, \tilde{\mathbf{F}}, \dot{\tilde{\mathbf{T}}}, \dot{\tilde{\mathbf{F}}}) = \mathbf{0},$$

where $\tilde{\mathbf{T}} = \tilde{\mathbf{R}}^{\mathrm{T}}\tilde{\mathbf{T}}\tilde{\mathbf{R}}$ and $\dot{\tilde{\mathbf{T}}}$ is the Green-Naghdi stress rate.

Defining $\tilde{\mathbf{L}} = \dot{\tilde{\mathbf{F}}}\tilde{\mathbf{F}}^{-1}$ and the vectors $\mathsf{X} = \begin{pmatrix} \tilde{\mathbf{T}} \\ \tilde{\mathbf{F}} \end{pmatrix}$ and $\mathsf{V} = \begin{pmatrix} \dot{\tilde{\mathbf{T}}} \\ \tilde{\mathbf{L}} \end{pmatrix}$ the implicit constitutive relation is rewritten as:

$$\mathfrak{F}(\mathsf{X}, \mathsf{V}) = \mathbf{0}.$$

It is assumed that $\mathfrak{F}$ is rate-independent, then $\mathfrak{F}(\lambda\mathsf{X}, \lambda\mathsf{V}) = \lambda\mathfrak{F}(\mathsf{X}, \mathsf{V})\ \forall\lambda > 0$.

In [168] the following special case has been studied

$$\mathfrak{F}(\mathsf{X}, \mathsf{V}) = \begin{cases} \mathscr{A}[\mathsf{V}] = \mathbf{0} & \text{if} \quad \mathbf{N}\cdot\mathsf{V} > 0 \\ \mathscr{B}[\mathsf{V}] = \mathbf{0} & \text{otherwise} \end{cases}, \tag{289}$$

where $\mathscr{C}[\mathbf{H}]$ is equivalent to $\mathscr{C}_{ijkl}H_{kl}$ in index notation and Cartesian coordinates. In the above expression $\mathscr{A} = \mathscr{A}(\mathsf{X})$ and $\mathscr{B} = \mathscr{B}(\mathsf{X})$. The vector $\mathbf{N} = \mathbf{N}(\mathsf{X})$ is called the local 'loading' direction. In the case of rate-independent hysteretic material the following particular case of (289) is proposed:

$$\mathscr{A}[\mathsf{V}] - <\lambda> \boldsymbol{\mu} = \mathbf{0}, \tag{290}$$

where $\lambda = \mathbf{N}\cdot\mathsf{V}$ and $<\lambda>= \frac{1}{2}(\lambda + |\lambda|)$. Then, for the particular case of rate-independent plasticity it is assumed that $\mathscr{A} = \begin{pmatrix} -\mathscr{I} \\ \mathscr{C} \end{pmatrix}$, where $\mathscr{I}$ is a 6×6 identity matrix and $\mathscr{C} = \mathscr{C}(\mathsf{X})$ is a 6×6 matrix with the elastic tagent moduli. As for $\mathbf{N}$ it is assumed $\mathbf{N} = \{\mathbf{0}, \mathbf{n}\}$, where in this case $\mathbf{0}$ is a 6×1 vector with components filled with 0, and $\mathbf{n}$ is a 6×1 vector defined as $\mathbf{n} = \frac{\mathscr{C}[\partial\phi/\partial\tilde{\mathbf{T}}]}{\left|\frac{\partial\phi}{\partial\tilde{\mathbf{T}}}\mathscr{C}\frac{\partial\phi}{\partial\tilde{\mathbf{T}}}\right|}$. On the other hand, it is assumed that $\boldsymbol{\mu}$ is a 6×1 vector defined as $\boldsymbol{\mu} = H[\phi]\mathbf{n}$. Using the above expressions in (290) we obtain

$$\dot{\tilde{\mathbf{T}}} = \begin{cases} \mathscr{C}[\tilde{\mathbf{L}}] & \text{if} \quad \phi(\tilde{\mathbf{T}}) < 0 \quad \text{or if} \quad \phi = 0, \quad \text{and} \quad \mathbf{N}\cdot\mathbf{V} < 0 \\ (\mathscr{C} - H(\phi)\mathbf{n}\otimes\mathbf{N})[\tilde{\mathbf{L}}] & \text{otherwise} \end{cases},$$

where $H(\phi) = 0$ if $\phi < 0$ and $H(\phi) = 1$ if $\phi \geqslant 0$. The Heaviside step function H can be replaced by the function $f = f(\phi) = \frac{1}{2}(1 + \tanh(a\phi))$ for a smooth transition between elastic and plastic behaviour.

In [168] and implicit relation for Mullins effect is also proposed but for brevity is not shown here.

10.1.2 An implicit constitutive relation for inelastic deformations based on the use of a 'replacement' stress

In this section we show the main elements of an implicit constitutive relation presented in [129, 182] for inelastic deformations, whose main concept is the use of a 'replacement stress'. The implicit relation proposed in that paper is of the form

$$\mathfrak{F}(\mathbf{T}, \mathbf{T}^{\mathrm{R}}, \boldsymbol{\eta}, \rho, \theta) = \mathbf{0},$$

where we recall that $\boldsymbol{\eta}$ is the Hencky strain tensor and $\mathbf{T}^{\mathrm{R}}$ is called the replacement stress. Let us explain briefly the meaning of the above stress. If such a body deforms from $\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})$ to $\boldsymbol{\kappa}_{\mathrm{t}}(\mathscr{B})$, and if there is dissipation of energy, then when all external loads disappear due to the inelastic deformations caused by dissipation of energy that configuration is not $\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})$ but something else, say $\boldsymbol{\kappa}_{\mathrm{d}}(\mathscr{B})$. The replacement stresses $\mathbf{T}^{\mathrm{R}}$ are the stresses necessary for $\boldsymbol{\kappa}_{\mathrm{d}}(\mathscr{B})$ to become $\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})$. Such stresses are defined in the reference configuration, thus the following equations must be satisfied

$$\mathfrak{F}(\mathbf{T}^{\mathrm{R}}, \mathbf{T}^{\mathrm{R}}, \mathbf{0}, \rho_{\mathrm{r}}, \theta_{\mathrm{r}}) = \mathbf{0}, \quad \mathrm{Div}\,\mathbf{T}^{\mathrm{R}} + \rho_{\mathrm{r}}\mathbf{b} = \mathbf{0}, \tag{291}$$

where we notice that for that configuration $\mathbf{T}$ must be equal to $\mathbf{T}^{\mathrm{R}}$, and θ_{r} is the temperature in the reference configuration. We recall that the Cauchy stress $\mathbf{T}$ must satisfy $(4)_1$ in the current configuration.

Taking the total time derivative of (291) we obtain $\frac{\partial\mathfrak{F}}{\partial\mathbf{T}}\cdot\dot{\mathbf{T}} + \frac{\partial\mathfrak{F}}{\partial\mathbf{T}^{\mathrm{R}}}\cdot\dot{\mathbf{T}}^{\mathrm{R}} + \frac{\partial\mathfrak{F}}{\partial\boldsymbol{\eta}}\cdot\dot{\boldsymbol{\eta}} + \frac{\partial\mathfrak{F}}{\partial\rho}\dot{\rho} + \frac{\partial\mathfrak{F}}{\partial\theta}\cdot\dot{\theta} = \mathbf{0}$ from where if $\left(\frac{\partial\mathfrak{F}}{\partial\mathbf{T}}\right)^{-1}$ exists we can calculate

$$\dot{\mathbf{T}} = \mathscr{A}_1\cdot\dot{\boldsymbol{\eta}} + \mathscr{A}_2\cdot\dot{\mathbf{T}}^{\mathrm{R}} + \mathscr{A}_3\dot{\theta},$$

where $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}_3$ are fourth order tensors and a second order tensor, respectively, defined in terms of $\left(\frac{\partial\mathfrak{F}}{\partial\mathbf{T}}\right)^{-1}$, $\frac{\partial\mathfrak{F}}{\partial\mathbf{T}^{\mathrm{R}}}$, $\frac{\partial\mathfrak{F}}{\partial\boldsymbol{\eta}}$ and $\frac{\partial\mathfrak{F}}{\partial\theta}$, which for the sake of brevity are not shown here. In the above the following form of the balance of mass has been used: $\dot{\rho} = -\rho \mathbf{I} \cdot \dot{\boldsymbol{\eta}}$.

The second law of thermodynamics written in terms of $\mathbf{T}$ and $\boldsymbol{\eta}$ is

$$\rho\theta\dot{s} - \rho\dot{u} + \mathbf{T}\cdot\dot{\boldsymbol{\eta}} + (\mathbf{T}\boldsymbol{\eta} - \boldsymbol{\eta}\mathbf{T})\cdot(\dot{\mathbf{Q}}^{\mathrm{T}}\mathbf{Q}) + (\mathbf{V}\mathbf{T}\mathbf{V}^{-1})\cdot(\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}) = \xi \geqslant 0, \quad (292)$$

where ξ is the rate of dissipation and $\mathbf{Q}$ is a second order tensor, whose columns in matrix form are formed with the components of the eigenvalues of $\mathbf{B} = \mathbf{V}^2$. In [182] the Gibbs potential $\mathcal{G}$ is used[47], where $\mathcal{G} = U - s\theta - (\mathbf{T}\cdot\boldsymbol{\eta})/\rho$, as well as this, it is assumed that $\mathcal{G} = \hat{\mathcal{G}}(\mathbf{T}, \mathbf{T}^{\mathrm{R}}, \boldsymbol{\eta}, \rho, \theta)$. Using the above Gibbs potential in (292) along with the balance of mass (which for brevity is not shown here), it is obtained (compare with $(109)_{1,2}$):

$$\boldsymbol{\eta} = -\rho\frac{\partial\mathcal{G}}{\partial\mathbf{T}}, \quad s = -\frac{\partial\mathcal{G}}{\partial\theta}, \quad (293)$$

and (from now assuming isothermal processes)

$$\dot{\boldsymbol{\eta}}\cdot\mathbf{M}_1 - \dot{\mathbf{T}}^{\mathrm{R}}\cdot\mathbf{M}_2 + (\mathbf{T}\boldsymbol{\eta} - \boldsymbol{\eta}\mathbf{T})\cdot(\dot{\mathbf{Q}}^{\mathrm{T}}\mathbf{Q}) + (\mathbf{V}\mathbf{T}\mathbf{V}^{-1})\cdot(\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}) = \xi \geqslant 0, \quad (294)$$

where the second order tensors $\mathbf{M}_1$, $\mathbf{M}_2$ are defined in terms of the derivatives of $\mathcal{G}$ in ρ, $\boldsymbol{\eta}$ and $\mathbf{T}^{\mathrm{R}}$, and the term $(\mathbf{T}\cdot\boldsymbol{\eta})\mathbf{I}$, and which for the sake of brevity are not shown here.

