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PARAMETRICALLY DRIVEN INSTABILITY IN QUASI-REVERSAL SYSTEMS

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Parametric instability of quasi-reversal system — i.e. time reversible systems perturbed with injection and dissipation of energy — is studied in a unified manner. We infer and characterize an adequate amplitude equation, which is the parametrically driven damped nonlinear Schrödinger equation, corrected with higher order terms. This model exhibits rich dynamical behavior which are lost in the parametrically driven damped nonlinear Schrödinger equation such as: uniform states, fronts and coherent states. The dynamical behavior of a simple parametrically driven system, the vertically driven chain of pendula, exhibits quite good agreement with the amended amplitude equation.

Keywords: Nonequilibrium systems; bifurcations; particle type solutions.

1. Introduction

Stationary or steady states of equilibrium systems, in nature, are characterized by spatial and temporal invariance. Instead, systems driven far from equilibrium often lead to the formation of spatially homogeneous states, oscillatory states, patterns, waves, spatio-temporal chaos states or localized structures with nontrivial dynamical behaviors [Nicolis & Prigogine, 1977]. A simple way to maintain a system out of equilibrium is by means of an external forcing, which can be either static or periodic in time. When forcing is periodic in time and depends on the system variables, it is denominated *parametric*. The generation of standing waves at the surface of a vertically vibrated Newtonian fluid — Faraday waves — is one of the classical parametrically driven hydrodynamic instabilities [Faraday, 1831], where the injection of energy is made through vertical oscillations. These surface waves are subharmonic, that is, the system responds at half the forcing frequency [Arnold, 1983]. This type of parametric resonance is commonly termed *strong resonance* [Hale & Kocak, 1991]. Theoretically, close to the resonance, this system is modeled by the parametrically driven damped nonlinear Schrödinger equation [Miles, 1984; Zhang & Viñals, 1995]. This model has been derived in several physical frameworks to describe pattern formation, such as nonlinear lattices [Denardo *et al.*, 1992], optical fibers [Kutz *et al.*, 1993], Kerr-type optical parametric oscillators [Longhi, 1996], magnetization in an easy-plane ferromagnetic exposed to an oscillatory magnetic field [Barashenkov *et al.*, 1991], and parametrically driven damped chain of pendula [Alexeeva *et al.*, 2000]. However, we have recently shown that the conventional approach is not adequate to describe a family of localized states that link asymptotically two zones of uniform oscillations in phase-opposition, which are observed in the validity region of the parametrically driven damped nonlinear Schrödinger equation [Clerc *et al.*, 2008].

The aim of this work is to study in a unified manner the strong parametric resonance observed in quasi-reversal system–time reversible systems perturbed with injection and dissipation of energy [Clerc *et al.*, 1999a, 1999b, 2001]. We derive an adequate amplitude equation, which is the parametrically driven damped nonlinear Schrödinger equation with the addition of higher order terms. The bifurcation diagram of this model is characterized. In particular, this diagram exhibits rich dynamical behaviors which are lost by the parametrically driven damped nonlinear Schrödinger equation such as: uniform states, fronts and coherent states. The dynamical behavior of a simple parametrically driven system — vertically driven chain of pendula — is in good agreement with the unified amended amplitude equation.

The manuscript is organized as follows: in Sec. 2, we introduce and characterize the dynamical behavior of parametrically driven damped nonlinear Schrödinger equation. In Sec. 3, an amended amplitude equation is proposed, which describes the dynamical behaviors exhibited by parametrically driven quasi-reversal system. In Sec. 4, the pertinence of parametrically driven damped nonlinear Schrödinger equation is analyzed. Finally, conclusions are presented in Sec. 5.

