# Lorenz Pendulum and Poincaré chaos

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**Abstract:**

A simple mechanical system which exhibits Lorenz chaos is studied both theoretically and experimentally. Such a system is shown to present a generic bifurcation of a quasi-reversible system whose amplitude equation are equivalents to the Lorenz model. The experimental realization of such a pendulum exhibited a very simple example of homoclinic Poincaré chaos, which arises because of some imperfection of the apparatus.

**Résumé:**

Nous étudions, théoriquement et expérimentalement un système mécanique simple qui met en évidence des comportements chaotiques du type de ceux observés dans le modèle de Lorenz. En fait nous montrons que ce système présente une bifurcation quasi-reversible générique dont les équations d'amplitudes sont équivalentes aux équations de Lorenz. En faisant l'expérience, nous avons observé des comportements chaotiques d'une nature différente, liés à une imperfection de l'appareillage utilisé. Ceci nous a permis de mettre en évidence une manifestation très simple et pédagogique du chaos homocline tel que Poincaré l'a décrit.

The Lorenz model of the dynamics of thermal convection is famous because it displays chaotic behaviors which have been extensively studied in the 80's. Lorenz chaos has never been observed in convection, mainly because the model was based upon a drastic truncation of the Boussinesq equations [1, 2, 3]. Later, the Maxwell-Bloch equation describing the interaction of an assembly of two level atoms and an electromagnetic field have been shown to be identical to the Lorenz equation [5], when the detuning between a cavity and electromagnetic mode vanishes. Several attempts to observe Lorenz chaos in lasers have been done, but the experimental

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ifficulty there is to achieve the appropriate parameter ranges and in particular o make zero the detuning. Although Lorenz equations were also derived using symptotic analysis for a large class of dispersive instabilities [8], including the aroclinic instabilities [4], to our knowledge, no experiments in this context have een performed. The only experiment we are aware of was performed by W. Malkus [2]. It consists in a leaky waterwheel apparatus whose modelization leads o the Lorenz equations. In this short paper, we intend to describe a very simple nechanical system, a pendulum, whose dynamics is indeed described by the Lorenz nodel. The idea of such a system comes from a general theory of bifurcations in quasi-reversible systems [9], in which the Lorenz equations appear as the amplitude equations of a generic instability. A surprising chaotic behavior was observed while we were constructing our first experimental realization of the pendulum. The behavior observed arises because of an experimental imperfection which turns out o be a very interesting example of how the lost of a conserved quantity can lead o homoclinic Poincaré chaos [6].

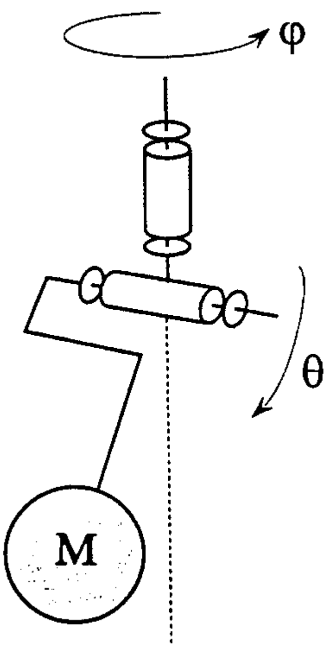


Figure 1: Sketch of the rotating pendulum.

One of the simple example of a pitchfork bifurcation is given by the rotating pendulum (see Fig. 1). When the angular velocity $\Omega_0$ of the pendulum exceeds a critical one, the stationary vertical position of the mass looses its stability, when the centrifugal force overcomes the gravity. The dimensionless equation describing this system, in the absence of viscous damping is given by

$$\frac{\partial^2\theta}{\partial t^2} + \sin(\theta)(1 - \Omega_0^2\cos(\theta)) = 0,$$

where the time scale has been chosen as the period of the small oscillations of the pendulum $T = \sqrt{l/g}$, where $g$ represents the gravity and $l$ the length of the pendulum. The instability sets in when $\Omega_0 > 1$. Close to the instability, this equation reduces to

$$\frac{\partial^2 A}{\partial t^2} + \varepsilon A - A^3 = 0$$

where $\epsilon \equiv \Omega_0^2 - 1$, $A \equiv \sqrt{2}\theta$.