In [129] in order to simplify the problem the authors consider the special case, wherein it is assumed that the principal directions of $\boldsymbol{\eta}$ are the same as of $\mathbf{T}$, thus $(\mathbf{T}\boldsymbol{\eta} - \boldsymbol{\eta}\mathbf{T})\cdot(\dot{\mathbf{Q}}^{\mathrm{T}}\mathbf{Q}) + (\mathbf{V}\mathbf{T}\mathbf{V}^{-1})\cdot(\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}) = 0$ and (294) becomes

$$\dot{\boldsymbol{\eta}}\cdot\mathbf{M}_1 - \dot{\mathbf{T}}^{\mathrm{R}}\cdot\mathbf{M}_2 - \xi = 0. \quad (295)$$

Regarding ξ for isothermal processes it is assumed that $\xi = \xi(\mathbf{T}, \mathbf{T}^{\mathrm{R}}, \dot{\boldsymbol{\eta}})$. In [129, 182] it is assumed that (295) is satisfied if $\mathbf{M}_1 = \mathbf{0}$ and $\dot{\mathbf{T}}^{\mathrm{R}}\cdot\mathbf{M}_2 + \xi = 0$. The entropy production is maximized such that $\dot{\mathbf{T}}^{\mathrm{R}}\cdot\mathbf{M}_2 + \xi = 0$ holds. For the particular case of aluminium alloy, in [129] from such maximization for ξ, it is found an evolution equation for $\mathbf{T}^{\mathrm{R}}$ of the form $\dot{\mathbf{T}}^{\mathrm{R}} = \mathfrak{g}(\mathbf{T}^{\mathrm{R}}, \boldsymbol{\eta})$. For such a material (aluminium alloy) the following expression is proposed for $\xi = \bar{\xi}\sqrt{\dot{\boldsymbol{\eta}}\cdot\dot{\boldsymbol{\eta}}}$, where $\bar{\xi} = s_1\{\tanh[s_2(\sigma_{\mathrm{v}} - \sigma')] + \tanh[-s_2(\sigma_{\mathrm{v}} + \sigma')] + 2\tanh(s_2\sigma')\}$ and $\sigma_{\mathrm{v}} = \sqrt{K_2 - \frac{K_1^2}{6}}$, $\sigma' = s_3 + s_4\frac{K_1}{\sqrt{2K_2}}\frac{K_5}{K_4}$, $K_1 = \operatorname{tr}\mathbf{T}$, $K_2 =$

[47] The Gibbs potential used here is defined as minus the Gibbs potential used, for example, in Section 6.3

$\frac{1}{2}\mathrm{tr}(\mathbf{T}^2)$, $K_4 = \mathrm{tr}\,\mathbf{T}^{\mathrm{R}}$, $K_5 = \frac{1}{2}\mathrm{tr}[(\mathbf{T}^{\mathrm{R}})^2]$ and σ_{v} is the Von Misses stress. To make further progress for $\mathcal{G} = \hat{\mathcal{G}}(\mathbf{T}, \mathbf{T}^{\mathrm{R}}, \boldsymbol{\eta}, \rho)$ it is assumed that $\mathcal{G} = \mathcal{G}(K_1, K_2, K_4, K_5, K_7, J_1)$, where $K_7 = \mathrm{tr}(\mathbf{T}\mathbf{T}^{\mathrm{R}})$ and $J_1 = \mathrm{tr}\,\boldsymbol{\eta}$. Using this in (293) and considering (295) and explicit expression for $\mathcal{G}$ is proposed, from where a constitutive equation for $\boldsymbol{\eta}$ is found, which we do not show explicitly here. All these elements and relations are used in [129] for the fitting of experimental data for aluminium alloy obtained from a cyclic uniaxial test protocol. In [182] the above implicit relation has been used for the fitting of experimental data for some metals, such as steel, aluminium and titanium, to study the plasticity with cycles considering 1D deformations and stresses. For the sake of brevity here we do not show the numerical values for the material constants.

10.1.3 An inelastic constitutive relation by Cichra and Prusa

From [63] an implicit constitutive relation for the inelastic response of solids has been proposed. That work uses the definitions $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, $\mathbf{D} = \frac{1}{2}(\mathbf{L}+\mathbf{L}^{\mathrm{T}})$ and $\mathbf{W} = \frac{1}{2}(\mathbf{L}-\mathbf{L}^{\mathrm{T}})$ and the objectives derivatives for a second order tensor $\mathbf{A}$:

$$\overset{\triangledown}{\overline{\mathbf{A}}} = \dot{\mathbf{A}} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^{\mathrm{T}}, \quad \overset{\circ}{\overline{\mathbf{A}}} = \dot{\mathbf{A}} - \mathbf{W}\mathbf{A} + \mathbf{A}\mathbf{W}^{\mathrm{T}}. \tag{296}$$

In [63] it has been noticed that for the left Cauchy-Green tensor $\overset{\triangledown}{\overline{\mathbf{B}}} = \mathbf{0}$ and $\overset{\circ}{\overline{\mathbf{B}}} = \mathbf{D}\mathbf{B} + \mathbf{B}\mathbf{D}$, then a Helmholtz free energy ψ is assumed to be given as $\psi = \psi(\theta, \mathbf{B}_{\mathrm{e}})$, where $\mathbf{B}_{\mathrm{e}}$ is found from the 'evolution' equation

$$\overset{\circ}{\overline{\mathbf{B}}}_{\mathrm{e}} = \mathbf{D}\mathbf{B}_{\mathrm{e}} + \mathbf{B}_{\mathrm{e}}\mathbf{D} + \mathbf{M}, \tag{297}$$

where $\mathbf{M}$ for inelastic deformations is different from zero, and for elastic deformations $\mathbf{M} = \mathbf{0}$, thus $\mathbf{B}_{\mathrm{e}} = \mathbf{B}$.

The second order symmetric tensor $\mathbf{M}$ is found in the following way. From the first law of thermodynamic considering the above expression for ψ we have

$$\rho\theta\dot{s} = \mathrm{tr}(\mathbf{T}\mathbf{D}) - \rho\frac{\partial\psi}{\partial\mathbf{B}_{\mathrm{e}}} \cdot \dot{\mathbf{B}}_{\mathrm{e}} - \mathrm{div}\,\mathbf{Q}, \tag{298}$$

where in this case the authors of [63] use a definition for the heat flux $\mathbf{Q}$ that is the negative of the definition considered in previous sections. From $(296)_2$ we have $\dot{\mathbf{B}}_{\mathrm{e}} = \overset{\circ}{\overline{\mathbf{B}}}_{\mathrm{e}} + \mathbf{W}\mathbf{B}_{\mathrm{e}} + \mathbf{B}_{\mathrm{e}}\mathbf{W}$, and since $\frac{\partial\psi}{\partial\mathbf{B}_{\mathrm{e}}}$ is a symmetric second order tensor, it is possible to show that $\rho\frac{\partial\psi}{\partial\mathbf{B}_\mathrm{e}}\cdot\dot{\mathbf{B}}_\mathrm{e} = \rho\frac{\partial\psi}{\partial\mathbf{B}_\mathrm{e}}\cdot \overset{\circ}{\overline{\mathbf{B}}}_\mathrm{e}$. On the other hand the symmetric tensor $\mathbf{D}$ can be found from (297) as $\mathbf{D} = \int_{\varsigma=0}^{\infty} e^{-\varsigma\mathbf{B}_\mathrm{e}}\left(\overset{\circ}{\overline{\mathbf{B}}}_\mathrm{e} - \mathbf{M}\right)e^{-\varsigma\mathbf{B}_\mathrm{e}}\,\mathrm{d}\varsigma$. Replacing all these elements in (298) we get

$$\begin{aligned}\rho\theta\dot{s} = {} & \left[\int_{\varsigma=0}^{\infty} e^{-\varsigma\mathbf{B}_\mathrm{e}}\left(\mathbf{T} - 2\rho\mathbf{B}_\mathrm{e}\frac{\partial\psi}{\partial\mathbf{B}_\mathrm{e}}\right)e^{-\varsigma\mathbf{B}_\mathrm{e}}\,\mathrm{d}\varsigma\right]\cdot\overset{\circ}{\overline{\mathbf{B}}}_\mathrm{e} \\ & -\mathbf{T}\cdot\left(\int_{\varsigma=0}^{\infty} e^{-\varsigma\mathbf{B}_\mathrm{e}}\mathbf{M}e^{-\varsigma\mathbf{B}_\mathrm{e}}\,\mathrm{d}\varsigma\right) - \operatorname{div}\mathbf{Q}. \end{aligned} \tag{299}$$

In the case $\mathbf{M} = \mathbf{0}$ the above equation (299) is satisfied if $\mathbf{T} = 2\rho\mathbf{B}_\mathrm{e}\frac{\partial\psi}{\partial\mathbf{B}_\mathrm{e}}$, where $\mathbf{B}_\mathrm{e} = \mathbf{B}$ that is the classical constitutive equation for Green elastic bodies. In (299) the term $-\mathbf{T}\cdot\left(\int_{\varsigma=0}^{\infty} e^{-\varsigma\mathbf{B}_\mathrm{e}}\mathbf{M}e^{-\varsigma\mathbf{B}_\mathrm{e}}\,\mathrm{d}\varsigma\right)$ produces entropy along with $-\operatorname{div}\mathbf{Q}$.

In [63] it is assumed that $\mathbf{M}$ is a function of $\mathbf{T}$, $\mathbf{D}$ and $\mathbf{B}_\mathrm{e}$, and must be prescribed such that the second law of thermodynamics is always satisfied. For example, for the case of elastic-perfectly plastic solids the expression proposed for $\mathbf{M}$ is $\mathbf{M} = -H(\mathbf{T}\cdot\mathbf{D})H(|\mathbf{T}| - \sigma_Y)(\mathbf{D}\mathbf{B}_\mathrm{e} + \mathbf{B}_\mathrm{e}\mathbf{D})$, where H is the Heaviside step function and σ_Y is the yielding stress for the body. Using the above expression for $\mathbf{M}$ in (297) we obtain the evolution equation for $\mathbf{B}_\mathrm{e}$:

$$\overset{\circ}{\overline{\mathbf{B}}}_\mathrm{e} = [1 - H(\mathbf{T}\cdot\mathbf{D})H(|\mathbf{T}| - \sigma_Y)](\mathbf{D}\mathbf{B}_\mathrm{e} + \mathbf{B}_\mathrm{e}\mathbf{D}).$$

The above is an implicit rate-type constitutive relation.

10.1.4 An implicit constitutive relation for stress softening

Using as a starting point (7) for the case of inelastic deformations, in [50] the following implicit relation has been proposed:

$$\mathscr{A}\cdot\dot{\mathbf{S}} + \mathscr{B}\cdot\dot{\mathbf{E}} + \mathscr{P}\dot{\theta} = \boldsymbol{\zeta},$$

where $\mathscr{A} = \mathscr{A}(\mathbf{S},\mathbf{E},\theta)$, $\mathscr{B} = \mathscr{B}(\mathbf{S},\mathbf{E},\theta)$ are fourth order tensors, $\mathscr{P} = \mathscr{P}(\mathbf{S},\mathbf{E},\theta)$ is a second order tensor, and $\boldsymbol{\zeta} = \boldsymbol{\zeta}(\mathbf{E},\dot{\mathbf{E}},\mathbf{S},\dot{\mathbf{S}},\theta,\dot{\theta})$ is a vector field. In [50] the authors assumed isothermal processes.

