Strong Nonlocal Coupling Stabilizes Localized Structures: An Analysis Based on Front Dynamics

C. Fernandez-Oto,[1] M. G. Clerc,[2] D. Escaff,[3] and M. Tlidi[1]

[1]*Faculté des Sciences, Université Libre de Bruxelles (U.L.B.), CP 231, Campus Plaine, B-1050 Bruxelles, Belgium*
[2]*Departamento de Física, Universidad de Chile, Blanco Encalada 2008, Santiago, Chile*
[3]*Complex Systems Group, Facultad de Ingeniería y Ciencias Aplicadas, Universidad de los Andes, Avenida San Carlos de Apoquindo 2200, Santiago, Chile*

(Received 21 February 2013; published 22 April 2013)

We investigate the effect of strong nonlocal coupling in bistable spatially extended systems by using a Lorentzian-like kernel. This effect through front interaction drastically alters the space-time dynamics of bistable systems by stabilizing localized structures in one and two dimensions, and by affecting the kinetics law governing their behavior with respect to weak nonlocal and local coupling. We derive an analytical formula for the front interaction law and show that the kinetics governing the formation of localized structures obeys a law inversely proportional to their size to some power. To illustrate this mechanism, we consider two systems, the Nagumo model describing population dynamics and nonlinear optics model describing a ring cavity filled with a left-handed material. Numerical solutions of the governing equations are in close agreement with analytical predictions.

DOI: 10.1103/PhysRevLett.110.174101 PACS numbers: 05.45.−a, 42.65.Hw, 42.65.Pc, 87.23.Cc

The emergence of localized structures (LSs), often called dissipative solitons or cavity solitons, has attracted considerable attention in many areas of natural science, such as chemistry, physics, plant ecology, and optics [1,2]. They attract growing interest in optics due to potential applications for all-optical control of light, optical storage, and information processing [3]. These stable solutions arise in a dissipative environment and belong to the class of dissipative structures found far from equilibrium [4]. In most cases, the spatial coupling is local for which transport processes like diffraction, dispersion, or diffusion are described by the Laplace operator. The coupling between this process and nonlinearity in dissipative environment, leads to a self-organization phenomenon that is responsible for the formation of either extended or localized patterns. This behavior also occurs in many natural dissipative systems with nonlocal coupling such as firing of cells [5,6], propagation of infectious diseases [7], chemical reactions [8], population dynamics [9,10], nonlinear optic [11], granular [12], neural science [13], and vegetation patterns [14–17]. This issue has been abundantly discussed and is by now fairly well understood. So far, however, far from any pattern forming instability, fronts dynamics leading to formation of LSs in these systems has received only scant attention [9,18]. These nonlinear waves can be seen as a solution that connects two stable steady states spatially. They are characterized by a continuous parameters—the front position—which accounts for the position of the largest spatial variation of the front.

Nonlocal functions, also known as influence or kernel functions, can be classified into two types, depending on whether this function decays asymptotically to infinity faster or slower than an exponential function, which correspond to a weak or strong influence function, respectively [19]. Front interaction is usually characterized by the behavior of the tail of one front around the position of the other front. This situation corresponds to the case of weak influence functions. However, for strong influence functions, the interaction is controlled by the whole influence function and not only by the asymptotic behavior of the fronts, front tails. More importantly, the nature of the interaction between fronts is affected by the choose of the influence function. When considering a weak nonlocal coupling, the asymptotic behavior of front solutions is characterized by either exponential decay or damping oscillation. In the former case, front interaction is always attractive and decays exponentially with the distance between the fronts. Therefore, bound states resulting from fronts interaction are unstable. However, in the case of damping oscillation, fronts interaction alternates between attractive and repulsive with an intensity that decays exponentially with the distance between fronts [9,18]. For a fixed value of parameters, a family of stable one dimensional localized structures with different sizes has been reported by these authors.