This integrable Hamiltonian system possesses periodic solutions, a double homoclinic solution and two stable fixed points $A_s = \pm\sqrt{\varepsilon}$ (see Fig. 2).

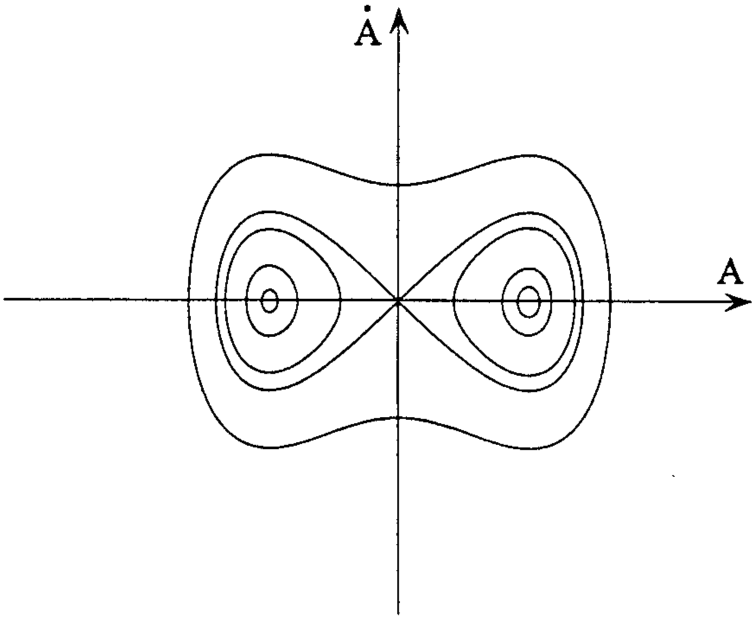


Figure 2: Phase portrait of the small oscillations of the rotating pendulum.

The mechanical pendulum considered so far is indeed a two degree of freedom system. It is described by two angles $\theta$ and $\phi$ and the two angular velocities $\dot{\theta}$ and $\Omega = \dot{\phi}$.

The equations of the motion are given by

$$\frac{\partial^2 \theta}{\partial t^2} + \sin(\theta) - \frac{\sin(2\theta)}{2\left(I + \sin(\theta)^2\right)^2} P^2 = 0\,,$$

$$\frac{dP}{dt} = 0\,,$$

where $I$ represents the inertia ratio of the rotating body and $P = (I + \sin(\theta)^2)\Omega$ is the total angular momentum with respect to the vertical axis which is of course conserved. The instability corresponds to a pitchfork bifurcation, in the presence of a neutral mode associated with the conservation of the vertical component of the angular momentum. For each value of the conserved quantity, the problem reduces to the single degree of freedom Hamiltonian system. The dissipation enters in the problem in two different ways. First, the viscous damping of the fluid in which the pendulum is swinging has to be included. The Stokes law has been used to model this damping. It has both an effect on the oscillation and the rotation. Second, the damping of the rotation axis is assumed to be viscous, i.e. proportional to the angular velocity. The "energy" is injected in the system through a constant torque $\tau$. With these assumptions, the mechanical system is described by the following equations

$$\frac{\partial^2\theta}{\partial t^2} + \sin(\theta) - \frac{\sin(2\theta)}{2}\Omega^2 + \nu\,\dot{\theta} = 0\,,$$
$$\frac{dP}{dt} = -\mu(\Omega - \Omega_0) - \nu\Omega\sin(\theta)^2\,,$$

where $\mu$ and $\nu$ represents the two damping constants. It is important to note that the dissipative forced dynamical system is three dimensional while the reversible system is only two dimensional for each value of the conserved angular momentum. In particular the Stokes damping provides a non trivial feedback of the oscillation on the rotation. Close to the instability ($\Omega_0 \sim 1$), these equations reduces to the amplitude equations