From the second law of thermodynamics we have

$$-\dot{\psi} + \frac{1}{\rho_\mathrm{r}}\operatorname{tr}(\mathbf{S}\dot{\mathbf{E}}) \geqslant 0. \tag{300}$$

Assuming that $\psi = \psi(\mathbf{S}, \mathbf{E})$ from (300) we obtain $-\frac{\partial \psi}{\partial \mathbf{S}} \cdot \dot{\mathbf{S}} - \frac{\partial \psi}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} + \mathrm{tr}\,(\mathbf{S}\dot{\mathbf{E}}) = \xi \geqslant 0$, where ξ is the rate of entropy production and ρ_{r} has been absorbed in the definition of ψ. Following the method presented in Section 3.2, taking the derivative of (300) in $\mathbf{S}$ we obtain

$$\frac{\partial^2 \psi}{\partial \mathbf{S} \partial \mathbf{S}} \cdot \dot{\mathbf{S}} + \left[\frac{1}{2} \left(\frac{\partial^2 \psi}{\partial \mathbf{E} \partial \mathbf{S}} + \frac{\partial^2 \psi}{\partial \mathbf{S} \partial \mathbf{E}} \right) - \boldsymbol{\mathscr{I}} \right] \cdot \dot{\mathbf{E}} = -\frac{\partial \xi}{\partial \mathbf{S}}, \tag{301}$$

where we identify $\boldsymbol{\mathscr{A}} = \frac{\partial^2 \psi}{\partial \mathbf{S} \partial \mathbf{S}}$, $\boldsymbol{\mathscr{B}} = \frac{1}{2} \left(\frac{\partial^2 \psi}{\partial \mathbf{E} \partial \mathbf{S}} + \frac{\partial^2 \psi}{\partial \mathbf{S} \partial \mathbf{E}} \right) - \boldsymbol{\mathscr{I}}$ and $\boldsymbol{\zeta} = -\frac{\partial \xi}{\partial \mathbf{S}}$. In [50] the following special case has been proposed as a starting point for the analysis of stress softening: $\psi(\mathbf{S}, \mathbf{E}) = \varphi(\mathbf{S}) + \mathrm{tr}\,[\mathbf{S} \boldsymbol{\varpi}(\mathbf{E})]$, where $\boldsymbol{\varpi} = \boldsymbol{\varpi}(\mathbf{E})$ is a second order tensor. Replacing this in (301) we get $\frac{\partial^2 \varphi}{\partial \mathbf{S} \partial \mathbf{S}} \cdot \dot{\mathbf{S}} + \left(\frac{\partial \boldsymbol{\varpi}}{\partial \mathbf{E}} - \boldsymbol{\mathscr{I}} \right) \cdot \dot{\mathbf{E}} = -\frac{\partial \xi}{\partial \mathbf{S}}$, and now defining $\phi = \phi(\mathbf{E})$ such that $\frac{\partial^2 \phi}{\partial \mathbf{E} \partial \mathbf{E}} = \frac{\partial \boldsymbol{\varpi}}{\partial \mathbf{E}} - \boldsymbol{\mathscr{I}}$ we obtain $\overline{\frac{\partial \varphi}{\partial \mathbf{S}}}^{\boldsymbol{\cdot}} + \overline{\frac{\partial \phi}{\partial \mathbf{S}}}^{\boldsymbol{\cdot}} = -\frac{\partial \xi}{\partial \mathbf{S}}$. Let us assume that φ, ϕ and ξ are given implicity by $\frac{\partial \varphi}{\partial \mathbf{S}} = \ln(\boldsymbol{\mathscr{A}}_o \cdot \mathbf{S})$, $\frac{\partial \phi}{\partial \mathbf{E}} = \ln \mathbf{E}$ and

$$-\frac{\partial \xi}{\partial \mathbf{S}} = \begin{cases} \overline{\ln(\boldsymbol{\mathscr{A}}_o \cdot \mathbf{S})}^{\boldsymbol{\cdot}} - \overline{\ln(\mathbf{E}_o + \boldsymbol{\mathscr{B}}_o \cdot \mathbf{S})}^{\boldsymbol{\cdot}} H(\mathrm{tr}\,(\mathbf{E}\dot{\mathbf{S}})) & \text{if } \tilde{\sigma}_1 \geqslant \sigma^*_{Y_T} \text{ or } \tilde{\sigma}_3 \leqslant -\sigma^*_{Y_C}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\boldsymbol{\mathscr{A}}_o$, $\boldsymbol{\mathscr{B}}_o$ are fourth order tensors with constant components, $\mathbf{E}_o$ is a symmetric second order tensor with constants components, $\sigma^*_{Y_T}$, $\sigma^*_{Y_C}$ are material positive constants, H is the Heaviside step function, and $\tilde{\sigma}_1 \geqslant \tilde{\sigma}_2 \geqslant \tilde{\sigma}_3$ are the eigenvalues of $\tilde{\mathbf{S}} = \boldsymbol{\mathscr{C}} \cdot \mathbf{S} + \boldsymbol{\mathscr{D}} \cdot \mathbf{E}$, where $\boldsymbol{\mathscr{C}}$, $\boldsymbol{\mathscr{D}}$ are fourth order tensor with constant components. Combining all these elements the implicit constitutive relation for stress softening is:

$$\overline{\ln(\boldsymbol{\mathscr{A}}_o \cdot \mathbf{S})}^{\boldsymbol{\cdot}} - \overline{\ln \mathbf{E}}^{\boldsymbol{\cdot}} = \begin{cases} \overline{\ln(\boldsymbol{\mathscr{A}}_o \cdot \mathbf{S})}^{\boldsymbol{\cdot}} - \overline{\ln(\mathbf{E}_o + \boldsymbol{\mathscr{B}}_o \cdot \mathbf{S})}^{\boldsymbol{\cdot}} H(\mathrm{tr}\,(\mathbf{E}\dot{\mathbf{S}})) & \text{if } \tilde{\sigma}_1 \geqslant \sigma^*_{Y_T} \text{ or } \tilde{\sigma}_3 \leqslant -\sigma^*_{Y_C}, \\ 0 & \text{otherwise,} \end{cases}$$

In [51] the above constitutive relation has been used to solve some boundary value problems (considering small gradient of the displacement field), namely:

- The homogeneous compression of a cylinder without radial stresses, where $\mathbf{T} = \sigma_z(t) \mathbf{e}_z \otimes \mathbf{e}_z$ and $\boldsymbol{\varepsilon} = \varepsilon_r(t)(\mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\theta \otimes \mathbf{e}_\theta) + \varepsilon_z(t) \mathbf{e}_z \otimes \mathbf{e}_z$.

- The uniform compression of a cylinder with radial stress (triaxial test), where $\mathbf{T} = \sigma_r(t)(\mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\theta \otimes \mathbf{e}_\theta) + \sigma_z(t)\mathbf{e}_z \otimes \mathbf{e}_z$, where $\boldsymbol{\varepsilon}$ is of the same form as in the previous case.

- The homogeneous simple shear of a slab where $\mathbf{T} = \tau(t)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$ and $\boldsymbol{\varepsilon} = \varepsilon_1(t)\mathbf{e}_1 \otimes \mathbf{e}_1 + \varepsilon_2(t)\mathbf{e}_2 \otimes \mathbf{e}_2 + \varepsilon_{12}(t)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$.

- The inflation of a cylindrical annulus (for the cinematics see, for example, Section 8.1.2).

10.1.5 An implicit constitutive relation for elastic-perfectly plastic solids

In [82] an implicit constitutive relation has been proposed for the modelling of elastic and perfectly plastic deformations in solids. For the 1D case the constitutive relation is:

$$\dot{\sigma} = E\left[1 - H(\sigma\dot{\varepsilon})H(|\sigma| - \sigma_Y)\right]\dot{\varepsilon}, \tag{302}$$

where σ, ε are the 1D stress and strain (small), respectively, E is the Young's modulus and σ_Y is the yield stress.

In (302) if $|\sigma| = \sigma_Y$ we have that $\dot{\sigma} = 0$, thus $\sigma = \textit{constant}$, i.e., the solid is perfectly plastic. On the other hand if $\sigma\dot{\varepsilon} < 0$ from (302) we get $\dot{\sigma} = E\dot{\varepsilon}$, from where we get the elastic behaviour of the solid, valid for loading and unloading, including the residual strains that appear after the stress reaches the yielding stress.

For 3D problem in [82] the following counterpart of (302) has been proposed:

$$\frac{1}{E}[(1+\nu)\dot{\mathbf{T}} - \nu(\mathrm{tr}\,\dot{\mathbf{T}})\mathbf{I}] = \dot{\boldsymbol{\varepsilon}} - H(\mathbf{T}\cdot\dot{\boldsymbol{\varepsilon}})H(|\mathbf{T}_{\mathrm{D}}|^2 - \kappa^2)\dot{\boldsymbol{\varepsilon}}, \tag{303}$$

where $\dot{\mathbf{A}} = \frac{\partial}{\partial t}\mathbf{A}(\mathbf{X}, t)$ and $\mathbf{T}_{\mathrm{D}}$ is the deviatoric stress, ν is the Poisson ratio and κ is a material parameter.

Using (303) in the first law of thermodynamics considering the Fourier model for heat transfer $k\nabla\theta$, we obtain:

$$\rho_r c_V \dot{\theta} - \mathrm{div}(k\nabla\theta) = H(\mathbf{T}\cdot\dot{\boldsymbol{\varepsilon}})H(|\mathbf{T}_{\mathrm{D}}|^2 - \kappa^2)\mathbf{T}\cdot\dot{\boldsymbol{\varepsilon}}.$$

In [82] the authors also consider a regularized version for H namely H_ϵ defined as (see (7) therein) $H_\epsilon(s) = 1/2 + \arctan(s/\epsilon)/\pi$. A finite element formulation is presented, and using that the problem of a 2D plate with an elliptic hole under traction is solved and analyzed.

10.1.6 Elasto-plastic beams based on an implicit rate type model

In [215] an implicit constitutive relation is proposed for elasto-plastic 3D beams (see Sections 8.3 and 9.5.2 for similar models for elastic beams). In that paper the balance equations are written in a compact manner as:

$$\frac{\mathrm{d}\mathsf{X}}{\mathrm{d}s}(s,t) = \mathsf{g}(\mathsf{X}(s,t)), \tag{304}$$

where the vectors X and g are defined as

$$\mathsf{X} = (x, y, \theta, V_x, V_y, M), \quad \mathsf{g} = (\cos\theta, \sin\theta, \kappa, f_x, f_y, V_x \sin\theta - V_y \cos\theta),$$

respectively, where x, y are the coordinates of the position of the centerline of the deformed beam, θ is the angle of the centerline and the axis x, V_x, V_y are the net forces in the cross section of the beam in the directions x, y, respectively, M is the bending moment, s is the arc length in the reference configuration, $\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s}$ is the curvature, and f_x, f_y are the distributed external forces applied on the beam.

In the classical theory for beams a constitutive equation for M of the form $M = M(\kappa)$ is assumed. Here it is assumed the following implicit constitutive relation (compare with (5)):

$$\mathfrak{F}(M, \dot{M}, \kappa, \dot{\kappa}) = 0. \tag{305}$$

The following particular subclass of (305) is proposed for the modelling of beams made of aluminium alloys:

$$\dot{M} = \left\{1 - H(M\dot{\kappa})\left[1 + \tanh\left(\alpha H\left(1 - \left|\frac{M}{M_y}\right|\right)\left(\left|\frac{M}{M_y}\right| - 1\right)\right)\right]\right\} EI\dot{\kappa}, \tag{306}$$

where α is a material constant, $H(x)$ is the Heaviside step function, EI is the bending stiffness and M_y is a material parameter connected with the bending moment for which you have yielding.

Some boundary value problems are solved using the finite element method and a weak formulation applied on (304):

$$\int_0^L w(s)\left(\frac{\mathrm{d}\mathsf{X}}{\mathrm{d}s} - \mathsf{g}\right)\,\mathrm{d}s = 0, \tag{307}$$

where $w(s)$ is a weight function. The finite element method is implemented considering a discretization for X, g and also M and (306) and (307) are solved numerically.

10.2 Visco-elastic solids

In this section we present some constitutive relations for visco-elastic bodies of rate-type. For classical constitutive equations for visco-elastic solids such rate-type models are of the form stress equals some function of the strain and strain rate. Here we study constitutive relations that are generalizations of the above.

Let us consider an implicit constitutive relation of the form:

$$\mathfrak{F}(\mathbf{T}, \dot{\mathbf{T}}, \mathbf{B}, \mathbf{D}) = \mathbf{0}, \tag{308}$$

where $\dot{\mathbf{T}}$ should be replaced for some other time derivative that permits the implicit relation to be frame-indifferent, and $\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}})$, where $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$.