In this Letter, we show that front dynamics mediated by strong nonlocal coupling like Lorentzian type of kernel in bistable media leads the formation of stable localized structures. Indeed, strong nonlocal interaction could induce a repulsion between fronts, which decreases with the front separation. A balance between this interaction and tendency of an uniform state to invade the other one at constant speed is responsible for stabilizing localized states. Thus, a deformation of this domain returns to its equilibrium size. These structures have a fixed intrinsic width for a fixed value of parameters. A single localized structure possesses a fringe and a plateau. We show also that the kinetic laws governing front interaction obey a power law. To illustrate this mechanism, we consider two different models describing the population dynamics and cavity nonlinear optics. This mechanism is robust in one and in two dimensions. The generality of our analysis suggest that the strong nonlocal coupling leading to the stabilization of localized structures is a universal phenomenon which does not depend on a specific type of model equation. Our theoretical prediction should therefore be applicable to any spatially extended systems with strong nonlocal coupling.

Our analysis is based on the well-known nonlocal Nagumo equation that constitutes a prototype model to describe a population dynamics. The space-time dynamics evolution of the population density is modeled by the following integro-differential type of equation [9,10]

$$\partial_t u = u(\alpha - u)(u-1) + \nabla^2 u + \varepsilon u \int_{-\infty}^{\infty} u^2(\mathbf{r}+\mathbf{r}', t) K(\mathbf{r}') d\mathbf{r}', \tag{1}$$

where $u(\mathbf{r}, \mathbf{t})$ is a scalar field. α is a parameter related with the adversity of the environment that satisfies $0 < \alpha < 1$. The Laplace operator $\nabla^2 = \partial_{xx} + \partial_{yy}$ acts in the $\mathbf{r} = (x, y)$ plane, and t is time. The influence or the kernel function is $K(\mathbf{r}) = \delta(\mathbf{r}) - f_\sigma(\mathbf{r})$, with $\delta(\mathbf{r})$ is the delta function and

$$f_\sigma(\mathbf{r}) = \frac{N_n}{[1 + (|\mathbf{r}|/\sigma)^2]^n} \tag{2}$$

is a Lorentzian kernel which accounts for the interaction of the field with their environment, σ is the characteristic length of the nonlocal interaction, ε measures the nonlocal interaction intensity, n is the power of the Lorentzian that describes how the nonlocal interaction decays with the distance, and N_n is a normalization constant. For the sake of simplicity, hereinafter, we will treat this intensity as small parameter ($\varepsilon \ll 1$).

Another system which produces bistability and a nonlocal coupling is a ring cavity filled with a slab of a right-handed material and a slab of a left-handed material. Both layers have been assumed to be nonlinear Kerr media. This cavity is driven by an external coherent laser beam. Assuming the mean field approximation, the space-time evolution of the intracavity field is described by the following Lugiato-Lefever model [20] with nonlocal interaction [11]

$$\partial_t E = E_i - (1 + i\theta)E + i|E|^2 E + iD\nabla^2 E - i\gamma \int_{-\infty}^{\infty} E(\mathbf{r}+\mathbf{r}', t) K(\mathbf{r}') d\mathbf{r}', \tag{3}$$

where E is the normalized slowly varying complex envelope of the electric field. The input field amplitude E_i is real and constant. The detuning parameter is θ. The diffraction coefficient is denoted by D. The kernel function f_σ defined by Eq. (2), describes the nonlocal response of the linear left-handed material, effectively couples the electric field at different positions. The parameter γ measures the nonlocal interaction intensity.

Model Eq. (1) admits three homogeneous steady states: the unpupulated $u_s = 0$, the uniformly populated $u_s = 1$ states, and the unstable state $u_s = \alpha$. The uniformly populated state can undergo a Turing type of instability leading to the formation of both periodic and localized structures [9]. In what follows, we focus on a modulationally stable regime, i.e., a regime far from any Turing instability [21]. In this regime, Eq. (1) admits front solutions connecting the two stable homogeneous steady states $u_s = 0$ and $u_s = 1$. The Maxwell point, i.e., the point of the parameter space where both solutions are equally stable [22,23], corresponds to $\alpha = \alpha_M = 1/2$. At this point, front solutions are motionless. Numerical simulations of Eq. (1) show a stable single front connecting the two stable homogeneous steady states as illustrated in Fig. 1. Near the front position, and for $\varepsilon > 0$, the spatial profile exhibits a fringe corresponding to a peak of population density as shown in Fig. 1.