$$\ddot{A} + \varepsilon A + \nu\dot{A} + AB - A^3 = 0\,,$$
$$\dot{B} = -\tilde{\mu}B + \eta A^2\,,$$

where

$$\varepsilon = 2(\Omega_0 - 1)\,, \qquad \tilde{\mu} = \frac{\mu}{I}\,, \qquad \eta = \frac{4(\nu - \mu)}{I + 4}\,,$$

and

$$B = -2(\Omega - 1) - \frac{2}{I}\theta^2\,, \qquad A = \sqrt{\frac{I+4}{2I}}\,\theta\,.$$

The numerical simulation of the full equations (see Fig. 3), close to the instability threshold, displays behaviors which are reminiscent of the Lorenz chaos.

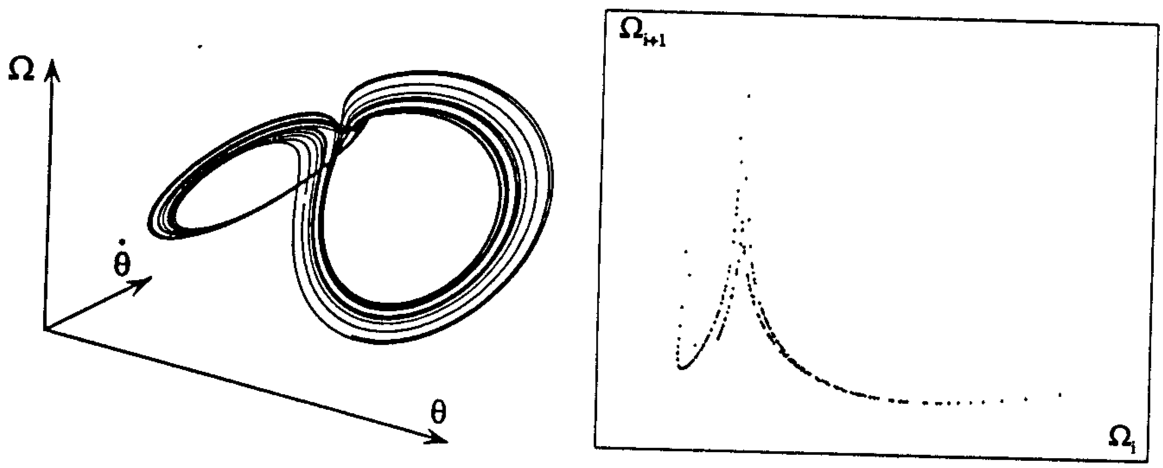


Figure 3: (a) Phase portrait of forced-dissipative rotating pendulum. (b) Corresponding Lorenz map.

Actually a simple change of variables allows one to reduce the following amplitude equations to the Lorenz model:

$$\begin{aligned} \partial_t x &= \sigma(y-x), \\ \partial_t y &= Rx - y - xz, \\ \partial_t z &= -bz + xy, \end{aligned}$$

with

$$A = \frac{\epsilon}{\sqrt{\sigma}}x, \qquad \dot{A} = \frac{\epsilon^2}{\sqrt{2}}(y-x), \qquad B = \epsilon^2\left(z - \frac{x^2}{2\sigma}\right),$$

where $t \equiv (\epsilon/\sqrt{\sigma})\,t$, $\nu = \epsilon\,(\sigma+1)/\sigma$, $\bar{\mu} = \epsilon\, b/\sqrt{\sigma}$, $\eta = \epsilon\,(2\sigma - b)/\sqrt{\sigma}$, and $\epsilon = \sqrt{r-1}$.