10.2.1 A model for thermo-visco-elastic solids using the Gibbs potential

In [160] a constitutive equation has been proposed for the modelling of thermo-visco-elastic solids using the Gibbs potential. Using the 1st Piola-Krichhoff stress tensor $\mathbf{P} = \mathbf{TF}$, the equation of motion $(4)_1$ (without body forces) and the first law of thermodynamics (see, for example, (42)) become

$$\rho_{\mathrm{r}}\ddot{\mathbf{x}} = \operatorname{Div}\mathbf{P}, \quad \rho_{\mathrm{r}}\dot{U} = -\operatorname{Div}\mathbf{Q}_{\mathrm{L}} + \mathbf{P}\cdot\dot{\mathbf{F}},$$

where $\mathbf{Q}_{\mathrm{L}}$ is the heat flux in the reference configuration. A two network model is proposed (for small gradient of the displacement field that implies small strains), wherein the stress tensor is decomposed as $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$. Assuming the Gibbs potential $\mathcal{G} = \mathcal{G}(\mathbf{T}_1, \mathbf{T}_2, \theta)$, from the first law of thermodynamics we obtain $s = -\frac{\partial\mathcal{G}}{\partial\theta}$, as well as this, $U = \mathcal{G} + \theta s - \frac{\partial\mathcal{G}}{\partial\mathbf{T}_1}\cdot\mathbf{T}_1 - \frac{\partial\mathcal{G}}{\partial\mathbf{T}_2}\cdot\mathbf{T}_2$, and the first law of thermodynamics becomes

$$\rho_{\mathrm{r}}\theta\dot{s} = -\operatorname{Div}\mathbf{Q}_{\mathrm{L}} + \xi,$$

where ξ is the rate of mechanical dissipation. The second law of thermodynamics means $\xi \geqslant 0$. In [160] it is assumed that

$$\xi = \sum_{i=1}^{2}\mathbf{T}_i\cdot\left[\dot{\boldsymbol{\varepsilon}} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{G}}{\partial\mathbf{T}_i}\right)_{\theta:\text{fixed}}\right].$$

One of the 'networks' is assumed to be elastic from where it is obtained:

$$\boldsymbol{\varepsilon} = -\frac{\partial \mathcal{G}}{\partial \mathbf{T}_1} \quad \text{thus} \quad \boldsymbol{\varepsilon}_{\mathrm{v}} = \boldsymbol{\varepsilon} + \frac{\partial \mathcal{G}}{\partial \mathbf{T}_2},$$

therefore $\xi = \mathbf{T}_2 \cdot \dot{\boldsymbol{\varepsilon}}_{\mathrm{v}}$, where $\boldsymbol{\varepsilon}_{\mathrm{v}}$ is the viscous part of the strain tensor, as a results from

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\mathrm{e}} + \boldsymbol{\varepsilon}_{\mathrm{v}},$$

(where $\boldsymbol{\varepsilon}_{\mathrm{e}}$ is the elastic part of the strain) we get $\boldsymbol{\varepsilon}_{\mathrm{e}} = -\frac{\partial \mathcal{G}}{\partial \mathbf{T}_2}$ and $\xi = \xi(\dot{\boldsymbol{\varepsilon}}_{\mathrm{v}}, \theta)$. In [160] after some manipulations, which are not shown here, a new dissipation function is proposed, which is called $\tilde{\xi} = \tilde{\xi}(\mathbf{T}_2, \theta)$, and the following expression for $\mathcal{G}$ is proposed:

$$\begin{aligned} \mathcal{G}(\mathbf{T}_1, \mathbf{T}_2, \theta) &= -\frac{1}{2}(\boldsymbol{\mathscr{C}}^{(1)}[\mathbf{T}_1] \cdot \mathbf{T}_1 + \boldsymbol{\mathscr{C}}^{(2)}[\mathbf{T}_2] \cdot \mathbf{T}_2 - \mathbf{A}_1 \cdot \mathbf{T}_1\theta - \mathbf{A}_2 \cdot \mathbf{T}_2\theta) + f(\theta), \\ \tilde{\xi}(\mathbf{T}_2) &= \mathbf{K}(\theta)[\mathbf{T}_2] \cdot \mathbf{T}_2, \end{aligned}$$

where $\boldsymbol{\mathscr{C}}^{(i)}$, $i = 1, 2$ are fourth order tensors with constant components, $\mathbf{K}$ is a second order tensor, and $\boldsymbol{\mathscr{C}}^{(i)}[\mathbf{T}]$ is equivalent in index notation and Cartesian coordinates to $\mathscr{C}^{(i)}_{jklm} T_{lm}$.

10.2.2 On a class of implicit constitutive relation for visco-elastic bodies

In [57] the following implicit constitutive relations is proposed that is a subclass of (308):

$$\mathfrak{F}(\mathbf{T}, \mathbf{B}, \mathbf{D}, \rho) = \mathbf{0}. \tag{309}$$

From [202] we have a list of 36 invariants if $\mathfrak{F}$ is isotropic (which for brevity are not shown here, see [57]). Assuming that there exist a scalar function $\Pi = \Pi(\mathbf{T}, \mathbf{B}, \mathbf{D}, \rho)$ such that (309) can be obtained as $\mathfrak{F} = \frac{\partial \Pi}{\partial \mathbf{T}} + \frac{\partial \Pi}{\partial \mathbf{B}} + \frac{\partial \Pi}{\partial \mathbf{D}} = \mathbf{0}$ we get

$$\begin{aligned} &\alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{T}^2 + \alpha_3\mathbf{B} + \alpha_4\mathbf{B}^2 + \alpha_5\mathbf{D} + \alpha_6\mathbf{D}^2 + \alpha_7(\mathbf{TB} + \mathbf{BT}) \\ &+\alpha_8(\mathbf{TD} + \mathbf{DT}) + \alpha_9(\mathbf{BD} + \mathbf{DB}) + \alpha_{10}(\mathbf{T}^2\mathbf{B} + \mathbf{BT}^2) + \alpha_{11}(\mathbf{B}^2\mathbf{T} + \mathbf{TB}^2) \\ &+\alpha_{12}(\mathbf{T}^2\mathbf{D} + \mathbf{DT}^2) + \alpha_{13}(\mathbf{D}^2\mathbf{T} + \mathbf{TD}^2) + \alpha_{14}(\mathbf{B}^2\mathbf{D} + \mathbf{DB}^2) + \alpha_{15}(\mathbf{D}^2\mathbf{B} \\ &+\mathbf{BD}^2) + \alpha_{17}(\mathbf{D}^2\mathbf{T}^2 + \mathbf{T}^2\mathbf{D}^2) + \alpha_{18}(\mathbf{B}^2\mathbf{D}^2 + \mathbf{D}^2\mathbf{B}^2) + \alpha_{19}(\mathbf{DBT} + \mathbf{DTB}) \\ &+\alpha_{20}(\mathbf{TDB} + \mathbf{BDT}) + \alpha_{21}(\mathbf{BTD} + \mathbf{DTB}) + \alpha_{22}(\mathbf{DBT}^2 + \mathbf{T}^2\mathbf{BD}) \\ &+\alpha_{23}(\mathbf{BDT}^2 + \mathbf{T}^2\mathbf{DB}) + \alpha_{24}(\mathbf{TDB}^2 + \mathbf{B}^2\mathbf{DT}) + \alpha_{25}(\mathbf{DTB}^2 + \mathbf{B}^2\mathbf{TD}) \\ &+\alpha_{26}(\mathbf{BTD}^2 + \mathbf{D}^2\mathbf{TB}) + \alpha_{27}(\mathbf{TBD}^2 + \mathbf{D}^2\mathbf{BT}) + \alpha_{28}(\mathbf{BT}^2\mathbf{D} + \mathbf{DT}^2\mathbf{B}) \\ &+\alpha_{29}(\mathbf{DB}^2\mathbf{T} + \mathbf{TB}^2\mathbf{D}) + \alpha_{30}(\mathbf{TD}^2\mathbf{B} + \mathbf{BD}^2\mathbf{T}) = \mathbf{0} \end{aligned} \tag{310}$$

where $\alpha_i = \alpha_i(\mathbf{T}, \mathbf{B}, \mathbf{D}, \rho)$, $i = 0, 1, 2, ..., 30$ are scalar functions defined in terms of the derivatives of Π, which for brevity are not shown here.

In the case the gradient of the displacement field is very small, and if $|\mathbf{D}|/D_o \sim O(\delta)$, $\delta \ll 1$ (where D_o is some characteristic value for $\mathbf{D}$) we have $\mathbf{D} \approx \dot{\boldsymbol{\varepsilon}}$ and from (310) we obtain

$$\gamma_0 \mathbf{I} + \gamma_1 \mathbf{T} + \gamma_2 \mathbf{T}^2 + \alpha_3 \boldsymbol{\varepsilon} + \gamma_4 \dot{\boldsymbol{\varepsilon}} + \gamma_5 (\mathbf{T}\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}\mathbf{T}) + \gamma_6 (\mathbf{T}\dot{\boldsymbol{\varepsilon}} + \dot{\boldsymbol{\varepsilon}}\mathbf{T}) + \alpha_7 (\mathbf{T}^2 \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}\mathbf{T}^2) + \alpha_8 (\mathbf{T}^2 \dot{\boldsymbol{\varepsilon}} + \dot{\boldsymbol{\varepsilon}}\mathbf{T}^2) = \mathbf{0}. \quad (311)$$

In the case that in the above constitutive relation the stresses and strains only appear linearly, (311) becomes:

$$\zeta_0 \mathbf{I} + \zeta_1 \mathbf{T} + \zeta_2 \boldsymbol{\varepsilon} + \zeta_3 \dot{\boldsymbol{\varepsilon}} + \zeta_4 (\mathbf{T}\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}\mathbf{T}) + \zeta_5 (\mathbf{T}\dot{\boldsymbol{\varepsilon}} + \dot{\boldsymbol{\varepsilon}}\mathbf{T}) = \mathbf{0}, \quad (312)$$

where $\zeta_0 = \beta_0^{(1)} \mathrm{tr}\, \mathbf{T} + \beta_0^{(2)} \mathrm{tr}\, \boldsymbol{\varepsilon} + \beta_0^{(3)} \mathrm{tr}\, \dot{\boldsymbol{\varepsilon}} + \beta_0^{(4)} \mathrm{tr}\, (\mathbf{T}\boldsymbol{\varepsilon}) + \beta_0^{(5)} \mathrm{tr}\, (\mathbf{T}\dot{\boldsymbol{\varepsilon}})$, $\zeta_1 = \beta_1 + \beta_1^{(2)} \mathrm{tr}\, \boldsymbol{\varepsilon} + \beta_1^{(3)} \mathrm{tr}\, \dot{\boldsymbol{\varepsilon}}$, $\zeta_2 = \beta_2 + \beta_2^{(1)} \mathrm{tr}\, \mathbf{T}$, $\zeta_3 = \beta_3 + \beta_3^{(1)} \mathrm{tr}\, \mathbf{T}$, where β_i, $\beta_j^{(k)}$ and ζ_4, ζ_5 are material constants.

In [57] some boundary value problems were studied considering (312), namely the uniaxial homogeneous tension/compression of a cylinder without lateral constraints, the triaxial tension/compression of a cylinder, the biaxial compression of a slab, the simple shear of a slab, and the inflation of a cylindrical annulus.

10.2.3 A subclass of implicit constitutive relation for visco-elastic solids with applications to large deformations

In [41] a special class of (309) and (310) was studied, where in particular $\mathfrak{F}$ was assumed to be of the form

$$\mathbf{T} + \varphi \mathbf{I} - \alpha \mathbf{B} - \mu \mathbf{D} = \mathbf{0}, \quad (313)$$

where $\mu = \mu(I_1, I_2) = \mu_o \exp(\delta I_1)(1 + 2\beta I_2)^n$, $\varphi = \varphi(I_1, \det \mathbf{F}) = (\gamma I_1 + \lambda)(\det \mathbf{F})^m$, where μ_o, δ, β, γ and λ are constants, and $I_1 = \mathrm{tr}\, \mathbf{T}$, $I_2 = \frac{1}{2} \mathrm{tr}\, (\mathbf{D}^2)$. Here the viscosity of the body μ depends on the mechanical pressure I_1 and the strain rate (through I_2). We do now show the numerical values of the above numerical constants, since they have not been proposed for actual viscoelastic solids.