2. Parametrically Driven Damped Nonlinear Schrödinger Equation

A classical set up that exhibits a strong parametric resonance is a vertically driven damped chain of pendula, which is described in the continuum limit by

$$\ddot{\theta}(z,t) = -[\omega_o^2 + \gamma\sin(\omega t)]\sin\theta - \mu\dot{\theta} + \kappa\partial_{zz}\theta, \quad (1)$$

where $\theta(z,t)$ is the angle formed by the pendulum and the vertical axis in the z-position at time t; ω_o is the pendulum natural frequency, $\{\mu,\kappa,\gamma,\omega\}$ are the damping coefficient, elastic coupling, amplitude and frequency of the parametric forcing, respectively. For zero forcing and damping ($\gamma = \mu = 0$), Eq. (1) describes a Hamiltonian system, that presents a time reversal invariance–Sine–Gordon model [Arnold, 1983]. Hence, the inclusion of injection and dissipation of energy can lead to complex spatio-temporal dynamics which are usually observed in nonequilibrium systems. Notice that in this limit Eq. (1) has also the reflection symmetry $\theta \to -\theta$.

A simple homogeneous state of Eq. (1) is $\theta(z,t) = 0$, which represents a uniform vertical oscillation of pendula. When the pendula chain is forced to be close to double the natural frequency — $\omega = 2(\omega_0 + \nu)$, where $\nu \ll 1$ is the detuning parameter i.e. ν accounts for the difference between ω_0 and the half of ω — the vertical solution becomes unstable at $\nu^2 + \mu^2/4 = \gamma^2/16$ for small $\{\nu,\gamma,\mu\}$. This curve in the parameter space is well-known as *Arnold's tongue*. Figure 2 depicts the Arnold tongue in $\nu\gamma$-space. Inside the Arnold tongue the uniform vertical solution is unstable and gives rise to a uniform attractive periodic solution, so the pendula chain oscillates uniformly ($\theta(z,t+T) = \theta(z,t)$ where $T \approx 2\pi/\omega_0$). Due to the reflection symmetry, there is another uniform oscillation out of phase in π. Thus, we expect from the coexistence of these different states that the system exhibits a rich dynamics.

In order to explain the above-mentioned instability, we consider the quasi-reversal limit, that is, the time reversal limit perturbed with small injection and dissipation of energy [Clerc *et al.*, 1999a, 1999b, 2001]. In this limit, Eq. (1) correspond to the perturbed Sine–Gordon equation with $\gamma \sim \nu \sim \mu \sim \varepsilon$, where ε is an arbitrary small scale $\varepsilon \ll 1$.

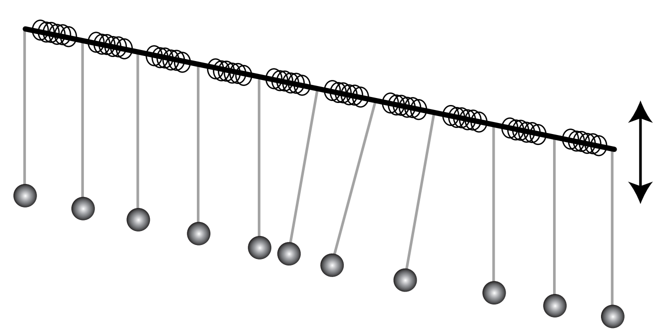

Fig. 1. Schematic representation of a vertically driven chain of pendula: all pendula are coupled with an ideal spring and the horizontal rod oscillates vertically in time.

Let us consider the following ansatz

$$\theta(z,t) = 2\sqrt{\frac{\varepsilon}{\omega_o}}A(x,\tau)e^{i(\omega_o+\nu)t} - 2\sqrt{\frac{\varepsilon}{\omega_o}}\left\{\frac{A^3(x,\tau)}{48} + \frac{i\gamma}{16\omega_0^2}A(x,\tau) - \frac{i\gamma\varepsilon}{8\omega_0^3}|A(x,\tau)|^2A(x,\tau)\right\} \times e^{3i(\omega_o+\nu)t} + \text{c.c.} + \text{h.o.t.}, \quad (2)$$

