International Journal of Bifurcation and Chaos, Vol. 18, No. 7 (2008) 1905–1915

SHILNIKOV BIFURCATION: STATIONARY QUASI-REVERSAL BIFURCATION

MARCEL G. CLERC, PABLO C. ENCINA* and ENRIQUE TIRAPEGUI
Departamento de Física, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 487-3, Santiago, Chile
**Departamento de Física, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile*

Received July 16, 2007; Revised August 28, 2007

A generic stationary instability that arises in quasi-reversible systems is studied. It is characterized by the confluence of three eigenvalues at the origin of complex plane with only one eigenfunction. We characterize the dynamics through the normal form that exhibits in particular, Shilnikov chaos, for which we give an analytical prediction. We construct a simple mechanical system, Shilnikov particle, which exhibits this quasi-reversal instability and displays its chaotic behavior.

Keywords: Bifurcation theory; instabilities; chaos.

1. Introduction

The study of instabilities plays a central role in the modern theory of dynamical systems, because the understanding of this can allow us to have a qualitative theory of dynamical systems [Guckenheimer & Holmes, 1983], that is, this study permits to describe in a universal way phenomena which belong to different fields [Coullet, 1985; Cross & Hohenberg, 1993]. In one parameter families of dissipative dynamical systems — in codimension one — only two local bifurcations occur generically for equilibrium points: the saddle-node and the Hopf bifurcations. The presence of symmetries changes these scenarios: for instance, the reflection symmetry transforms the saddle-node bifurcation in the pitchfork bifurcation. In time reversible systems, i.e. systems which are invariant under a time reversal transformation, linearization at a reversible equilibrium stable state gives a matrix with purely imaginary eigenvalues whose number is equal to the dimension of the system. In this kind of system the instabilities in one parameter families of equilibrium points are: (a) The stationary instability denoted by (0^2) in Arnold's notation [Arnold, 1980], which we use from now on, corresponding to a resonance at zero frequency; and (b) The confusion of frequencies $(i\Omega^2)$ or 1:1 resonance [Rocard, 1943], which corresponds to a resonance at a finite frequency. A natural way to observe these instabilities is to change a control parameter. However, for time reversible systems the observation of instabilities is generically through the change of the initial conditions, because this changes the values of conserved quantities and then the stability of relative equilibria. Hence, the injection of energy in a time reversible system is by means of a neutral mode related to a conserved quantity.

In recent years, the consequences of weakly breaking the symmetry of time reversal $t \to -t$ in the bifurcation theory of time reversible systems — quasi-reversible systems — have been considered [Clerc *et al.*, 1999a, 2000, 2001]. This circumstance occurs frequently in macroscopic systems. In their usual versions, the fundamental physical laws are time reversible, but this symmetry disappears in the macroscopic description due to separation of time scales that allows a description in terms of the slowly varying macroscopic variables, which satisfy a dissipative dynamics. The dissipation can vary from weak perturbations — quasi-reversibility — to strong dominant effects. Some well-known examples of quasi-reversible behaviors in mechanics, fluid mechanics and optics are the motion of planets in celestial mechanics, surface waves in water and laser, respectively. The stationary quasi-reversible instabilities in the presence of a neutral mode with reflexion symmetry — $(0^2 0)$ in Arnold's notation — have been studied in [Clerc *et al.*, 1999a, 2001], where it is shown that the asymptotic normal form is equivalent to the set of the real Lorenz equations, which then turn out to be universal equations. A simple mechanical pendulum — which we have called the Lorenz pendulum — is shown to be an example of this instability, and experimental results agree with the theoretical predictions. A similar dynamic instability is exhibited by the period doubling bifurcation of a closed orbit in quasi-reversible systems [Clerc *et al.*, 2000]. The confusion of frequencies in presence of a neutral mode- $((i\Omega)^2 0))$ in Arnold's notation has also been studied by [Clerc *et al.*, 1999b] and it is shown that the asymptotic normal form in this case is the well-known set of Maxwell–Bloch equations which describe the laser effect. A double pendulum with orthogonal oscillations — which we can call the Rocard pendulum — was shown to obey the universal Maxwell–Bloch equations by [Clerc & Marsden, 2001] and is then a mechanical analogue of the laser. However, the stationary quasi-reversible instability in the presence of a neutral mode without reflexion symmetry — (0^3) instability — has not been studied. This is one of the most generic instabilities and it studied here. We call it *Shilnikov bifurcation* for reasons which will become clear below.