Using (313) two boundary value problems were studied, namely the homogeneous axial deformation and shear of a slab, and the movement of a slab on a inclined plane under the influence of gravity. In [46] a generalization of (313) is used to study the circumferential shear of an annulus. Some details of the above are shown here.

For the homogeneous axial deformation and shear of the slab we assume the presence of the stress tensor $\mathbf{T} = \tau(t)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + \sigma_2(t)\mathbf{e}_2 \otimes \mathbf{e}_2$, which it is assumed to produce the deformation $x(t) = \aleph(t)X + \kappa(t)Y$, $y = \wp(t)Y$, $z = \ell(t)Z$ (using the notation X, Y, Z and x, y, z instead X_i, x_i, $i = 1, 2, 3$ in Cartesian coordinates). Calculating $\mathbf{F}$, $\mathbf{B}$ and $\mathbf{D}$ (assuming that the equation of motion is satisfied approximately if $\rho\ddot{\mathbf{x}}$ can be neglected), from (313) we obtain the four nonlinear ordinary differential equations, from where we can find $\aleph(t)$, $\kappa(t)$, $\wp(t)$ and $\ell(t)$ as function of $\tau(t)$ and $\sigma_2(t)$:

$$\lambda(\aleph^2 + \kappa^2) + \mu\frac{\dot{\aleph}}{\aleph} = \varphi, \quad \lambda\wp^2 + \mu\frac{\dot{\wp}}{\wp} = \sigma_2(t) + \varphi, \tag{314}$$

$$\lambda\ell^2 + \mu\frac{\dot{\ell}}{\ell} = 0, \quad \lambda\kappa\wp + \frac{\mu}{2\wp}\left(\dot{\kappa} - \frac{\kappa\dot{\aleph}}{\aleph}\right) = \tau. \tag{315}$$

In [41] numerical solutions of (314), (315) are found assuming the initial conditions $\aleph(0) = 1$, $\kappa(0) = 0$, $\wp(0) = 1$ and $\ell(0) = 1$. The main interest was to study the influence of the material constants m and n mentioned above in the numerical results.

In [42] Eq. (313) is used to study the movement of a slab moving on an inclined plane under the influence of gravity. In the reference configuration the slab is defined as $-\infty \leqslant X \leqslant \infty$, $0 \leqslant Y \leqslant H$ and $-\infty \leqslant Z \leqslant \infty$, where H is the thickness of the slab, and θ is the angle of the inclined plane with respect to X (the gravity acts in the direction y, thus $\mathbf{b} = g\sin\theta\mathbf{e}_1 - g\cos\theta\mathbf{e}_2$). We assume that the stress tensor is given by $\mathbf{T} = \sum_{i=1}^{2}\sigma_i(y,t)\mathbf{e}_i \otimes \mathbf{e}_i + \tau(y,t)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$, and the deformation is $x = X + u(Y,t)$, $y = Y + v(Y,t)$, $z = Z$. Calculating the deformation gradient, the left Cauchy-Green tensor and the rest of the quantities, from the equation of motion (in the reference configuration) and (313) we obtain the six nonlinear partial differential equations:

$$\frac{\partial \tau}{\partial Y} + \rho_{\mathrm{r}} \left(g \sin\theta - \frac{\partial^2 u}{\partial t^2} \right) = 0, \quad \frac{\partial \sigma_2}{\partial Y} - \rho_{\mathrm{r}} \left(g \cos\theta + \frac{\partial^2 v}{\partial t^2} \right) = 0,$$
$$\sigma_1 + \varphi - \lambda \left[1 + \left(\frac{\partial u}{\partial Y} \right)^2 \right] = 0, \quad \sigma_2 + \varphi - \lambda \left(1 + \frac{\partial v}{\partial Y} \right)^2 - \mu \frac{\frac{\partial^2 v}{\partial Y \partial t}}{\left(1 + \frac{\partial u}{\partial Y}\right)} = 0,$$
$$\sigma_3 + \varphi - \lambda = 0, \quad \tau - \lambda \frac{\partial u}{\partial Y} \left(1 + \frac{\partial v}{\partial Y} \right) - \frac{\mu}{2} \frac{\frac{\partial^2 u}{\partial Y \partial t}}{\left(1 + \frac{\partial v}{\partial Y}\right)} = 0.$$

In [42] the above equations are solved numerically in order to find $\sigma_1(Y,t)$, $\sigma_2(Y,t)$, $\sigma_3(Y,t)$, $\tau(Y,t)$, $u(Y,t)$ and $v(Y,t)$, assuming the boundary conditions $v(0,t) = 0$, $\sigma_2(H,t) = \hat{\sigma}_2(t)$ and $\tau(H,t) = \hat{\tau}(t)$. Some exact solutions have been found for the case the gradient of the displacement field is very small, but are not shown here.

In [46] a generalization of (313) is considered, namely

$$\varpi \mathbf{T} + \varphi \mathbf{I} - \alpha \mathbf{B} - \mu \mathbf{D} = \mathbf{0}, \tag{316}$$

where $\varpi = \varpi(I_2) = (1+2\zeta I_2)^q$, where ζ is a material constant, and φ, α and μ are the same as for (313) (see the paragraph after that equation). The implicit constitutive relation (316) is used in order to analyze the deformation of the annulus (described in cylindrical coordinates in the reference configuration) $R_{\mathrm{i}} \leqslant R \leqslant R_{\mathrm{o}}$, $0 \leqslant \Theta \leqslant 2\pi$, $0 \leqslant Z \leqslant L$. The stress tensor and the deformation field are assumed to be: $\mathbf{T} = \mathbf{T}(r,t) = \sigma_r(r,t)\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_\theta(r,t)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_z(r,t)\mathbf{e}_z \otimes \mathbf{e}_z + \tau_{r\theta}(r,t)(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r)$ and $r = f(R,t)$, $\theta = \Theta + h(R,t)$, $z = Z$, respectively. Calculating the deformation gradient and the rest of the variables, from (316) and the equation of motion (in the reference configuration) we obtain the six nonlinear partial differential equations:

$$\frac{R\rho_{\rm r}}{f}\left[\frac{\partial^2 f}{\partial t^2}-f\left(\frac{\partial h}{\partial t}\right)^2\right]=\frac{\partial\sigma_r}{\partial R}+\frac{1}{f}\frac{\partial f}{\partial R}(\sigma_r-\sigma_\theta),$$

$$\frac{R\rho_{\rm r}}{f}\left(2\frac{\partial f}{\partial t}\frac{\partial h}{\partial t}-\frac{\partial^2 h}{\partial t^2}\right)=\frac{\partial\tau_{r\theta}}{\partial R}+\frac{2}{f}\frac{\partial f}{\partial R}\tau_{r\theta},$$

$$\varpi\sigma_r+\varphi-\lambda\left(\frac{\partial f}{\partial R}\right)^2-\frac{\mu}{\frac{\partial f}{\partial R}}\frac{\partial^2 f}{\partial R\partial t}=0,$$

$$\varpi\sigma_\theta+\varphi-\lambda f^2\left[\left(\frac{\partial h}{\partial R}\right)^2+\frac{1}{R^2}\right]-\frac{\mu}{f}\frac{\partial f}{\partial t}=0,$$

$$\varpi\sigma_z+\varphi-\lambda=0,$$

$$\varpi\tau_{r\theta}-\lambda f\frac{\partial h}{\partial R}\frac{\partial f}{\partial R}-\frac{\mu}{2}\frac{f}{\frac{\partial f}{\partial R}}\frac{\partial^2 h}{\partial R\partial t}=0.$$

In [46] these equations are solved numerically in order to find $\sigma_r(R,t)$, $\sigma_\theta(R,t)$, $\sigma_z(R,t)$, $\tau_{r\theta}(R,t)$, $f(R,t)$ and $h(R,t)$, considering the boundary conditions $f(R_{\rm i},t)=R_{\rm i}$, $f(R_{\rm o},t)=R_{\rm o}$, $h(R_{\rm i},t)=0$ and $\tau_{r\theta}(R_{\rm o},t)=\tau_{\rm o}(t)$.

10.2.4 A viscoelastic model by Erbay et al and an analysis of one boundary value problem

In [76] the following implicit viscoelastic body as been proposed (see (308)):

$$\mathbf{B}+\alpha(\mathbf{A}_2-\mathbf{A}_1^2)=\beta_0\mathbf{I}+\beta_1\mathbf{T}+\beta_2\mathbf{T}^2, \tag{317}$$

where $\mathbf{A}_1=2\mathbf{D}$, $\mathbf{A}_2=\dot{\mathbf{A}}_1+\mathbf{A}_1\mathbf{L}+\mathbf{L}^{\rm T}\mathbf{A}_1$ are the Rivlin-Ericksen tensors, the scalar functions β_i, $i=0,1,2$ depend on the invariants of $\mathbf{T}$. In [76] the case of small gradient of the displacement field is assumed, as well as this, $\beta_2=0$ and isochoric motions are considered, thus ${\rm tr}\boldsymbol{\varepsilon}=0$ and as a result $3(\beta_0-1)+\beta_1{\rm tr}\,\mathbf{T}=0$, then from (317) we get

$$2\boldsymbol{\varepsilon}+2\alpha\ddot{\boldsymbol{\varepsilon}}=\beta_1\mathbf{T}_{\rm D}, \tag{318}$$

where $\mathbf{T}_{\rm D}$ is the deviatoric stress and $\beta_1=\beta_1(\mathbf{T}_{\rm D})$.

For the above implicit relation (318) the following deformation was assumed: $x_1=X_1+u(X_3,t)$, $x_2=X_2+v(X_3,t)$ and $x_3=X_3$, with the stress distribution: $\mathbf{T}=-p\mathbf{I}+T_{13}(x_3,t)(\mathbf{e}_1\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\mathbf{e}_1)+T_{23}(x_3,t)(\mathbf{e}_2\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\mathbf{e}_2)$.

Replacing the above stress tensor in the equation of motion $(4)_1$ (without body forces) we obtain

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial T_{13}}{\partial x_3} - \frac{\partial p}{\partial x_1}, \quad \rho\frac{\partial^2 v}{\partial t^2} = \frac{\partial T_{23}}{\partial x_3} - \frac{\partial p}{\partial x_2}, \quad \frac{\partial p}{\partial x_3} = 0. \tag{319}$$

On the other hand from the deformation and using the stress in (318) considering the above definitions, and using the notation z instead x_3 and β instead β_1 we get

$$\frac{\partial u}{\partial z} + \alpha\frac{\partial^3 u}{\partial z\partial t^2} = \beta(\Omega^2)T_{13}, \quad \frac{\partial v}{\partial z} + \alpha\frac{\partial^3 v}{\partial z\partial t^2} = \beta(\Omega^2)T_{23}, \tag{320}$$

where $\Omega^2 = T_{13}^2 + T_{23}^2$. It is assumed that $p = f_1(t)x_1 + f_2(t)x_2$ then $(319)_3$ is satisfied automatically. Defining U and V through $\frac{\partial U}{\partial t} = \frac{\partial u}{\partial z}$ and $\frac{\partial V}{\partial t} = \frac{\partial v}{\partial z}$, respectively, taking the derivative of $(319)_{1,2}$ in z we have $\rho\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 T_{13}}{\partial z^2}$, $\rho\frac{\partial^2 V}{\partial t^2} = \frac{\partial^2 T_{23}}{\partial z^2}$, while with the above notation (320) become: $U + \alpha\frac{\partial^2 U}{\partial t^2} = \beta(\Omega^2)T_{13}$ and $V + \alpha\frac{\partial^2 V}{\partial t^2} = \beta(\Omega^2)T_{23}$. Taking the derivative of these equations in t twice, using the previous expressions for the equation of motion, we can eliminate the variables U, V and by defining the complex stress $\sigma(z,t) = T_{13}(z,t) + \hat{I}T_{23}(z,t)$ (where $\hat{I}$ is the unit imaginary number) the final partial differential equation is:

$$\frac{\partial^2 \sigma}{\partial z^2} + \alpha\frac{\partial^2 \sigma}{\partial z^2\partial t^2} = \rho\frac{\partial^2}{\partial t^2}[\beta(\Omega^2)\sigma]. \tag{321}$$

In [76] the above partial differential equation is transformed into an ordinary differential equation for the problem of travelling wave, where it is assumed (see, for example, Sections 8.2.2 and 9.3) that $T_{13} = f(\xi)$ and $T_{23} = g(\xi)$, where in this case ξ is defined as $\xi = (z - ct)/\sqrt{\alpha c^2}$, where c is the speed of such waves. It is possible to see that $\Omega^2 = f^2 + g^2$. If the notation $(\ \ )'$ is used to denote the derivative in ξ Eq. (321) becomes

$$f'' + f^{(4)} = \rho c^2[\beta(\Omega^2)f]'', \quad g'' + g^{(4)} = \rho c^2[\beta(\Omega^2)g]''. \tag{322}$$

In [76] the authors assume that β is given as $\beta(\Omega^2) = 1/(\mu\sqrt{1+\kappa\Omega^2})$, where μ, κ are constants. For f and g they assume the solutions $f(\xi) = A\cos(kz - wt)$ and $g(\xi) = \pm A\sin(kz - wt)$ and from (322) they obtain $c^2 = 1/[\rho\beta(A^2) + \alpha k^2)$, where $c = w/k$.