As we shall see, the strong nonlocal coupling drastically modified the nature of interaction between fronts with respect to weak nonlocal and local coupling. To simplify further the analysis, we consider one dimensional setting and we assume that front positions are located, respectively, at the points $-\Delta/2$ and $\Delta/2$ along the x direction, where Δ is the distance between the two front positions [see Fig. 2(a)]. To investigate the interaction, we add a small perturbation to the linear superposition of the two front solutions as

$$u = u_0(x + \Delta/2) + u_0(-x + \Delta/2) + \varepsilon\phi(x, \Delta(t)), \tag{4}$$

where $u_0(x) = [1 + \tanh(x/2\sqrt{2})]/2$ is the well-known motionless front solution for $\varepsilon = 0$ at the Maxwell point. We assume that the distance between the two fronts evolves on the slow time scale $\Delta = \Delta(\eta t)$, where η is a small parameter which measures the distance from the Maxwell point, i.e., $\eta \equiv \alpha_M - \alpha$. Inserting Eq. (4) in the

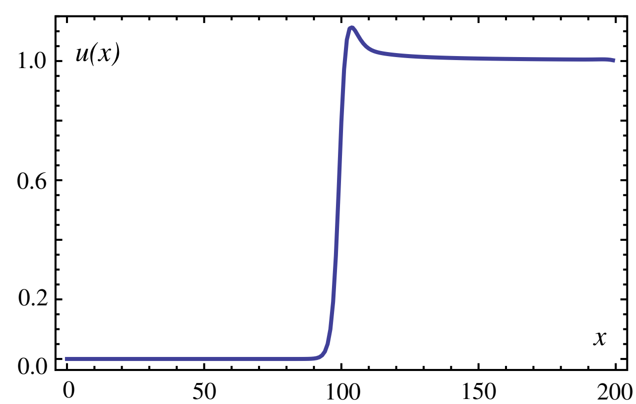

FIG. 1 (color online). Front profile obtained by numerical simulation of model Eq. (1) for $\alpha = 0.5$, $\sigma = 2$, $\varepsilon = 0.35$, and $n = 1$.

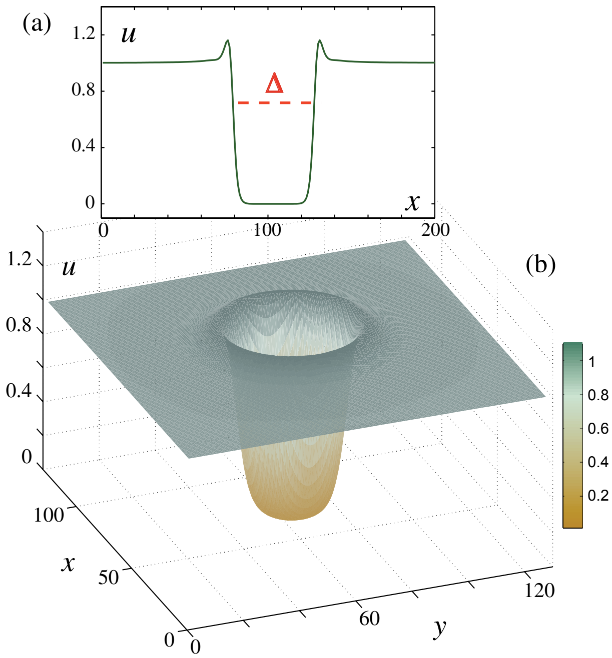

FIG. 2 (color online). Stable localized structures obtained by numerical simulations of Eq. (1) with the Lorentzian type kernel Eq. (2), using a suitable pseudospectral method. (a) One-dimensional localized state observed for $\alpha = 0.492$, $\sigma = 0.7$, $\varepsilon = 1.0$, and $n = 1.0$. Δ accounts for the distance between the fronts. (b) Two-dimensional localized state observed for $\alpha = 0.38$, $\sigma = 0.2$, $\varepsilon = 1.0$, and $n = 1.1$.