where $A(x,\tau)$ is the envelope of the uniform oscillation, $\tau \equiv \varepsilon t$, $x \equiv \sqrt{2\varepsilon\omega_0/\kappa z}$ are slow variables, c.c. and h.o.t. mean complex conjugate and high order terms, respectively. Introducing the above ansatz in Eq. (1) and after straightforward calculations, the amplitude A satisfies at dominant order (parametrically driven damped nonlinear Schrödinger equation),

$$\partial_\tau A = -i\nu A - i|A|^2A - i\partial_x^2 A - \tilde{\mu}A + \tilde{\gamma}\overline{A}, \quad (3)$$

where $\tilde{\mu} \equiv \mu/2$, $\tilde{\gamma} \equiv \gamma/4$, and $\overline{A}$ stands for the complex conjugate of A. The terms of the above equation are of order $\varepsilon^{3/2}$ and the higher order terms are at least of order $\varepsilon^{5/2}$. This model has been used intensively to describe patterns and disipative solitons in several systems such as: vertically oscillating layer of water [Zhang & Viñal, 1995], localized structures in nonlinear lattices [Denardo *et al.*, 1992], and the Kerr-type optical parametric oscillators [Longhi, 1996], just to mention a few. We remark that the above model has played an important role in the understanding of the pattern formation in the afore-mentioned physical systems.

2.1. *Phase diagram*

The parametrically driven damped nonlinear Schrödinger equation (3) has an homogeneous state, $A(x,\tau) = 0$, that represents the vertical solution $\theta(z,t) = 0$. Moreover, this equation has uniform states

$$A_{\pm,\pm} = \pm\left(1 \pm i\sqrt{\frac{\tilde{\mu}-\tilde{\gamma}}{\tilde{\mu}+\tilde{\gamma}}}\right)x_0 = R_0 e^{i\varphi},$$

where

$$x_0 \equiv \sqrt{\frac{(\tilde{\gamma}-\tilde{\mu})(-\nu+\sqrt{\tilde{\gamma}^2-\nu^2})}{2\tilde{\gamma}}},$$

$R_0 = \sqrt{-\nu - \tilde{\gamma}\sin 2\varphi}$ and $\cos 2\varphi = \tilde{\mu}/\tilde{\gamma}$. In order to characterize the different bifurcations exhibited by Eq. (3), we realize a trajectory $\Gamma(\nu)$ in the $\nu\tilde{\gamma}$-space, decreasing the detuning parameter as it is illustrated in Fig. 2. For large detuning the zero amplitude state is stable ($A = 0$). Nevertheless, when the detuning decreases and the forcing amplitude increases this state becomes unstable at $\tilde{\gamma} \equiv \tilde{\gamma}_F = \tilde{\mu}$, further increasing the forcing amplitude $\tilde{\gamma}$, giving rise to the appearance of pattern state. Hence, at $\tilde{\gamma} \equiv \tilde{\gamma}_F$ the system exhibits a spatial instability. It is important to note that this spatial instability is of order quintic in A and supercritical, hence the pattern amplitude increases proportionally to the power law $(\tilde{\gamma} - \tilde{\gamma}_F)^{1/4}$. This spatial bifurcation is represented by A in Figs. 2 and 3. In addition, this pattern state corresponds to a standing subharmonic wave in the vertically driven damped pendula chain [Eq. (1)].