The aim of this article is then to describe the generic (0^3) stationary instability of equilibrium points which arises in quasi-reversible systems. It is characterized by the confluence of three eigenvalues at the origin of the complex plane with only one eigenfunction. We characterize the normal form of this bifurcation and its dynamics. In particular, we describe the chaotic behavior — homoclinic Shilnikov chaos — exhibited by the normal form and we present analytic predictions for the appearance of Shilnikov chaos. We construct a simple mechanical system — which we call the Shilnikov particle — composed by a ring sliding on an inclined rotating rod which exhibits this quasi-reversible instability and displays the Shilnikov chaotic behavior.

The manuscript is organized as follows. In Sec. 2, we characterize the unified description of 0^3-bifurcation for dissipative, time reversible and quasi-reversible systems. In Sec. 3, we describe the dynamics exhibited by the quasi-reversible normal form. In particular, we characterize the different steady states and bifurcation diagram exhibited by the normal form. In Sec. 4, an analytical condition for chaos is obtained by means of the Melnikov condition. A simple mechanical system that exhibits quasi-reversible (0^3) bifurcation is described in Sec. 5 and finally in Sec. 6 we draw our conclusions.

2. Normal Form of the Shilnikov Bifurcation

Stationary instabilities in strictly time reversible systems are associated with resonances which occur at zero frequency. This resonance takes place when one modifies a parameter or a constant of motion (neutral mode). The last mechanism is the physical way to create an instability in a relative equilibrium, that is, this equilibrium is a fixed point of the Routhian. The stationary quasi-reversible instability of a relative equilibrium in the presence of a neutral mode without reflection symmetry (0^3), is characterized by the confluence of three eigenvalues at the origin of the complex plane with only one eigenfunction. The reduced linear operator $\mathcal{L}$ is then given by a simple Jordan block,

$$\mathcal{L} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and the homological operator that characterizes the central manifold is [Arnold, 1980; Elphick *et al.*, 1987]

$$\mathcal{H}(\mathcal{L}) = \mathcal{L} - \mathcal{L}_{ij} A^j \frac{\partial}{\partial A^i},$$

where $A = \{x, y, z\}$ are the variables that characterize the central manifold [Elphick *et al.*, 1987]. The above operator characterizes the dynamics around the bifurcation. It is simple to show that the other critical variables associated with the pure imaginary eigenvalues with finite frequencies can be eliminated when the dissipative irreversible unfolding terms are considered. The relevant variables will then be $\{x, y, z\}$ and from the global characterization of normal forms in [Elphick *et al.*, 1987], we have the normal form that can be written as (*minimal normal form*)

$$\begin{aligned}\dot{x} &= y + xf(x, 2zx - y^2),\\ \dot{y} &= z + yf(x, 2zx - y^2) + xg(x, 2zx - y^2),\\ \dot{z} &= zf(x, 2zx - y^2) + yg(x, 2zx - y^2)\\ &\quad + h(x, 2zx - y^2),\end{aligned}$$

where f, g and h are polynomial functions of their arguments and functions of the small parameters that characterize the dynamics around the bifurcation. Due to the non-uniqueness of the solvability condition applied to the homological operator, by means of asymptotic polynomial changes of variables close to the identity, the above normal form can be rewritten (*Taken's type normal form*, see [Arneodo *et al.*, 1985]) as

$$\begin{aligned}\dddot{x} &= h(x, 2\ddot{x}x - \dot{x}^2) + \dot{x}g(x, 2\ddot{x}x - \dot{x}^2)\\ &\quad + \ddot{x}f(x, 2zx - y^2),\end{aligned}$$

or in the form (*Mechanical type normal form*)

$$\begin{aligned}\ddot{x} &= z + xF(x, 2zx - \dot{x}^2) + \dot{x}G(x, 2zx - \dot{x}^2),\\ \dot{z} &= H(x, 2zx - \dot{x}^2).\end{aligned} \tag{1}$$

All the above models are equivalent through non-singular asymptotic change of variables close to the identity [Elphick *et al.*, 1987]. However, the physical interpretation of the different terms is sensible to the form of the equations. For simplicity reasons henceforth, we shall consider the mechanical type normal form (1) which is the more adequate for our analysis below.

2.1. *Time reversible limit*

When these equations are invariant under the time reversal transformation $t \to -t$, $x \to x$, $z \to z$, the above model reads

$$\ddot{x} = z + xF(x, 2zx - \dot{x}^2),$$

$$\dot{z} = 0,$$

where z is constant of motion, and the system is described by a simple two-dimensional field. The asymptotic dimensionless reversible normal form — the normal form with the dominant terms — i.e. obtained from the previous equations is