Another solution is also considered, namely $T_{13} = \phi(z)\cos(wt) + \zeta(z)\sin(wt)$ and $T_{23} = \phi(z)\sin(wt) - \zeta(z)\cos(wt)$, in such a case $\Omega^2 = \phi^2 + \zeta^2$ and (322) become $(1-\alpha w^2)\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2}+\rho w^2\beta(\Omega^2)\phi=0$ and $(1-\alpha w^2)\frac{\mathrm{d}^2\zeta}{\mathrm{d}z^2}+\rho w^2\beta(\Omega^2)\zeta=0$. In [76] the above system of equations is further reduced but it is not solved exactly.

10.2.5 Analysis of circular polarized waves for a class of visco-elastic solid

In [162] there is a study of one boundary value problem for a class of constitutive relation for visco-elastic bodies of the form (compare with (318), (316) and (313)):

$$\boldsymbol{\varepsilon}+\nu\dot{\boldsymbol{\varepsilon}}=\beta_1\mathbf{T}_{\mathrm{D}}, \tag{323}$$

where ν, β_1 are constants and $\mathbf{T}_{\mathrm{D}}$ is the deviatoric part of the stress $\mathbf{T}_{\mathrm{D}}=\mathbf{T}-\frac{1}{3}\mathrm{tr}\,(\mathbf{T})\mathbf{I}$.

It is assumed that the stress tensor is of the form (using the notation x,y,z instead x_i, and X,Y,Z for X_i, $i=1,2,3$)

$$\mathbf{T}=-p(x,y,z)\mathbf{I}+T_{13}(Z,t)(\mathbf{e}_1\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\mathbf{e}_1)+T_{23}(Z,t)(\mathbf{e}_2\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\mathbf{e}_2). \tag{324}$$

For the motion in [162] the authors assume:

$$x=X+f(Z,t),\quad y=Y+g(Z,t),\quad z=Z. \tag{325}$$

The above motion represents the propagation of shear waves.

Replacing (324), (325) in $(4)_1$ in the case there is no body force we obtain

$$\rho\frac{\partial^2 f}{\partial t^2}=-\frac{\partial p}{\partial x}+\frac{\partial T_{13}}{\partial z},\quad \rho\frac{\partial^2 g}{\partial t^2}=-\frac{\partial p}{\partial y}+\frac{\partial T_{23}}{\partial z},\quad 0=-\frac{\partial p}{\partial z}. \tag{326}$$

Using (324) and (325) in (323) we get

$$\frac{\partial f}{\partial z}+\nu\frac{\partial^2 f}{\partial z\partial t}=\beta_1 T_{13},\quad \frac{\partial g}{\partial z}+\nu\frac{\partial^2 g}{\partial z\partial t}=\beta_1 T_{23}. \tag{327}$$

From $(326)_3$ we have $p(z,y,z)=p_o(t)+p_1(t)x+p_2(t)y$. Defining $F(z,t)=\frac{\partial f}{\partial z}$ and $G(z,t)=\frac{\partial g}{\partial z}$, after some manipulations equations $(326)_{1,2}$ become $\rho\frac{\partial^2 F}{\partial t^2}=\frac{\partial^2 T_{13}}{\partial z^2}$ and $\rho\frac{\partial^2 G}{\partial t^2}=\frac{\partial^2 T_{23}}{\partial z^2}$. Using the above definitions in (327) we get $F+\nu\frac{\partial F}{\partial z}=\beta_1(T_{13}^2+T_{23}^2)T_{13}$ and $G+\nu\frac{\partial G}{\partial z}=\beta_1(T_{13}^2+T_{23}^2)T_{23}$. In [162] the solutions of the above equations are:

$$T_{13}=A\cos(kz-wt),\quad T_{23}=\pm A\sin(kz-wt),$$

where A, k and w are constants.

10.2.6 A summary of some studies on analysis and existence of solutions, for some boundary value problems, considering some implicit constitutive relations for viscoelastic solids

In this section we show a summary of different works on some implicit relations for visco-elastic solids, which are dedicated to the analysis of existence of solutions for some boundary value problems.

- In [15, 16, 99] we can see some studies on existence of solutions for some boundary value problems for (309) and in particular the subclass

  $$\tilde{\nu}\mathbf{D} + \tilde{\alpha}\mathbf{B} = \bar{\beta}_0\mathbf{I} + \bar{\beta}_1\mathbf{T} + \bar{\beta}_2\mathbf{T}^2,$$

  where $\bar{\beta}_i = \bar{\beta}_i(\mathbf{T})$.

  In the case of small gradient of the displacement field, where we have $\mathbf{B} \approx \mathbf{I} + 2\boldsymbol{\varepsilon}$ from the above we obtain

  $$\alpha\boldsymbol{\varepsilon} + \nu\dot{\boldsymbol{\varepsilon}} = \mathfrak{g}(\mathbf{T}), \quad \nu > 0, \tag{328}$$

  where for strain limiting behaviour in [15] it is proposed $\mathfrak{g}(\mathbf{T}) = \frac{1}{(1+|\mathbf{T}|^a)^{1/a}}\mathbf{T}$.

  In [73] for 1D problems the equation of motion and (328) becomes

  $$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial\sigma}{\partial x}, \quad \alpha\varepsilon + \nu\frac{\partial\varepsilon}{\partial t} = \mathfrak{g}(\sigma). \tag{329}$$

  In [73] a travelling wave motion is analyzed for (329) in the case $\alpha = 1$ for a 1D problem, where the 1D stress σ is assumed to be of the form $\sigma(x,t) = \sigma(\xi)$, where $\xi = x - ct$ (c is the speed of the travelling waves). Eqs. (329) are manipulated using the notation $(\quad)'$ for $\frac{\mathrm{d}}{\mathrm{d}\xi}$, getting

  $$\sigma - \nu c\sigma' = c^2\mathfrak{g}(\sigma) + a_o,$$

  where a_o is a constant[48].

  In [99] for the case of (329), when $\alpha = 1$ the authors study weak formulations and provide some existence theorems, assuming that $\mathfrak{g}(\mathbf{T}) = \frac{\beta\mathbf{T}}{(1+\kappa|\mathbf{T}|^s)^{1/s}}$, where β, κ and s are positive constants.

[48] In [73] the authors considered different examples for $\mathfrak{g}(\sigma)$, such as: $\mathfrak{g}(\sigma) = \beta\sigma + \alpha(1 + \alpha\sigma^2/2)^n\sigma$, $\mathfrak{g}(\sigma) = \sigma/(1 + |\sigma|^2)^(1/r)$ and $\mathfrak{g}(\sigma) = \alpha\{[1 - \exp(-\beta\sigma/(1 + \delta|\sigma|))] + \gamma\sigma/(1 + |\sigma|)\}$. In [74] a similar analysis was considered, but studying an equation for σ written in a different manner, namely $\frac{\partial^2\sigma}{\partial x^2} + \nu\frac{\partial^3\sigma}{\partial x\partial t^2} = \frac{\partial^2}{\partial t^2}[\mathfrak{h}(\sigma)]$. Existence of local solutions is considered therein.

- In [184] some exact solutions are found for the travelling wave (1D problems), which are found for a special case of (328), namely for the implicit relation $\boldsymbol{\varepsilon}+\nu\dot{\boldsymbol{\varepsilon}} = \mathfrak{g}(\mathbf{T})$. From the equation of motion in 1D and the above implicit relation the following partial differential equation is found

$$\frac{\partial^2\sigma}{\partial x^2} + \nu\frac{\partial^2\sigma}{\partial x^2\partial t} = \frac{\partial^2}{\partial t^2}[\mathfrak{g}(\sigma)].$$

Assuming that $\sigma = \sigma(\xi)$ where $\xi = x - ct$ (see the previous point and also [39] for some problems about that procedure to reduce (329)) the above equation becomes

$$\sigma'' - \nu c\sigma''' = c^2[\mathfrak{g}(\sigma)]'', \tag{330}$$

where $(\quad)' = \frac{\mathrm{d}}{\mathrm{d}\xi}$. For the solution of (330) it is assumed that $\lim_{\xi\to\pm\infty}\sigma(\xi) = \sigma_\infty^\pm$ and that $\mathfrak{g}(\sigma) = \aleph\arctan(\vartheta\sigma)$, where $\aleph$ and ϑ are constants. The solutions are found for some approximations in Taylor series of the above expression for $\mathfrak{g}(\sigma)$, for example: $\mathfrak{g}(\sigma) \approx \sigma - \sigma^2/3$ and $\mathfrak{g}(\sigma) \approx \sigma - \sigma^3/3 + \sigma^5/5$. One of such exact solutions is (in implicit form):

$$\frac{\sigma(\xi)^(1/2)[1-\sigma(\xi)^2]^{1/2}}{[2-3\sigma(\xi)^2]^{3/4}} = Ce^{-\frac{\xi}{13\nu c}}.$$

In [185] a viscoelastic model of the form $\boldsymbol{\varepsilon}+\nu\dot{\boldsymbol{\varepsilon}} = \mathfrak{g}(\mathbf{T})$ is considered for the case $\mathfrak{g}(\sigma) = \beta_0 + \frac{\sigma}{\alpha_0[1+\gamma_0(\sigma^2)^{r/2}]^{1/r}}$, where β_0, α_0, γ_0 and r are material constants. This model and $\mathfrak{g}(\sigma) = \aleph\arctan(\vartheta\sigma)$ are both proposed to obtain limiting strain behaviour. In that paper [185] a review of elastic and visco-elastic relations is also presented.