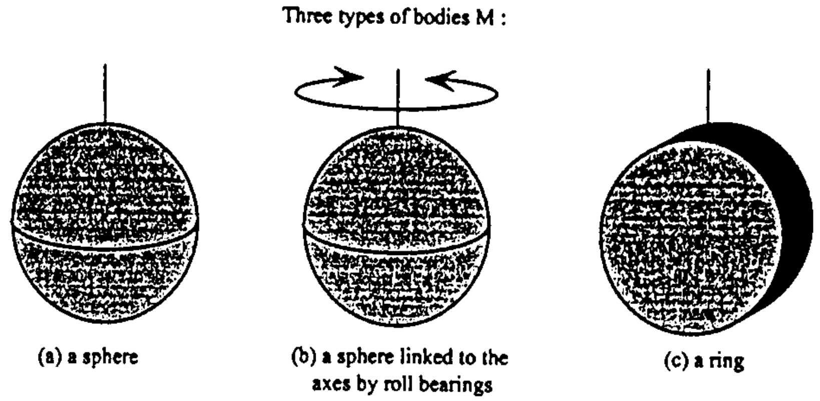


Figure 4: Type of bodies used in the experiment. $M$ refers to the body shown in figure 1.

We have built a preliminary experiment with two perpendicular, low friction, ball bearing axis (see Fig. 1). A constant torque is applied on the vertical axe by using a simple DC motor with constant current intensity control. The rotation speed $\Omega$ is measured with a tachometer dynamo delivering a DC voltage proportional to it. The time dependent voltage proportional to $\Omega$ is directly analyzed on a computer. The Poincaré map experimentally constructed is the famous map introduced by Lorenz by plotting the successive maximas of $\Omega$. It will be named the Lorenz map thereafter. As seen in figure Fig. 4, three different massive bodies M are used:

- In case (a), only metastable chaotic-like behavior is qualitatively observed.
- In case (b), the massive body is immersed in a viscous liquid and a ball bearing is introduced in order to suppress the local rotation of the massive body. This allows the change of the damping coefficient ratio $\mu/\nu$: in that case a chaotic like behavior is observed but the Lorenz map extracted remain very noisy.

- In case (c), an asymmetrical damping is introduced by the ring shape of the massive body and the Lorenz map (see Fig. 5) is obtained: here, the model equations must be slightly modified, giving rise to a larger space parameter domain undergoing chaotic behavior, as supported by simulations and theory.

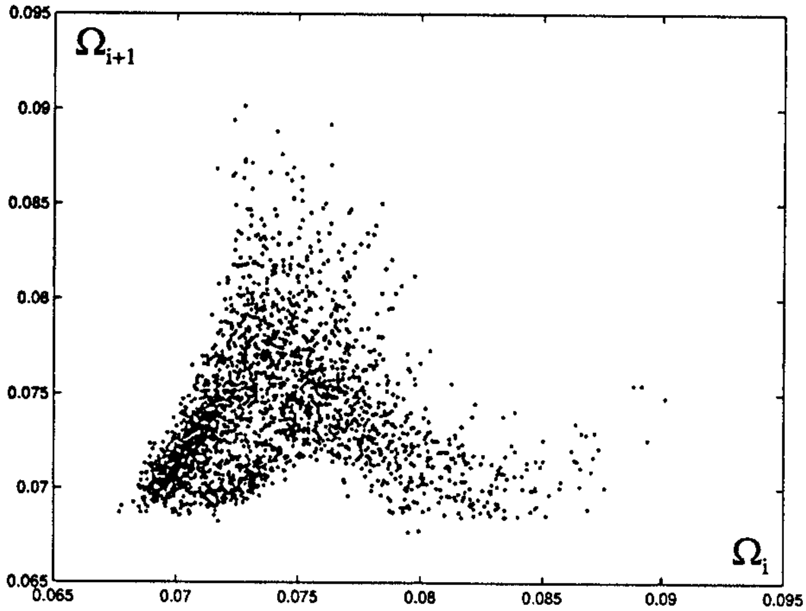


Figure 5: Experimental Lorenz map.