$$\begin{aligned}\ddot{x} &= z \pm x^2,\\ \dot{z} &= 0,\end{aligned} \tag{2}$$

The other higher order terms are neglected by means of the scaling $x \sim z^{1/2}$, $\partial_t \sim z^{1/2}$ and $z \ll 1$. This set of equations describes a saddle-node bifurcation in the presence of a conservative quantitative z. Depending on the initial condition one can rule the value of z. Due to the transformation $x \to -x$, and $z \to -z$, we will consider without loss of generality henceforth the minus sign in the quadratic nonlinear term. Hence, for negative z the system does not have fixed points and any initial condition gives rise to a trajectory that moves to infinity. For positive z, the system has two fixed points $\{(\sqrt{z}, z), (-\sqrt{z}, z)\}$, one is a center and the other is a hyperbolic point, respectively. Hence, the fixed point $(\sqrt{z}, z)$ exhibits a stationary instability in the presence of a neutral mode when $z = 0$, which is characterized by the confluence of three eigenvalues at the origin of the complex plane with only one eigenfunction. Therefore for each z-plane in the phase space we have a typical saddle-node phase space diagram. In Fig. 1 this phase diagram is displayed. Note that this dynamical system has a family of homoclinic curves, which is represented by a hard curve in the z-plane and has the analytical expression

$$x_h(z, t) = -\sqrt{z}\left[1 - 3\sec h^2\left(\sqrt[4]{\frac{z}{4}}(t - t_0)\right)\right]. \tag{3}$$

This solution will play a central role in the generation of complex behaviors like chaos, when one considers the terms that break the time reversible symmetry as we shall see below.

2.2. *Quasi-reversible normal form*

The inclusion of small terms of dissipation and injection of energy — breaking the time

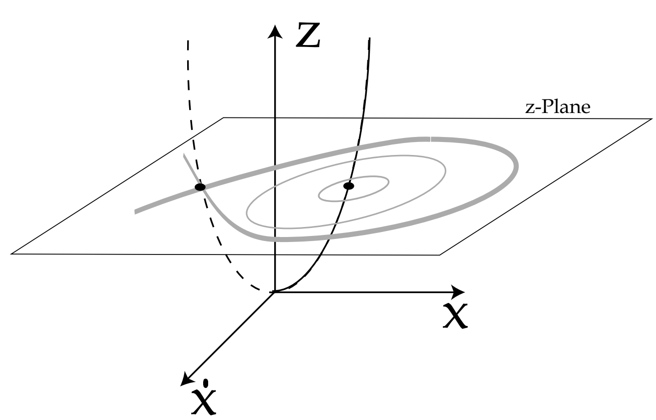


Fig. 1. Phase space diagram of time reversal normal form (2). The dashed and continuous curves represent the stable and unstable fixed points, respectively. The outer curve is for the homoclinic solution.

reversibility — in the unfolding of Eq. (2) gives

$$\begin{aligned}\ddot{x} &= z - x^2 - \mu\dot{x},\\ \dot{z} &= \delta + \nu x + \beta x^2,\end{aligned} \tag{4}$$

where the other higher order terms are ruled out through the scaling $x \sim z^{1/2}$, $\partial_t \sim z^{1/4}$, $\mu \sim z^{1/4}$, $\delta \sim z^{5/4}$, $\nu \sim z^{3/4}$, $\beta \sim z^{1/4}$ and $z \ll 1$. The parameter μ, ν and δ account for the dissipation and injection of energy, respectively, and βx^2 is a nonlinear saturation term. Depending on the different combinations of sign of $\{\nu, \beta\}$ the last two terms of the z-equation are dissipation or injection of energy. A similar model with an extra nonlinear dissipative term has been studied in the context of the 0^3 dissipative instability in the context of the thermally excited nonlinear oscillator [Moore & Spiegel, 1966; Baker *et al.*, 1971; Marzec & Spiegel, 1980]. An extra cubic nonlinearity has also been studied in the same context [Pismen, 1987].

3. Bifurcation Diagram of the Quasi-Reversible Normal Form

The quasi-reversible normal form (4) has the fixed points

$$z = x_\pm^2,$$

$$x_\pm = \frac{-\nu \pm \sqrt{\nu^2 - 4\delta\beta}}{2\beta},$$

where $\nu^2 - 4\delta\beta > 0$. When this inequality is not satisfied all the trajectories go to infinity, that is, infinity is the only attractor. The two fixed points (x_+, x_+^2) and (x_-, x_-^2) appear by saddle-node bifurcation. In Fig. 2 we display the different bifurcation diagrams of model (4), and the saddle-node bifurcation is represented by the point a. Close to this bifurcation the fixed points (x_+, x_+^2) and (x_-, x_-^2) are stable and unstable, respectively. The stable fixed point loses its stability by a Hopf–Andronov–Poincaré bifurcation at the curve in the parameter space defined by

$$\nu = (\mu + \beta)\frac{\nu - \sqrt{\nu^2 - 4\delta\beta}}{\beta}.$$

This instability is represented by the points b and c in Fig. 2. As we see, this bifurcation may be supercritical (point b) or subcritical (point c).