Eq. (1), assuming $\eta \sim \varepsilon/\Delta^{(2n-1)} \sim \dot{\Delta} \ll 1$ and $\Delta \gg 1$, expanding in terms of small parameter η and by applying the solvability condition at the dominant order, we obtain the following front interaction law

$$\dot{\Delta} = \varepsilon c_n \left(\frac{\sigma}{\Delta}\right)^{2n-1} - 2\sqrt{2}\eta, \tag{5}$$

where c_n is a numerical constant (for instance, $c_1 = 6\sqrt{2}/\pi$). The first term of the right-hand side of this equation shows that depending on the sign ε, the interaction between fronts can be either attractive ($\varepsilon < 0$) or repulsive ($\varepsilon > 0$). This term is inversely proportional to the distance Δ to the power $2n - 1$. Note, however, that this term is originated from the strong nonlocal coupling mediated by the Lorentzian type of function. In the case of week nonlocal [9] or even local coupling [24], this term does not exist, and the interaction is rather governed by an exponential law; i.e., $\dot{\Delta} \propto -\exp(-\mu\Delta)$ where μ is a constant that characterizes the asymptotic behavior of the front tail. With a weak nonlocal or a local coupling, the interaction between fronts is always attractive, and therefore, localized structures are not stable. The second term in Eq. (5) accounts for the shift from the Maxwell point which expresses the tendency of fronts to propagate at the most favorable state with constant speed. From the front interaction law, Eq. (5), we can infer the existence of a single-length stable localized state for $\varepsilon > 0$ and $\alpha < \alpha_M$. The equilibrium size is $\Delta_{\rm eq}$. The linear stability analysis shows that this equilibrium is stable, with the eigenvalue equal to $-\varepsilon c_n(2n-1)\sigma^{(2n-1)}/\Delta_{\rm eq}^{2n} < 0$. As an example, for $n = 1$, the equilibrium length size is

$$\Delta_{\rm eq} = \frac{3\varepsilon\sigma}{\pi(\alpha_M - \alpha)}. \tag{6}$$

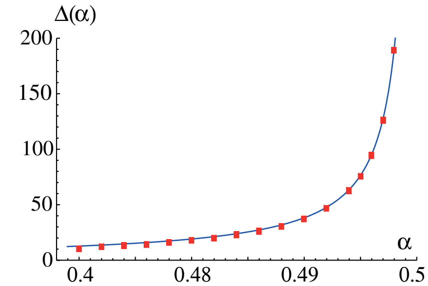

FIG. 3 (color online). Localized structures size as a function of adversity. The points are obtained from numerical simulations of model Eq. (1) with the Lorentzian type kernel Eq. (2) with $\sigma = 0.8$, $\varepsilon = 0.5$, and $n = 1.0$. The solid curve is obtained using formula Eq. (6), which has no adjustable parameters.

The equilibrium length resulting from the strong interaction mediated between fronts by strong nonlocal coupling is plotted in Fig. 3 as a function of the parameter α. Numerical simulations of the model Eq. (1) are in perfect agreement with the one obtained from analytical predictions. This comparison is shown in Fig. 3. Localized structure resulting for nonlocal coupling posses a width Δ that significantly increases with the increase of the parameter α as shown in Fig. 3. In a two dimensional setting, the same behavior has been obtained numerically as shown in Fig. 2(b).

In order to illustrate the above described mechanism leading to the stabilization of localized structures out of the pattern forming regime, we instigate numerically the so called Lugiato-Lefever model, Eq. (3), with strong nonlocal coupling with a Lorentzian type kernel Eq. (2). In the absence of nonlocal coupling, this model exhibits both periodic [20,25] and localized structures [26] even in the monostable regime resulting from a subcritical pattern forming regime. Localized structures are not necessarily stationary, they could undergo a self-pulsating behavior [27]. In what follows, we focus on a regime far from any pattern forming instability. In addition, we consider the system that operates in the negative diffraction regime by placing a nonlinear left-handed material in an optical cavity together with a traditional Kerr nonlinear material. This configuration allows engineering the diffraction strength [28]. Considering the negative diffraction $D < 0$ and bistable regime $\theta > \sqrt{3}$, numerical simulations of the model Eq. (3) shows indeed that stable bright localized structures resulting from front interaction between the two