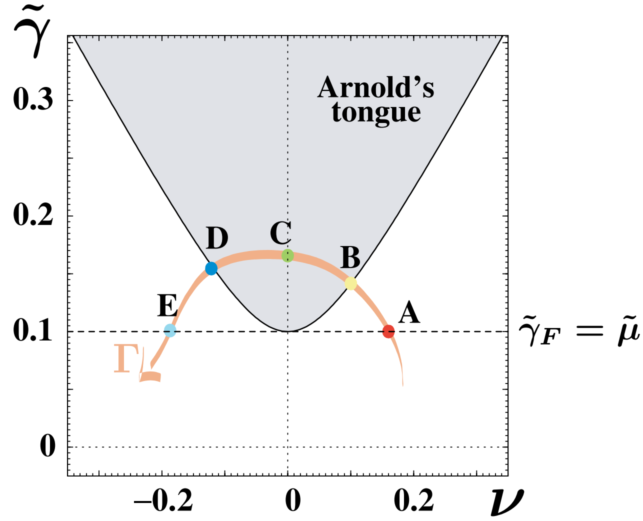

Fig. 2. Bifurcation diagram of the parametrically driven damped nonlinear Schrödinger equation (3). This diagram is also valid for the vertically driven pendula chain Eq. (1). The gray area stands for Arnold's tongue [Coullet *et al.*, 1994; Arnold, 1983]. A, B, C, D and E points represent different bifurcation points, when one realizes a trajectory Γ in $\nu\tilde{\gamma}$-space.

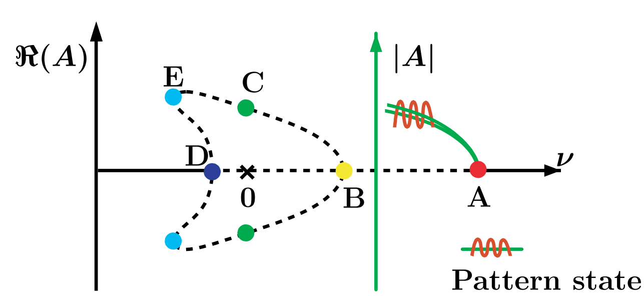

Fig. 3. Bifurcation diagram of parametrically driven damped nonlinear Schrödinger equation (3) obtained from the trajectory Γ in the $\nu\tilde{\gamma}$-space represented in Fig. 2. The vertical axis is the real part of A for uniform states and the modulus for pattern states, depicted by the continuous green curve. The continuous and dashed black curves are stable and unstable uniform states, respectively.

When the detuning decreases and the forcing amplitude increases, the unstable zero amplitude state suffers a secondary stationary pitchfork bifurcation at $\tilde{\gamma} = \sqrt{\tilde{\mu}^2 + \nu^2}$, resulting in two uniform states with amplitude

$$A_{\pm,+} = \pm \left(1 + i\sqrt{\frac{\tilde{\mu} - \tilde{\gamma}}{\tilde{\mu} + \tilde{\gamma}}}\right) x_0.$$

This bifurcation is represented by point B in Figs. 2 and 3. Close to this bifurcation all these three states are unstable. Here, the nonzero amplitude state represents uniform oscillation for the vertically driven damped pendula chain. Decreasing ν further through the Γ trajectory, for negative detuning, the zero amplitude state suffers another secondary instability at $\tilde{\gamma} = \sqrt{\tilde{\mu}^2 + \nu^2}$, which gives rise to unstable uniform states

$$A_{\pm,-} = \pm \left(1 - i\sqrt{\frac{\tilde{\mu} - \tilde{\gamma}}{\tilde{\mu} + \tilde{\gamma}}}\right) x_0.$$

Consequently, the zero amplitude state $A = 0$ becomes stable and the instability is an inverted pitchfork bifurcation as it is depicted in Fig. 3 by point D. The D and B points are part of the border of Arnold tongue. Decreasing further the detuning and forcing amplitude, the zero amplitude state becomes stable.

The nonzero states $\{A_{\pm,+}, A_{\pm,-}\}$ merge together and disappear by saddle-node bifurcation at $\tilde{\gamma} = \tilde{\mu}$. This saddle-node bifurcation is represented by point E in Figs. 2 and 3. To grasp the stability of these states, we introduce the linear perturbation $A = R_0 e^{i\varphi} + \delta A$, where the equation for δA the perturbation reads

$$\delta A_\tau = (i\nu - 2iR_0^2 - \tilde{\mu})\delta A + (\tilde{\gamma} - iR_0^2 e^{2i\varphi})\overline{\delta A} - \partial_{xx}\delta A.$$