In order to study the dynamics around this instability and based on the normal form theory [Elphick *et al.*, 1987], we introduce the following

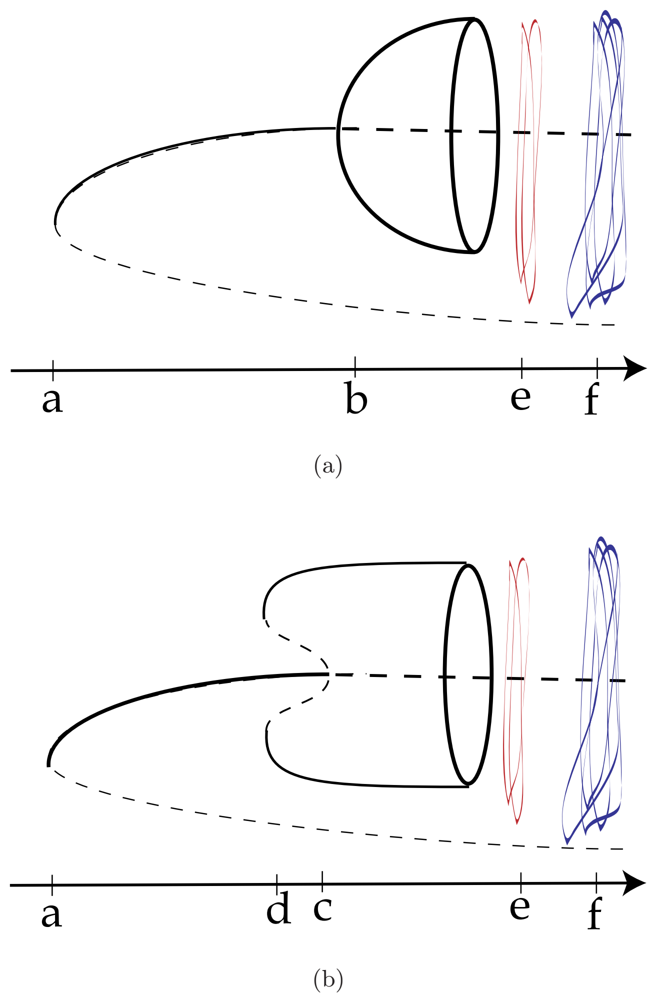


Fig. 2. Bifurcation diagrams of model (4): (a) saddle-node bifurcation, (b) supercritical Hopf–Andornov–Poincaré bifurcation, (c) subcritical Hopf–Andornov–Poincaré bifurcation, (d) saddle-node bifurcation of limit cycle, (e) periodic double bifurcation, and (f) Shilnikov chaos.

asymptotic change of variables

$$\begin{aligned}\begin{pmatrix} x \\ \dot{x} \\ z \end{pmatrix} &= A\begin{pmatrix} 1 \\ i\sqrt{2x_+} \\ i\mu\sqrt{2x_+} \end{pmatrix} + \overline{A}\begin{pmatrix} 1 \\ -i\sqrt{2x_+} \\ -i\mu\sqrt{2x_+} \end{pmatrix}\\ &\quad + a_0\left[A^2 - 2|A|^2 - \frac{\overline{A}^2}{3}\right]\begin{pmatrix} 1 \\ i\sqrt{2x_+} \\ i\mu\sqrt{2x_+} \end{pmatrix}\\ &\quad + a_0\left[\overline{A}^2 - 2|A|^2 - \frac{A^2}{3}\right]\begin{pmatrix} 1 \\ -i\sqrt{2x_+} \\ -i\mu\sqrt{2x_+} \end{pmatrix}\\ &\quad + \left[dA^2 + d\overline{A}^2 + e|A|^2\right]\begin{pmatrix} 1 \\ \mu \\ 2x_+ \end{pmatrix} + \text{h.o.t},\end{aligned}$$

with

$$a_0 \equiv \frac{\left(2\sqrt{2x_+} - \beta\mu\sqrt{\dfrac{2}{x_+}}\right) + 2i(\beta + \mu)}{\sqrt{2x_+}(4\mu^2 + 8x_+)},$$

$$d \equiv \frac{(\beta + \mu)(\mu - 2i\sqrt{2x_+})}{(\mu^2 + 2x_+)(\mu^2 + 8x_+)},$$

$$e \equiv \frac{2(\beta + \mu)}{\mu(\mu^2 + 2x_+)},$$

in the quasi-reversible model (4). The amplitude A satisfies

$$\partial_t A = \left[(\mu + \beta)\frac{\nu - \sqrt{\nu^2 - 4\delta\beta}}{\beta} + i\sqrt{2x_+}\right]A + g|A|^2 A, \quad (5)$$