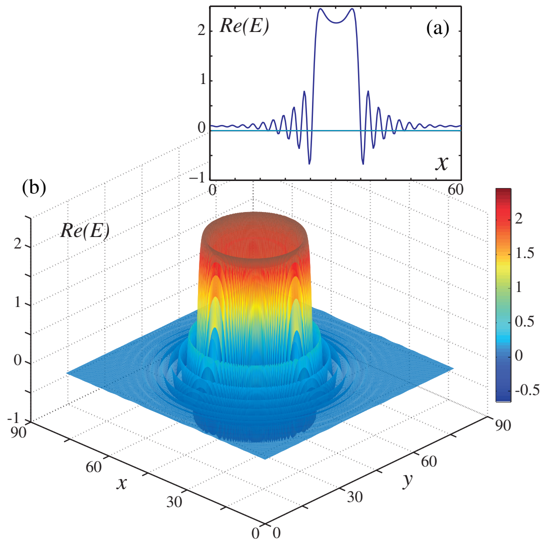

FIG. 4 (color online). Stable localized structures obtained by numerical simulations of Eq. (3) with the Lorentzian type kernel Eq. (2). (a) One-dimensional localized state observed for $\theta = 6$, $E_i = 3$, $\gamma = 1$, $\sigma = 0.7$, $D = -1.0$, and $n = 1$. (b) Two-dimensional localized state observed for $\theta = 5.93$, $E_i = 3$, $\gamma = 1$, $\sigma = 0.4$, $D = -1.0$, and $n = 1.1$.

homogeneous steady states are indeed stable for this model (see Fig. 4). This figure shows stable bright localized structure with damping oscillations. We stress again the fact that without strong nonlocal coupling, bright localized structures will be unstable. However, dark localized structures with damped oscillations may be stable even without nonlocal coupling in this model.

To summarize, we presented a mechanism of generation of stable localized structures based on strong nonlocal coupling mediated by a Lorentzian-like kernel. This nonlocal coupling modifies the nature of front interaction between two homogeneous steady states, and allows for the stabilization of localized structures. Without strong nonlocal coupling localized structures are instable. They either shrink or expand. An analytical expression of front interaction law is provided. This simple mechanism applies to two different systems: population dynamics described by the Nagumo equation and the mean field model describing the driven nonlinear cavity filled with a left-handed medium. This generic mechanism is robust in one and in two spatial dimensions and could be applied to large class of far from equilibrium systems with strong nonlocal coupling.

Fruitful discussions with S. Coulibaly and R. Lefever are gratefully acknowledged. M. T. is a Research Associate with the Fonds de la Recherche Scientifique F.R.S.-FNRS, Belgium. M. G. C. acknowledges the financial support of FONDECYT Project No. 1120320 and C. F. O. acknowledges the financial support of Becas Chile. This research was supported in part by the Interuniversity Attraction Poles program of the Belgian Science Policy Office under Grant No. IAP P7-35.