In [186] starting again with $\boldsymbol{\varepsilon}+\nu\dot{\boldsymbol{\varepsilon}} = \mathfrak{g}(\mathbf{T})$ defining $\omega = \varepsilon+\nu\dot{\varepsilon}$ assuming that $\mathfrak{g}$ has an inverse we can have $\sigma = \mathfrak{h}(\omega)$, where $\mathfrak{h}$ is the inverse of $\mathfrak{g}$. With such inverse Eq. (329) can be rewritten as

$$\frac{\partial^2\omega}{\partial t^2} = \frac{\partial^2}{\partial x^2}[\mathfrak{h}(\omega)] + \nu\frac{\partial^3}{\partial x^2\partial t}[\mathfrak{h}(\omega)]. \tag{331}$$

Letting $\eta(x,t) = \int_{-\infty}^{x}\omega(y,t)\,\mathrm{d}y$ Eq. (331) becomes

$$\frac{\partial^2\eta}{\partial t^2} = \frac{\partial}{\partial x}\left[\mathfrak{h}\left(\frac{\partial\eta}{\partial x}\right)\right] + \nu\frac{\partial^2}{\partial x\partial t}\left[\mathfrak{h}\left(\frac{\partial\eta}{\partial x}\right)\right]. \tag{332}$$

Assuming the existence of a function ϕ such that $\frac{\partial \eta}{\partial t} = \phi$ Eq. (332) becomes $\frac{\partial \phi}{\partial t} + A\phi = G + F$, where $A = 1 - \nu \frac{\partial}{\partial x}\left[\mathfrak{h}'\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial}{\partial x}\right]$, $G = \frac{\partial}{\partial x}\mathfrak{h}\left(\frac{\partial \eta}{\partial x}\right)$ and $F = \frac{\partial \eta}{\partial t}$. In [186] a theorem on global existence of solution for the above problem is provided.

- In [6] an analysis of existence solution for a 1D problem is provided for

$$\boldsymbol{\varepsilon} = \mathfrak{h}(\mathbf{T}) - \gamma \dot{\mathbf{T}},$$

for a 1D problem, where $(329)_1$ is valid, as well as this (recall that $\varepsilon = \frac{\partial u}{\partial x}$)

$$\frac{\partial u}{\partial x} = \mathfrak{h}(\sigma) - \gamma \frac{\partial \sigma}{\partial t}. \tag{333}$$

From $(329)_1$ we get $\rho \frac{\partial^3 u}{\partial x \partial t^2} = \frac{\partial^2 \sigma}{\partial x^2}$ and from (333) we have $\frac{\partial^3 u}{\partial x \partial t^2} = \frac{\partial^2}{\partial t^2}[\mathfrak{h}(\sigma)] - \gamma \frac{\partial^3 \sigma}{\partial t^3}$, then combining the above equations we get

$$\rho \left\{ \frac{\partial^2}{\partial t^2}[\mathfrak{h}(\sigma)] - \gamma \frac{\partial^3 \sigma}{\partial t^3} \right\} = \frac{\partial^2 \sigma}{\partial x^2}.$$

This partial differential equation is analyzed in [6].

- In [70] there is a study of travelling wave solutions for 1D problem for a subclass of (308), wherein the authors consider $\boldsymbol{\mathfrak{F}}(\mathbf{T}, \dot{\mathbf{T}}, \mathbf{B}) = \mathbf{0}$ and in particular the subclass $\mathbf{B} = \boldsymbol{\mathfrak{H}}(\mathbf{T}, \dot{\mathbf{T}})$, which when $\boldsymbol{\mathfrak{H}}$ is isotropic becomes[49]:

$$\begin{aligned}\mathbf{B} = {} & \alpha_0 \mathbf{I} + \alpha_1 \mathbf{T} + \alpha_2 \dot{\mathbf{T}} + \alpha_3 \mathbf{T}^2 + \alpha_4 \dot{\mathbf{T}}^2 + \alpha_5(\mathbf{T}\dot{\mathbf{T}} + \dot{\mathbf{T}}\mathbf{T}) + \alpha_6(\mathbf{T}^2\dot{\mathbf{T}} + \dot{\mathbf{T}}\mathbf{T}^2) \\ & + \alpha_7(\dot{\mathbf{T}}^2\mathbf{T} + \mathbf{T}\dot{\mathbf{T}}^2) + \alpha_8(\mathbf{T}^2\dot{\mathbf{T}}^2 + \dot{\mathbf{T}}^2\mathbf{T}^2). \end{aligned} \tag{334}$$

In [70] the above constitutive equation is studied for a 1D problem (travelling wave solution), and if in such a case the 1D stress is denoted σ, a particular expression for the right side of (334) is considered, namely $\mathfrak{h}(\sigma) - \gamma(\sigma)\dot{\sigma}$. An additional assumption in [70] is that the gradient of the displacement field is small, then if ε denotes the 1D strain from (334) we have

$$\varepsilon = \mathfrak{h}(\sigma) - \gamma(\sigma)\dot{\sigma}. \tag{335}$$

[49] It is necessary to indicate that in [70] there is no analysis if (334) is frame-indifferent, in particular regarding the term $\dot{\mathbf{T}}$, however as shown in this section, in [70] the actual interest is in the analysis of a special 1D boundary value problem, for which that is not an issue.

If u is the 1D displacement, from $\varepsilon = \frac{\partial u}{\partial x}$ and the 1D equation of motion $\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}$ (without body forces and incorporating the density that is assumed to be constant in the definition of u) combining them the following nonlinear partial differential equation is found:

$$\frac{\partial^2 \sigma}{\partial x^2} = \frac{\partial^2}{\partial t^2}[\mathfrak{h}(\sigma)] - \frac{\partial^2}{\partial t^2}\left[\gamma(\sigma)\frac{\partial \sigma}{\partial t}\right].$$

Travelling waves are studied for the above partial differential equation.

- In [75] a constitutive equation for 1D problems for viscoelastic bodies was proposed for the case of large deformations. If the tensor $\mathbf{A}$ is defined as $\mathbf{A} = \boldsymbol{\varepsilon} - \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^{\mathrm{T}}$ then that is equivalent to say $\mathbf{A} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})$.

  The Clausius-Duhem inequality in terms of the complementary energy ϕ_c in the 1D case is $\rho\dot{\phi}_c + \sigma A\frac{\dot{\rho}}{\rho} - \dot{\sigma}A - \rho s\dot{\theta} - \frac{q}{\theta}\frac{\partial \theta}{\partial x} \geqslant 0$, from where we get $A = \rho\frac{\partial \phi_c}{\partial \sigma}(\sigma, \rho)$. In the viscoelastic case the strain A is decomposed as $A = A_e - A_d$, where A_d can be considered as a sort of viscoelastic part of A, where $A_d\dot{\sigma} \geqslant 0$. In [75] it is assumed that $A_d = A_d(\rho, \sigma, \dot{\sigma})$, and in particular the subclass $A_d = \gamma(\rho, \sigma)\dot{\sigma}$ is studied. Considering the above we finally get

  $$A(\rho, \sigma, \dot{\sigma}) = h(\rho, \sigma) - \gamma(\rho, \sigma)\dot{\sigma}, \tag{336}$$

  which is a large deformation counterpart of (335).

  A subclass of (336) has been considered in [75]. Using the equation of motion and (336) it is possible to obtain one differential equation for $\sigma(x, t)$, which is then simplified assuming that $\sigma(x, t) = \sigma(\xi)$, where $\xi = x - ct$. The resultant nonlinear ordinary differential equation is not solved explicitly in that paper.

- In [100] a constitutive equation for a visceolastic solids (showing limiting strains) is presented, and a study of existence of solutions for the boundary value problem for a body with a crack is considered. The model (small strains) is (the stress and strain tensors below can also depend on $\mathbf{x}$):

  $$\boldsymbol{\varepsilon}(t) = \mathfrak{J}(t)\boldsymbol{\mathfrak{F}}(\mathbf{T}(0)) + \int_0^t \mathfrak{J}(t-s)\frac{\mathrm{d}}{\mathrm{d}s}[\boldsymbol{\mathfrak{F}}(\mathbf{T}(s))]\,\mathrm{d}s, \tag{337}$$

where $\mathfrak{J} > 0$ is called the kernel, which in [100] is assumed to be of the form $\mathfrak{J}(t) = \mathfrak{J}(0) + \sum_{n=1}^{N} \mathfrak{J}_n[1 - \exp(-t/\tau_n)]$, where $\mathfrak{J}(0)$, $\mathfrak{J}_1$, $\mathfrak{J}_2$,...,$\mathfrak{J}_N$, τ_1, τ_2,...,$\tau_N \geqslant 0$ are constants (for the generalized creep).

In the case $\mathfrak{J}(t) = 1$ from (337) we obtain $\boldsymbol{\varepsilon}(t) = \boldsymbol{\mathfrak{F}}(\mathbf{T}(t))$ that is a constitutive equation for elastic solids of the type (29).

A model for $\boldsymbol{\mathfrak{F}}$ is proposed for strain limiting behaviour, namely:

$$\boldsymbol{\mathfrak{F}}(\mathbf{T}) = \frac{1}{2\mu} \frac{\mathbf{T}}{(1 + \kappa|\mathbf{T}|^r)^{(2-p)/r}},$$

where κ, μ, $r > 0$ and $1 \leqslant p < \infty$.

- A second constitutive relation for visco-elastic bodies has been proposed by Itou et al. in [105], where the linearized strain is decomposed as

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\mathrm{e}} + \boldsymbol{\varepsilon}_{\mathrm{v}}, \tag{338}$$

where $\boldsymbol{\varepsilon}_{\mathrm{e}}$ would be the elastic part of the strain, and $\boldsymbol{\varepsilon}_{\mathrm{v}}$ would be the viscous part of the strain, where:

$$\boldsymbol{\varepsilon}_{\mathrm{e}} = \mathfrak{f}(\mathbf{T}) = E_1 \mathbf{T}_{\mathrm{D}} + E_3 \mathfrak{b}(\mathrm{tr}\,\mathbf{T})\mathbf{I}, \tag{339}$$

where the function $\mathfrak{b} = \mathfrak{b}(\mathrm{tr}\,\mathbf{T})$ is defined as

$$\mathfrak{b}(\mathrm{tr}\,\mathbf{T}) = \begin{cases} \frac{\mathrm{tr}\,\mathbf{T}}{\underline{\mathrm{M}}} & \text{if} \quad 1 + \lambda_3 \mathrm{tr}\,\mathbf{T} \leqslant \underline{\mathrm{M}} \\ \frac{\mathrm{tr}\,\mathbf{T}}{1+\lambda_3 \mathrm{tr}\,\mathbf{T}} & \text{if} \quad \underline{\mathrm{M}} \leqslant 1 + \lambda_3 \mathrm{tr}\,\mathbf{T} \leqslant \overline{M} \\ \frac{\mathrm{tr}\,\mathbf{T}}{\overline{M}} & \text{if} \quad 1 + \lambda_3 \mathrm{tr}\,\mathbf{T} > \overline{M}, \end{cases}$$

where E_1, E_3, λ_3, $\underline{\mathrm{M}}$ and $\overline{M}$ are material constants. On the other hand for $\boldsymbol{\varepsilon}_{\mathrm{v}}$ it is found that (compare with (337)):

$$\boldsymbol{\varepsilon}_{\mathrm{v}} = \int_0^t \mathfrak{J}'(t-s)\mathfrak{f}(\mathbf{T}(s))\,\mathrm{d}s,$$

where $\mathfrak{J}$ is the kernel. In [105] there is a study on restrictions on the material constants such that there exists a unique solution for some boundary value problems.

- In [104] another implicit constitutive relation for visco-elastic bodies was proposed by Itou et al, similar to what has been presented in (309) (see also (338) and (339)), wherein

$$\alpha_1 \boldsymbol{\varepsilon}_{\mathrm{D}} + \alpha_2 \dot{\boldsymbol{\varepsilon}}_{\mathrm{D}} = \mathbf{T}_{\mathrm{D}}, \quad \frac{1}{a}(1 + b\mathrm{tr}\,\mathbf{T})(\alpha_1 \mathrm{tr}\,\boldsymbol{\varepsilon} + \alpha_2 \mathrm{tr}\,\dot{\boldsymbol{\varepsilon}}) = \mathrm{tr}\,\mathbf{T},$$

  where α_1, α_2, a and b are material constants, and $\boldsymbol{\varepsilon}_{\mathrm{D}}$ is the deviatoric part of the linearized strain tensor.

  The above implicit relation and other generalizations (showing limiting strain behaviour), which for brevity are not shown here, are used to study well-possednes, existence and uniqueness of solution for some boundary value problems.