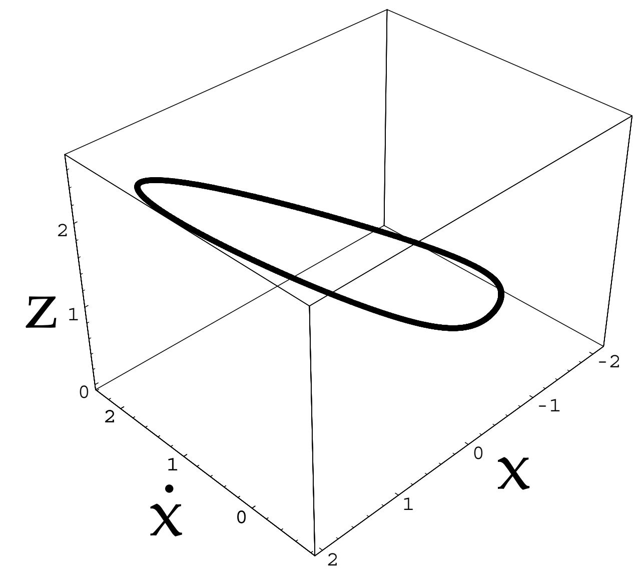
where

$$g \equiv (-2(\beta + \mu) + i(2\sqrt{2x_+} - \beta\mu\sqrt{2/x_+})) \times \frac{[\Re(-a_0 2) + e + a_o - \bar{a}_0/3 + d]}{(2\mu^2 + 4x_+)},$$

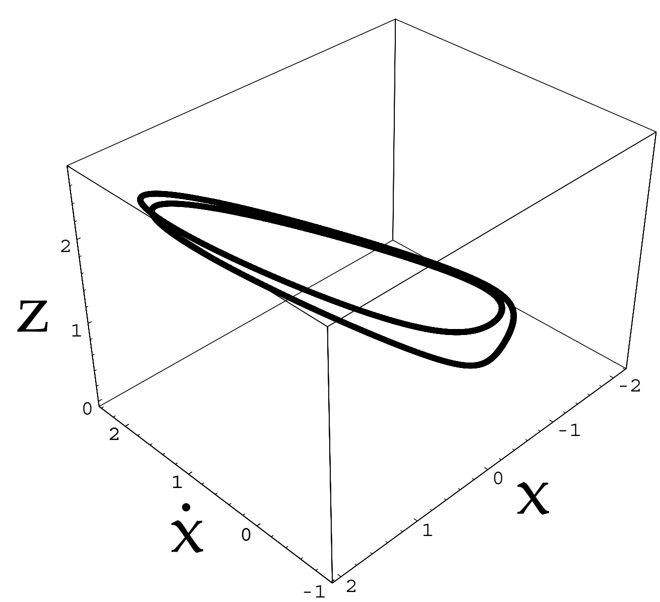
where $\Re$ stand for real part. If $\beta \approx -\mu$ the Hopf–Andronov–Poincaré bifurcation is supercritical (cf. Fig. 2(a)) or subcritical (cf. Fig. 2(b)), when $(2\sqrt{2x_+} - \beta\mu\sqrt{2/x_+})$ is negative or positive, respectively. In Fig. 2 we illustrate these two bifurcations. Hence from model (5), we deduce that the Hopf–Andronov–Poincaré bifurcation gives rise to the appearance of a limit cycle solution with frequency $\sqrt{2x_+}$. The typical limit cycle observed in this system is shown in Fig. 3. In the subcritical Hopf–Andronov–Poincaré bifurcation, the unstable limit cycle disappears by a saddle-node bifurcation that is represented by point d in Fig. 2(b).

Changing the parameters a the stable limit cycle persists, and it moves in the direction of the unstable fixed point (x_+, x_+^2). Hence, the limit cycle may exhibit a homoclinic bifurcation [Andronov *et al.*, 1987]. In Fig. 4 we represent schematically a homoclinic solution, where the hyperbolic fixed point is represented by point O, which is the origin of the coordinate system.

The dynamics close to a hyperbolic fixed point is characterized by the eigenvalues $\{-\lambda_- + i\omega, -\lambda_- - i\omega, \lambda_+\}$. The eigenvectors associated to $-\lambda_- \pm i\omega$ and λ_+ eigenvalues characterize the stable and the unstable manifolds, respectively. In Fig. 4, the unstable and stable manifolds of the hyperbolic point O are represented by $\{v, w\}$-plane and u-line, respectively. Hence, the dynamics around homoclinic solutions is characterized, after



(a)



(b)

Fig. 3. Limit cycle solution exhibits by model (4) by $\beta = -0.814$, $\delta = 1.0$, $\mu = 0.6$, (a) $\nu = 0.31$, and (b) $\nu = 0.295$.