In the first experimental realization of the rotating pendulum, the axis of rotation had a slight tilt from vertical. Very surprising complicated behaviors were then observed even without an external torque. They have been interpreted in the frame of conservative dynamical systems. As we have mentioned above, the full system is a two degrees of freedom Hamiltonian system whose integrability is a consequence of the existence of two conserved quantities: the energy and the vertical component of the angular momentum. The later being directly related to the axial symmetry of the mechanical system. Any small tilt breaks this invariance and the associated conserved quantity is lost. The non-integrability generally manifests itself by the existence of chaotic behavior, as first shown by H. Poincaré for the three bodies problem. Our inclined pendulum is in some sense one of the simplest manifestation of the chaotic homoclinic Poincaré behavior. The equation of motion of the tilted spherical pendulum are given by

$$\begin{aligned}\ddot{\theta} &= -(\sin\theta\cos\alpha + \sin\alpha\sin\phi\cos\theta) + \sin\theta\cos\theta\,\dot{\phi}^2\,, \\ \left(I + \sin^2\theta\right)\ddot{\phi} &= -2\sin\theta\cos\theta\,\dot{\phi}\,\dot{\theta} + \kappa\sin\alpha\sin\theta\cos\phi\,.\end{aligned}$$

One simple way to analyze the chaotic behaviors associated is to consider the limit of very small tilt. For parameters and initial values where the pendulum is rotating, the full equations can be reduced to the classical periodically forced Duffing equations, which are known to exhibit homoclinic Poincaré chaos [7]:

$$\ddot{x} = \left(\Omega^2 - 1\right) x - x^3 + \gamma \sin(\Omega(t - t_0)),$$

where $P/I = \Omega > 1$, $x = \theta\sqrt{\frac{P^2}{I^2}\left(\frac{2}{I} + \frac{2}{3}\right) - \frac{1}{6}}$, and $\gamma = \alpha\sqrt{\Omega^2\left(\frac{2}{I} + \frac{2}{3}\right) - \frac{1}{6}}$.

Coming back to the dissipative-forced system, the presence of a tilt increases the dimension of the dynamical system from three to four. Consequently, the dimension of the attractor can be higher. A Lorenz map of the slightly tilted dissipative forced pendulum exhibits a thickness which is consistent with the increase of the Lorenz attractor dimension by one (see Fig. 6). This thickness could well be at the origin of the thickness of the experimental Lorenz map.

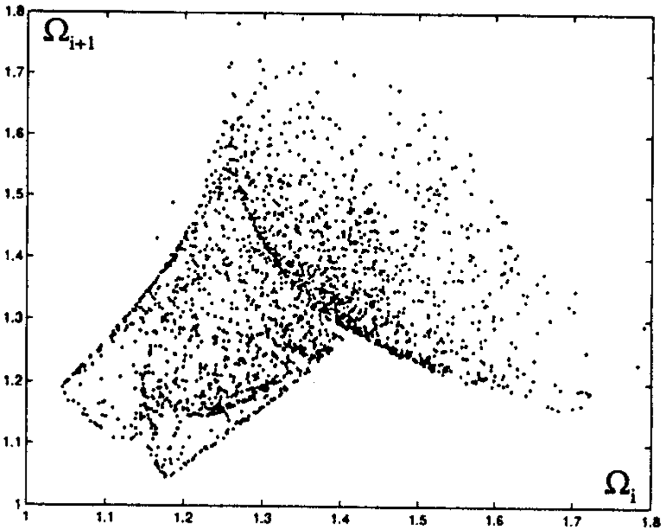


Figure 6: Lorenz map of the tilted dissipative-forced rotating pendulum.

The pendulum we have been considering has strong similarities with the flyball governer which has been used for speed control. This instrument was used by G. B. Airy in the nineteenth century[10] to observe fixed star for extended periods and he remarked that the mechanism was not always stable. In his own words: "... and the machine (if I may so express myself) became perfectly wild". If the motion observed by Airy was indeed chaotic, we believe that our analysis gives a direct interpretation.

As a matter of fact this simple problem illustrates well what Pierre Bergé always stressed to us: the irreducible relevance of experiments in Physics. In other words, without attempting to build the pendulum, we will have never introduced the small tilt, which is at the origin of a very pedagogical example of Poincaré's chaos.

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