We finish this section on some models and results for visco-elastic solids, mentioning that in [2] a numerical technique has been proposed, to solve 2D boundary value problems for some classes of implicit visco-elastic solids, considering small strains (small gradient of the displacement field), and the use of a stress potential, which is valid for quasi-static problems.

10.3 Application of implicit constitutive relations for other types of problems

In this section we show two constitutive equations that deal with applications for problems, which are differen to the cases considered in the previous sections.

10.3.1 A constitutive relation for ice creep

In [3] and implicit constitutive relation for fluid-like media, with applications for ice creep, has been proposed which is of the form:

$$\mathfrak{G}(\mathbf{T}, \mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_n) = \mathbf{0}$$

where $\mathbf{A}_i$ are the Rivlin-Ericksen tensors.

The following special subclass is studied in detail:

$$\mathbf{T} = \mathfrak{g}(p, \mathbf{A}_1, \mathbf{A}_2),$$

where $p = -\frac{1}{3}\mathrm{tr}\,\mathbf{T}$. The interested reader can see the references therein for the influence of pressure on the behaviour of ice and viscosity. Other potential applications of the above implicit constitutive relation can be found in the study of volcanic soil and graphite/epoxy composites.

10.3.2 A constitutive equation for heterogeneous elastic solid. A multiscale model

In [213] a constitutive equation of the form (75) is proposed for the development of a multiscale model for the study of heterogeneous elastic solids. The function $\mathfrak{h}(\mathbf{T})$ in (75) is of the form:

$$\mathfrak{h}(\mathbf{T}) = \psi_1(I_1)\mathbf{I} + \psi_2(|\mathbf{T}|)\mathbf{T}, \tag{340}$$

where $I_1 = \operatorname{tr}\mathbf{T}$ and $|\mathbf{T}| = \sqrt{\mathbf{T}\cdot\mathbf{T}}$. The following particular subclass of (340) is considered:

$$\mathfrak{h}(\mathbf{T}) = \frac{\mathscr{K}[\mathbf{T}]}{\left\{1 + \beta^\alpha \left|(\mathscr{K}[\mathbf{T}])^{1/2}\right|^a\right\}^{1/a}}, \tag{341}$$

where $\mathscr{K}[\mathbf{T}]$ is equivalent in index notation and Cartesian coordinates to $\mathscr{K}_{ijkl}T_{kl}$, and where a, α and β are material constants. It is assumed that $\mathscr{K}[\mathbf{T}] = \frac{1}{2\mu}\mathbf{T} - \frac{\lambda}{2\mu\left(\lambda+\frac{2\mu}{d}\right)}(\operatorname{tr}\mathbf{T})\mathbf{I}$, where μ, λ are material constants, d is equal to 2 or 3 depending if we study 2D or 3D problems.

In [213] it is shown that for (341) there exists a positive constant M such that $|\mathfrak{h}(\mathbf{T})| \leqslant M$, $\forall$ $\mathbf{T}$, as well as this, $[\mathfrak{h}(\mathbf{T_1}) - \mathfrak{h}(\mathbf{T_2})]\cdot(\mathbf{T}_1 - \mathbf{T}_2) > 0$, $\forall$, $\mathbf{T}_1$, $\mathbf{T}_2$.

In this paper the above properties for $\mathfrak{h}$ are used to invert (341), to express[50] $\mathbf{T} = \mathfrak{h}^{-1}(\boldsymbol{\varepsilon})$. That inverted expression for the constitutive equation is used for the application to the modelling of 'monolithic materials', wherein we have a composite made of a soft matrix filled with stiff inclusions. The main subdomain is denoted Ω_1, on the other hand Ω_2 denotes the subdomain of the inclusions, which are assumed to be circular. For the material constants μ, λ that appear in the expression for $\mathscr{K}$ above are assumed to be given as:

$$\mu(\mathbf{x}) = \frac{E(\mathbf{x})}{2(1+\nu)}, \quad \lambda(\mathbf{x}) = \frac{\nu E(\mathbf{x})}{(1+\nu)(1-2\nu)},$$

where $E(\mathbf{x}) = \begin{cases} E_1 & \mathbf{x}\in\Omega_1 \\ E_2 & \mathbf{x}\in\Omega_2 \end{cases}$, and E_1 and E_2 are the 'ground' Young's modulus for the main domain and the subdomain, respectively. In that paper a study of existence and uniqueness of solution is provided as well.

[50]This inversion is only a mathematical device and from the physical point of view this cannot be done, as explained in [169].

11 Final remarks

In this review, most of the papers on implicit constitutive relations and their subclasses for solid media have been reviewed in detail, which have been published from the year 2003 until 2024. The aim of this work is to provide to the potential readers, not only a detailed account of all such communications, but also a work where such papers are organized in a simple manner, such that it is possible to understand the relations between all the results presented in those communications.

There are many open problems in this area of nonlinear mechanics, and a non-complete rather short list of them is given here. Although in some of the works reviewed here, it is possible to find some studies on existence of solutions for some specific boundary value problems, there is no general treatment on such issue for any of these new constitutive relations and equations. The development of a variational formulation for such relatively new constitutive theories may be useful for such research.

Even though for one dimensional problems there are clear potential applications for the implicit constitutive relations for the modelling of soft tissue, no such application has been explored for three-dimensional problems. This is because of the difficulties of obtaining such relations in terms of the many invariants for such problems. That is also an open problem.

In the case of stress concentration, so far no exact or semi-analytical solution has been found for the problems of a plate with a circular or elliptical hole, or the case of a plate with a crack in mode I, under the application of a uniform tension applied far away. If it would be possible to find such solutions, it could be possible to study the concept of stress concentration for some of such constitutive equations, in particular the cases wherein we have strain limiting behaviour.

For the problem of modelling inelastic deformations, there are still many potential applications that should be considered, for soft solids (considering large deformations), and for electro-active and magneto-active solids, wherein we can have not only inelastic deformations, but also electrical and magnetic hysteresis and energy dissipation (considering in particular small strains).

Finally, for many of the applications presented in this review, a future important work to do is the use of machine learning and similar tools (along with the finite element method), for the determination of constitutive constants that appear in the different models presented here, see for example, [121, 216, 217] and the references therein.

12 Appendix

The development of constitutive theories for electro-elastic and magneto-elastic bodies is not simple, since it requires some knowledge of electromagnetism, and about how electromagnetic fields should appear in the constitutive relations. Such a brief introduction is shown in this appendix. Let us define $\mathbb{E}$, $\mathbb{D}$ and $\mathbb{P}$ as the electric field, electric displacement and polarization, respectively, and $\mathbb{H}$, $\mathbb{B}$ and $\mathbb{M}$ as the magnetic field, magnetic induction and magnetization, respectively.

Simplified forms for the Maxwell equations

For simplicity let us consider only problems where there is no coupling or interaction between the magnetic and electric fields, and there is no distributed charges and no time dependence[51]. In such a case for electro-elastic and magneto-elastic bodies the simplified forms of the Maxwell equations are [115]

$$\text{curl}\mathbb{E} = \mathbf{0}, \quad \text{div}\mathbb{D} = 0, \quad \text{curl}\mathbb{H} = \mathbf{0}, \quad \text{div}\mathbb{B} = 0, \tag{342}$$

where curl is the curl operator in the current configuration, and for condensed matter we have $\mathbb{P} = \mathbb{D} - \epsilon_o \mathbb{E}$ and for vacuum $\mathbb{D} = \epsilon_o \mathbb{E}$, where ϵ_o is the electric permittivity for free space or vacuum, where for condensed matter we have $\mathbb{B} = \mu_o(\mathbb{H} + \mathbb{M})$ and for vacuum $\mathbb{B} = \mu_o \mathbb{H}$, where μ_o is the magnetic permeability for free space or vacuum.

Boundary or Continuity Conditions:

In the mathematical modelling of the behaviour of electro- and magneto-elastic bodies, it is necessary to model not only the bodies but also the surrounding space, which we assume is vacuum. In such a case for electro-elastic bodies the electric field and electric displacement satisfy the continuity conditions [115] $\mathbf{n} \times [[\mathbb{E}]] = \mathbf{0}$, $\mathbf{n} \cdot [[\mathbb{D}]] = 0$, and for magneto-elastic bodies $\mathbf{n} \times [[\mathbb{H}]] = \mathbf{0}$, $\mathbf{n} \cdot [[\mathbb{B}]] = 0$, where $[[\mathbf{a}]] = \mathbf{a}^{(\mathrm{o})} - \mathbf{a}^{(\mathrm{i})}$ is the difference of a

[51] There is one work (within the context of classical constitutive equations), where the full form of the Maxwell equations and time-effects has been considered [197]. Since the formulation presented there is far from simple, we do not show that here. In this review we only consider separately electro-elastic and magneto-elastic interactions. In [48] for quasi-static electromagnetic fields a constitutive equation has been proposed, but in [197] and also in [48] only the classical constitutive equations, wherein the stress is a function of the strains are analyzed, which is another reason they are not shown in this review.

quantity between the outside and inside of a body at the boundary and $\mathbf{n}$ is the outer normal vector to the body in the current configuration.

Equations of Equilibrium

When we assume $\ddot{\mathbf{x}} = \mathbf{0}$ the equation of equilibrium for electro- and magneto-elastic bodies, when $\mathbb{P}$ is considered as the independent electric variable, and $\mathbb{M}$ is chosen as the independent magnetic variable, respectively, take the form (here the notation $\boldsymbol{\sigma}$ is used to denote the Cauchy stress tensor)

$$\mathrm{div}\boldsymbol{\sigma} + (\mathrm{grad}\mathbb{E})^{\mathrm{T}}\mathbb{P} + \rho\mathbf{b} = \mathbf{0}, \quad \mathrm{div}\boldsymbol{\sigma} + \mu_o^{-1}(\mathrm{grad}\mathbb{B})^{\mathrm{T}}\mathbb{M} + \rho\mathbf{b} = \mathbf{0}, \qquad (343)$$

where in both cases $\mathbf{b}$ represents the non-electric and non-magnetic body forces and $\boldsymbol{\sigma}$ is in general non-symmetric. The terms $(\mathrm{grad}\mathbb{E})^{\mathrm{T}}\mathbb{P}$ and $\mu_o^{-1}(\mathrm{grad}\mathbb{B})^{\mathrm{T}}\mathbb{M}$ can be incorporated into the $\mathrm{div}\boldsymbol{\sigma}$ term, on the basis of which we can define a quantity called the total stress tensor $\boldsymbol{\tau}$ (not to be confused with the Kirchhoff stress tensor), and (343) becomes (see, for example, [68, 69])

$$\mathrm{div}\boldsymbol{\tau} + \rho\mathbf{b} = \mathbf{0}. \qquad (344)$$

When we use the total stress tensor, the mechanical boundary conditions are $\boldsymbol{\tau}\mathbf{n} = \hat{\mathbf{t}} + \boldsymbol{\tau}_{\mathrm{m}}\mathbf{n}$, where $\boldsymbol{\tau}_{\mathrm{m}}$ is the Maxwell stress tensor, which for electro-elastic problems takes the form $\boldsymbol{\tau}_{\mathrm{m}} = \mathbb{D}^{(\mathrm{o})} \otimes \mathbb{E}^{(\mathrm{o})} - \frac{1}{2}(\mathbb{D}^{(\mathrm{o})} \cdot \mathbb{E}^{(\mathrm{o})})\mathbf{I}$, and for magneto-elastic problems the Maxwell stress is given as $\boldsymbol{\tau}_{\mathrm{m}} = \mathbb{B}^{(\mathrm{o})} \otimes \mathbb{H}^{(\mathrm{o})} - \frac{1}{2}(\mathbb{B}^{(\mathrm{o})} \cdot \mathbb{H}^{(\mathrm{o})})\mathbf{I}$, and in both cases $\hat{\mathbf{t}}$ represents the external non-electric and non-magnetic traction.

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