straightforward calculations, by the first return map [Arneodo *et al.*, 1985]

$$u_{n+1} = k u_n^{\sigma} \sin[\omega \ln(u_n) + \phi] + \eta, \quad (6)$$

where u_n is the u-component of nth interception of a given trajectory with the Π-plane, where u-direction characterizes the unstable manifold of the hyperbolic fixed point. The Π-plane is an arbitrary plane orthogonal to the unstable direction, which is close to the hyperbolic point, that is, the

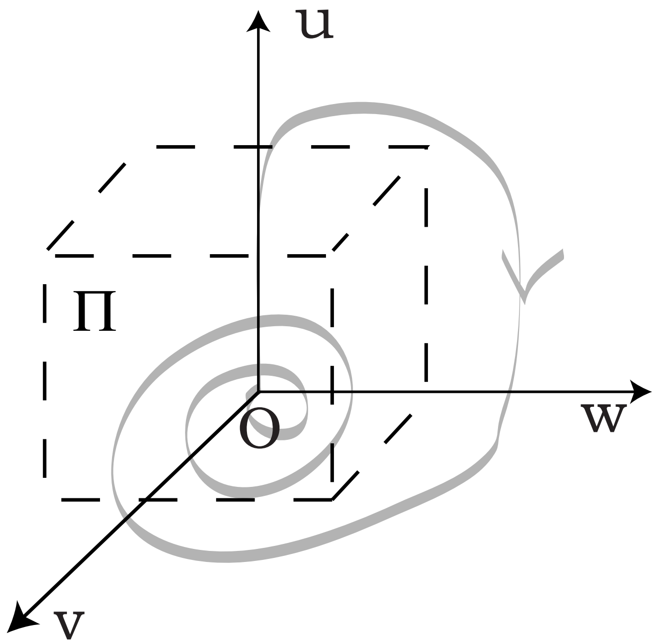


Fig. 4. Schematic representation of a homoclinic solution: the O-point is the hyperbolic fixed point, where the unstable and stable manifolds are represented by u-direction and $\{v, w\}$-directions, respectively. The Π-plane is parallel to unstable manifold.

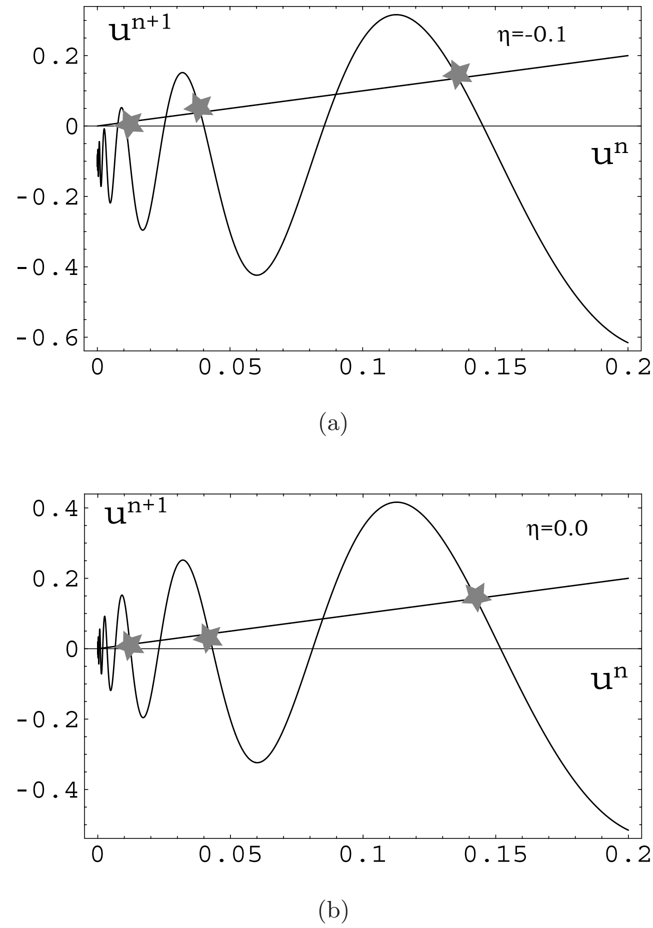


Fig. 5. Schematic representation of the first return map (6) for different values of η: (a) $\eta = -0.1$, and (b) $\eta = 0.0$. The stars stand for the equilibria points.

hyperbolic point does not belong to the Π-plane (cf. Fig. 4). The $\{k, \phi\}$-constant are parameters fixed by the election of the Π-plane, η is the homoclinic bifurcation parameter, that is, when $\eta = 0$ the system presents a homoclinic solution. The parameter $\sigma \equiv \lambda_-/\lambda_+$ characterizes the stability of the homoclinic curve, if $\sigma < 1$ ($\sigma \geq 1$) the homoclinic solution is unstable (stable). Shilnikov has shown that if a homoclinic solution has $\sigma < 1$ (unstable), then there is chaotical dynamics close to it [Shilnikov, 1965, 1968].