Introducing a solution of the form $\delta A(x,t) = A_0 e^{\sigma t + ikx}$, where A_0 is a constant complex number, we found that the relation between the growth rate and wave number is

$$\sigma^2 - 2\tilde{\mu}\sigma = 4R_0^2\tilde{\gamma}\sin 2\varphi + 2k^2(2R_0^2 - \nu) - k^4.$$

Figure 4 shows the typical growth rate as a function of the wave number. In order to compute the bifurcation condition for these states, we calculate the critical wave number with zero growth rate ($\sigma(k_c) = 0$)

$$4R_0^2\tilde{\gamma}\sin 2\varphi + 2k_c^2(2R_0^2 - \nu) - k_c^4 = 0, \qquad (4)$$

imposing that this critical wave number is a maximum, that is, the discriminant of the above bi-quadratic equation is zero, then after a straightforward calculation we obtain the condition

$$\nu = 0,$$

and $k_c^{(0)} = \pm\sqrt{2\sqrt{\tilde{\gamma}^2 - \tilde{\mu}^2}}$. However, the above equation (4) always has a solution given by $k_c = \pm\sqrt{-\nu + 2\sqrt{\tilde{\gamma}^2 - \tilde{\mu}^2}}$. Therefore, these nonzero uniform states are unstable and are only marginally stable at zero detuning. The marginal condition is represented by point C in Figs. 2 and 3. Numerical simulations of Eq. (3) for zero detuning show that the nonzero uniform amplitude state is unstable. Nevertheless, numerical simulations of vertically driven damped pendula chain disclose that the uniform oscillations are stable in a large region in the parameter space and also this system exhibits front solution between these uniform oscillations. Therefore, the parametrically driven damped nonlinear Schrödinger equation does not describe the dynamics exhibited by the primary system.

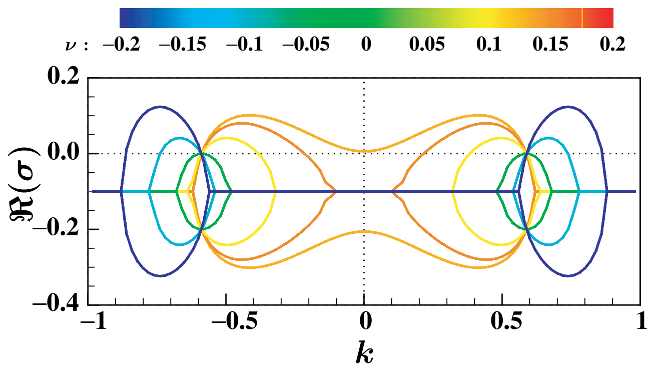

Fig. 4. Spectrum of Eq. (3) for $\tilde{\mu} = 0.10$ and $\tilde{\gamma} = 0.20$.

From the dynamical system theory, the prototype model [Eq. (3)] is structurally unstable [Arnold, 1983]. Hence the addition of higher order terms to Eq. (3) can modify the stability features of the nonzero amplitude states. In the next section, we shall amend the parametrically driven damped nonlinear Schrödinger equation with the higher order terms and we shall study the consequences of these terms in the dynamic behaviors.

3. Amended Parametrically Driven Damped Nonlinear Schrödinger Equation

To describe the parametric resonance of quasi-reversal system, it is required to consider higher order terms in the amplitude equation [Eq. (3)], since the addition of these terms may restore the features of the uniform states and particle type solutions such as fronts and localized structures. So, when we consider the dominant contributions of higher order terms, the amplitude equation reads

$$\partial_t A = -i\nu A - iA|A|^2 - i\partial_x^2 A - \tilde{\mu}A + \tilde{\gamma}\overline{A} + ib|A|^4 A - \alpha A|A|^2 + \eta\partial_x^2 A - \delta|A|^2\overline{A} + \beta A^3. \quad (5)$$