[1] V. S. Zykov, *Simulation of Wave Processes in Excitable Media* (Manchester University, Manchester, 1987); A. S. Mikhailov and K. Showalter, Phys. Rep. **425**, 79 (2006); L. A. Lugiato, IEEE J. Quantum Electron. **39**, 193 (2003); M. Tlidi and P. Mandel, J. Opt. B **6**, R60 (2004); G. Purwins, H. U. Bödeker, and Sh. Amiranashvili, Adv. Phys. **59**, 485 (2010).
[2] D. Mihalache, M. Bertolotti, and C. Sibilia, Prog. Opt. **27**, 229 (1989); K. Staliunas and V. J. Sanchez-Morcillo, *Transverse Patterns in Nonlinear Optical Resonators, Springer Tracts in Modern Physics* (Springer-Verlag, Berlin, 2003); Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fiber to Photonic Crystals* (Academic, New York, 2003); A. Malomed, D. Mihalache, F. Wise, and L. Torner, J. Opt. B **7**, R53 (2005); M. Tlidi, M. Taki, and T. Kolokolnikov, Chaos **17**, 037101 (2007); N. Akhmediev and A. Ankiewicz, *Dissipative Solitons: From Optics to Biology and Medicine* (Springer-Verlag, Berlin Heidelberg, 2008); O. Descalzi, M. Clerc, S. Residori, and G. Assanto, *Localized States in Physics: Solitons and Patterns* (Springer, New York, 2011).
[3] V. B. Taranenko, K. Staliunas, and C. O. Weiss, Phys. Rev. A **56**, 1582 (1997); Phys. Rev. Lett. **81**, 2236 (1998); S. Barland *et al.*, Nature (London) **419**, 699 (2002); X. Hachair, L. Furfaro, J. Javaloyes, M. Giudici, S. Balle, J. Tredicce, G. Tissoni, L. Lugiato, M. Brambilla, and T. Maggipinto, Phys. Rev. A **72**, 013815 (2005).
[4] P. Glansdorff and I. Prigogine, *Thermodynamic Theory of Structures, Stability and Fluctuations* (Wiley, New York, 1971).
[5] E. Hernandez-Garcia and C. Lopez, Physica (Amsterdam) **199D**, 223 (2004).
[6] J. D. Murray, *Mathematical Biology* (Springer-Verlag, Berlin, 1989).
[7] Shigui Ruan, in *Mathematics for Life Science and Medicine Biological and Medical Physics*, edited by Y. Takeuchi, Y. Iwasa, and K. Sato (Biomedical Engineering, Springer, 2007).
[8] Y. Kuramoto, D. Battogtokh, and H. Nakao, Phys. Rev. Lett. **81**, 3543 (1998); S. I. Shima and Y. Kuramoto, Phys. Rev. E **69**, 036213 (2004).
[9] M. G. Clerc, D. Escaff, and V. M. Kenkre, Phys. Rev. E **72**, 056217 (2005); **82**, 036210 (2010).
[10] D. Escaff, Int. J. Bifurcation Chaos Appl. Sci. Eng. **19**, 3509 (2009).
[11] L. Gelens, G. Van der Sande, P. Tassin, M. Tlidi, P. Kockaert, D. Gomila, I. Veretennicoff, and J. Danckaert, Phys. Rev. A **75**, 063812 (2007).
[12] I. Aranson and L. Tsimring, *Granular Patterns* (Oxford University Press, New York, 2009).
[13] S. Coombes, Biol. Cybern. **93**, 91 (2005).
[14] R. Lefever and O. Lejeune, Bull. Math. Biol. **59**, 263 (1997).
[15] O. Lejeune and M. Tlidi, Journal of vegetation science : official organ of the International Association for Vegetation Science **10**, 201 (1999).
[16] M. Tlidi, R. Lefever, and A. Vladimirov, Lect. Notes Phys. **751**, 381 (2008).
[17] E. Gilad, J. von Hardenberg, A. Provenzale, M. Shachak, and E. Meron, Phys. Rev. Lett. **93**, 098105 (2004).
[18] L. Gelens, D. Gomila, G. Van der Sande, M. A. Matias, and P. Colet, Phys. Rev. Lett. **104**, 154101 (2010).

[19] D. Escaff, Eur. Phys. J. D **62**, 33 (2011).
[20] L. A. Lugiato and R. Lefever, Phys. Rev. Lett. **58**, 2209 (1987).
[21] A. M. Turing, Phil. Trans. R. Soc. B **237**, 37 (1952).
[22] R. E. Goldstein, G. H. Gunaratne, L. Gil, and P. Coullet, Phys. Rev. A **43**, 6700 (1991).
[23] A. G. Vladimirov, R. Lefever, and M. Tlidi, Phys. Rev. A **84**, 043848 (2011).
[24] P. Coullet, Int. J. Bifurcation Chaos Appl. Sci. Eng. **12**, 2445 (2002).
[25] M. Tlidi, R. Lefever, and P. Mandel, Quantum Semiclass. Opt. **8**, 931 (1996).
[26] A. J. Scroggie, W. J. Firth, G. S. McDonald, M. Tlidi, R. Lefever, and L. A. Lugiato, Chaos Solitons Fractals **4**, 1323 (1994).
[27] D. Gomila, M. A. Matias, and P. Colet, Phys. Rev. Lett. **94**, 063905 (2005); D. Turaev, A. G. Vladimirov, and S. Zelik, Phys. Rev. Lett. **108**, 263906 (2012).
[28] P. Kockaert, P. Tassin, G. Van der Sande, I. Veretennicoff, and M. Tlidi, Phys. Rev. A **74**, 033822 (2006); P. Tassin, L. Gelens, J. Danckaert, I. Veretennicoff, G. Van der Sande, P. Kockaert, and M. Tlidi, Chaos **17**, 037116 (2007).