Numerically, we observe that the ratio λ_-/λ_+ of the eigenvalues for model (4) is lower than one. Therefore, the homoclinic bifurcation gives rise to unstable homoclinic curves. The first return map for $\sigma < 1$ is depicted in Fig. 5 for different values of η. The fixed points exhibited by this map represent limit cycle solutions for the quasi-reversible normal form (4). When η is changed, these fixed points have the typical period-doubling bifurcation of the logistic map that represents a period-doubling bifurcation for the respective limit cycle solution. In Fig. 3(b) we illustrate the typical limit cycle solution of model (4) just after the period-doubling. If we continue changing η, the first return map exhibits a cascade of bifurcations — the period-doubling route [Coullet & Tresser, 1978; Feigenbaum, 1978] — that gives rise to chaotical behavior. In Fig. 6 we show the strange attractor exhibited by model (4). Close to the homoclinic bifurcation — $\eta = 0$ — the first return map (6) has an infinite number of equilibrium points and each one presents the period doubling route when η is modified. The first period doubling bifurcation and the chaotical behavior are depicted by the points e and f in the bifurcation diagram of Fig. 2. Hence the dynamics around an unstable homoclinic bifurcation is characterized by a cascade of period-doubling bifurcations. The above dynamics is usually called *Shilnikov scenario*.

In brief, the dynamics of the quasi-reversible normal form (4) is characterized by having the infinite as attractor for a large domain of parameters. When the parameters are changed, two equilibrium appear by saddle-node bifurcation. If we continue modifying the parameters, the stable fixed point becomes unstable through a

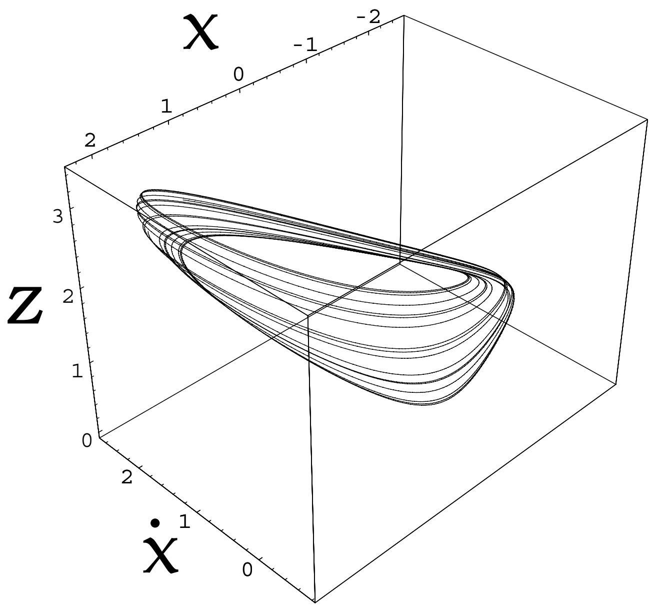


Fig. 6. Strange attractor exhibits by model (4), $\nu = 1.0$, $\beta = -1$, $\delta = 1.0$, and $\mu = 0.48$.

Hopf–Andronov–Poincaré bifurcation giving rise to limit cycle solutions. Increasing more the parameters, the limit cycle moves towards the hyperbolic fixed point giving rise to a homoclinic bifurcation. As a consequence of this bifurcation, the limit cycle presents successive period-doublings: Shinikov scenario.

4. Analytical Condition for Shilnikov Chaos

Although in dissipative dynamical system it is a thorny task to explicitly obtain homoclinic or heteroclinic solutions, this is quite possible in time reversible dynamical systems. Hence, in quasi-reversible systems by means of a persistence or Melmikov condition [Coullet & Elphick, 1987], one can grasp the homoclinic and heteroclinic solutions [Clerc *et al.*, 1999, 2000, 2001]. The persistence of the three-dimensional homoclinic solution guarantees the existence of chaos when the Shilnikov condition is satisfied ($\sigma < 1$).