All the terms of the second line are of order $\varepsilon^{5/2}$, and $\{\alpha, \eta, \delta, \beta\}$ are small parameters which are of order ϵ and b is of order one. The terms proportional to b, α, η and $\{\delta, \beta\}$ stand for the nonlinear response of the frequency as function of amplitude, nonlinear dissipation, diffusion and nonlinear parametric forcing, respectively. Let us denominate the above equation as *amended amplitude equation*. For example, in the parametrically driven damped pendula chain, these parameters are:

$$b \equiv \frac{1}{6\omega_0}, \quad \alpha \equiv 0, \quad \eta \equiv 0, \quad \delta \equiv \frac{\gamma}{2\omega_0^2}, \quad \text{and} \quad \beta \equiv \frac{\gamma}{6\omega_0^2}.$$

In the extreme limit $\tilde{\gamma} = b = \delta = \beta = 0$, Eq. (5) is the quasi-reversible Ginzburg–Landau equation [Clerc *et al.*, 2000], that is, the imaginary coefficients are of order one, and the real coefficients are small. This model, close to the threshold of the Benjamin–Feir–Newell–Kuramoto instability exhibits chaotical dynamic behavior [Cross & Hohenberg, 1993], which is described by a Lorenz type model [Clerc *et al.*, 2000].

For small detuning, we observe numerically that the amended amplitude equation has stable uniform solutions close to $A_{+,\pm}$. These states remain stable in a finite interval around zero detuning. The extremes of this interval are illustrated by

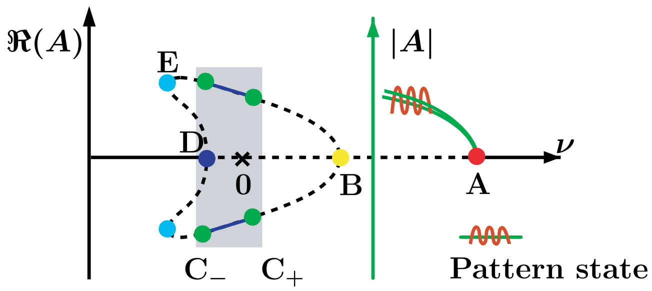

Fig. 5. Modified bifurcation diagram of the amended parametrically driven damped nonlinear Schrödinger equation (3). The gray region correspond to the stability domain of the nonzero uniform state.

C_- and C_+ points in Fig. 5. Hence, the bifurcation diagram of the amended parametrically driven damped nonlinear Schrödinger equation is depicted in Fig. 5. Consequently, this amended model has uniform nonzero amplitude states.

3.1. *Particle type solutions in amended parametrically driven damped nonlinear Schrödinger equation*

From the nonzero uniform states, we can build up states that connect asymptotically the uniform states, which are well known as kinks or wall solutions. Figure 6(a) shows these wall type solutions. The derivation of an analytical expression of these solutions is a thorny task in general. However, the dynamics of these kinks are characterized by the core of the front, that is, by the position in the space where the field $A = 0$. The core of the front is illustrated by x_0 in Fig. 6(a). Numerically, we observe that inside the Arnold tongue these walls are motionless for positive detuning and small forcing, since these kink solutions connect to symmetric states. Nevertheless in [Clerc *et al.*, 2009], we have shown that for negative detuning in a large region of parameters inside the Arnold tongue this interface becomes mobile wall, that is, the model (5) exhibits a nonvariational Ising–Bloch transition. This transition occurs in the $\nu < 0$ region for values of $\nu < \nu_c$, where ν_c is a threshold value that can be analytically computed. For values of $\nu < \nu_c$, the wall speed v follows the power law $v \sim |\nu + \nu_c|^{1/2}$.

The wall solutions have spatially damped oscillations or oscillatory tails, as it is displayed in Fig. 6(a). When wall solutions have this spatial behavior, it is well known that the nature of the kink and anti-kink interactions alternates between attractive and repulsive [Clerc *et al.*, 2005, and references therein]. Therefore, it is expected to find a family of localized states or horn solutions [Clerc *et al.*, 2005] with width Δ as shown in Fig. 6(c) roughly a multiple of the characteristic length of the damped spatial oscillation present in the kink state. Figure 6 shows the shortest localized states observed in parametrically driven damped nonlinear Schrödinger equation (5). Due to the reflection symmetry $A \to -A$, the system exhibits the opposite horn solutions. These horn solutions are coherent states characterized by two continuous parameters — position and width. The horn position is defined as the maximum of horn state or the

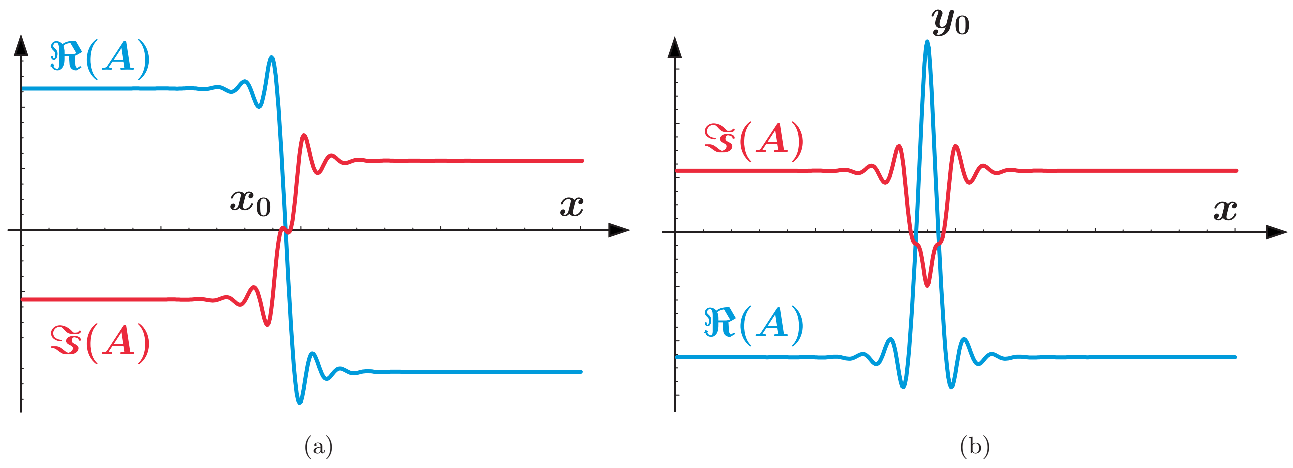

Fig. 6. Steady state particle type solutions of the parametrically driven damped nonlinear Schrödinger equation (3). (a) Wall solution, (b) smallest horn state and (c) horn state (where Δ is the horn state width), for $\nu = -0.03$, $\mu = 0.225$, $\gamma = 0.5$, $\alpha = 0.0$, $\delta = 0.01$, $b = 0.1$, $\beta = 0.01$ and $\eta = 0.0$.

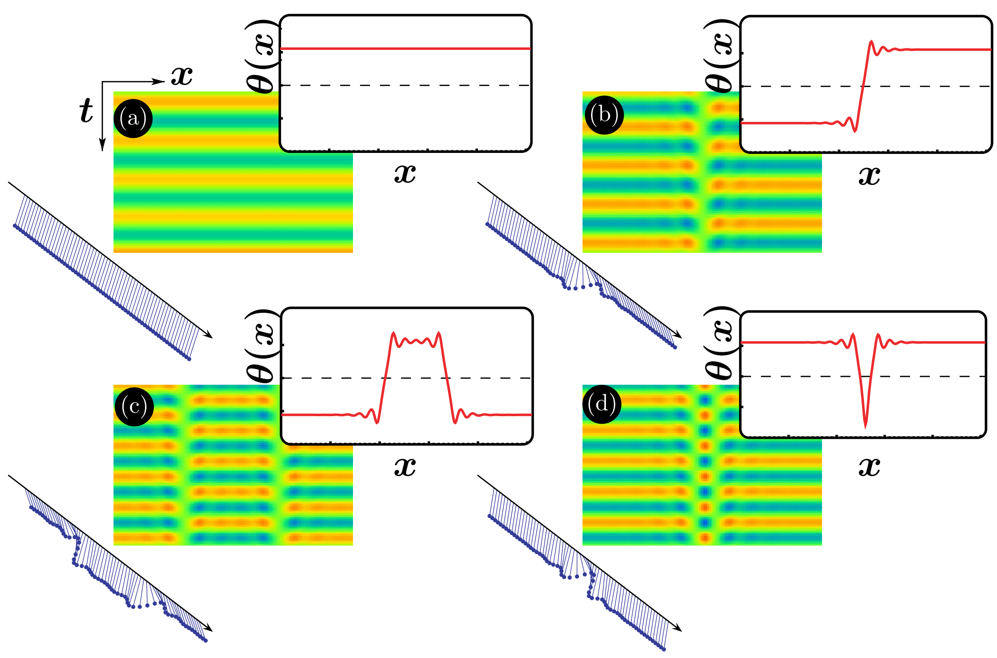

Fig. 7. Steady state particle type solutions of vertically driven pendula chain. (a) Uniform oscillation, (b) wall solution, (c) horn state and (d) smallest horn state by $\mu = 0.20$, $\gamma = 0.48$ and $\nu = -0.03$.

minimum of the opposite horn state, respectively. The horn position is represented by y_0 in Fig. 6(c) and its width is defined as the smallest distance between both uniform states that support this localized solution. The interaction of these particle type solutions is described by these two parameters. Further research in this direction are still in progress.

From the kink and anti-kink interaction theory [Kawasaki & Otha, 1982], we deduce that a horn state with small width is more stable than those with larger width. Figure 6(b) shows the smallest horn solution. Figure 7 displays the homogeneous oscillation, the kink and horn states.

4. Pertinence of Parametrically Driven Damped Nonlinear Schrödinger Equation

As we have mentioned in the afore sections the parametrically driven damped nonlinear Schrödinger equation (3), is structurally unstable. However, if we considered the nonlinear terms of the lowest order in Eq. (1), neglecting the fifth order in θ and also ignoring the nonlinear terms proportional to γ, that is, the vertically driven damped pendula chain is described by

$$\ddot{\theta}(z,t) = -\omega_o^2\left\{\theta - \frac{\theta^3}{6}\right\} - \gamma\sin(\omega t)\theta - \mu\dot{\theta} + k\partial_{zz}\theta.$$

Numerically, we observe that this extended parametrically driven nonlinear oscillator does not exhibit stable kinks nor families of localized states linking both uniform oscillations. The bifurcation diagram of the above equation is well described as shown in Fig. 3. Hence, parametrically driven damped nonlinear Schrödinger equation only describes a cubic nonlinear oscillator with small linear damping and linear parametric forcing. When we add the nonlinear terms of the fifth order and the nonlinear parametric forcing up to order three, the system presents the particle type solutions which connect the uniform oscillations. Therefore, the stability properties of these states are given by higher order nonlinearities.

5. Conclusions

Parametrically driven damped nonlinear Schrödinger equation has been used as a prototype model to describe pattern formation in parametric driven quasi-reversal oscillator. Nevertheless, this approach cannot describe the features of uniform oscillations and localized states that connect asymptotically uniform oscillating states. The parametrically driven damped nonlinear Schrödinger equation is structurally unstable. Then, amending this model with higher order terms, allows us to recuperate the robust dynamics observed in the vertically driven pendula chain. Hence, a parametric driven damped quasi reversal nonlinear oscillator close to the strong resonance is well described by model (5). The characterization of the dynamical behaviors and bifurcation diagram of the amended parametrically driven damped nonlinear Schrödinger equation is still in progress.

Acknowledgments

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