In order to study the persistence of the planar homoclinic solution (3), we consider the following change of variable in the quasi-reversible normal form (4)

$$w \equiv z - \mu\dot{x},$$
$$x = x,$$

and the equations take the form

$$\begin{aligned}\ddot{x} &= w - x^2,\\ \dot{w} &= \delta + vx + (\beta+\mu)x^2 - \mu w.\end{aligned} \tag{7}$$

When $\beta = -\mu$, $\nu = 0$, and rewriting $\delta = \mu w_0$ — for the sake of simplicity we assume that $w_0 > 0$ — the above model takes the form

$$\begin{aligned}\ddot{x} &= w - x^2,\\ \dot{w} &= -\mu(w - w_0).\end{aligned} \tag{8}$$

this system has the fixed points $\{(\sqrt{w_0}, w_0), (-\sqrt{w_0}, w_0)\}$, where $(\sqrt{w_0}, w_0)$ is an attractor and $(-\sqrt{w_0}, w_0)$ is a hyperbolic point. The above model has a planar homoclinic solution

$$x_h(t, t_0) = \sqrt{w_0}\left(-1 + 3\sec h\left[\sqrt[4]{\frac{w_0}{4}}(t - t_0)\right]\right),$$
$$w = w_0.$$

When we consider the perturbation of the condition $\{\beta = -\mu,$ and $\nu = 0\}$, the homoclinic becomes three-dimensional. However, to have chaotic behavior, we need that the attractive manifold has a focus as the attractor — the eigenvalues related to the hyperbolic fixed point are complex conjugate with negative real part — and the ratio of the modulus of real parts of eigenvalues related to unstable and stable manifolds of hyperbolic fixed point is lower than one.

The hyperbolic fixed point $(-\sqrt{w_0}, w_0)$ of the model (8) has three real eigenvalues $\{\sqrt[4]{4w_0}, -\sqrt[4]{4w_0}, -\mu\}$, the perturbation of the condition $\{\beta = -\mu,$ and $\nu = 0\}$ in general produces three new real eigenvalues. In order to satisfy the Shilnikov condition, we have to consider $\mu = \sqrt[4]{4w_0}$, and thus the negative eigenvalues are degenerate. For a small perturbation the eigenvalue takes approximative form

$$\sqrt[4]{4w_0} \to \sqrt[4]{4w_0} + \frac{[(\beta + \sqrt[4]{4w_0})\sqrt[4]{4w_0} - \nu]}{8\sqrt{w_0}},$$

$$-\sqrt[4]{4w_0} \to -\sqrt[4]{4w_0} \pm i\sqrt[2]{\frac{(\beta + \sqrt[4]{4w_0})\sqrt[4]{4w_0} - \nu}{\sqrt[4]{4w_0}}},$$

if $(\beta + \sqrt[4]{4w_0})\sqrt[4]{4w_0} > \nu$, then the eigenvalues become a pure real and two complex conjugates with negative real part as it is depicted in Fig. 7. Note that the ratio of the modulus of the real part of the complex conjugate eigenvalues over the real part of the pure real eigenvalue is lower than one. Hence, if the planar homoclinic solution persists in three-dimensions the system must exhibit chaotic behavior. After a straightforward calculation, we

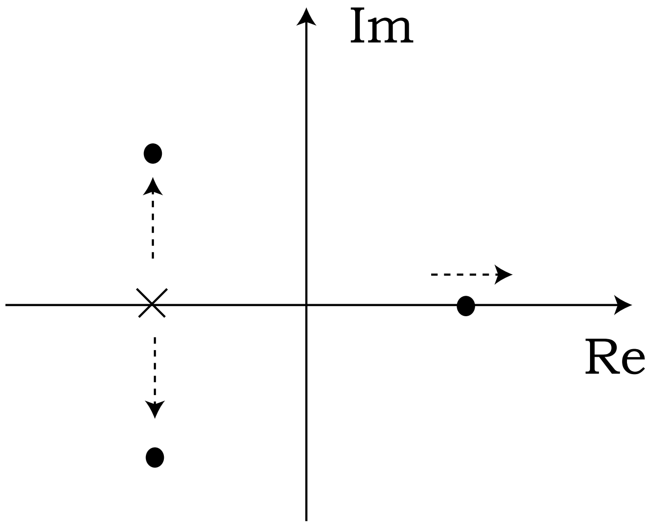


Fig. 7. Spectrum motion of model (7).

can integrate w in model (7) and the system reads

$$\ddot{x} = w_0 - x^2 + e^{-\sqrt[4]{4w_0}t}\int_{-\infty}^{t} dt' e^{\sqrt[4]{4w_0}t'} \times (\nu x + (\beta + \sqrt[4]{4w_0})x^2), \tag{9}$$

where the last term is taken as a perturbation. We consider the following ansatz

$$x(t) = x_h(t) + \zeta(t, x_h(t)), \tag{10}$$

where $\zeta(t, x_h(t))$ is a small corrective function of the order of the perturbation. Introducing the above ansatz in Eq. (9) and linearizing in ζ, we obtain the following condition for the persistence of the homoclinic solution (Melnikov condition)

$$\nu = -(\beta + \sqrt[4]{4w_0})f(w_0), \tag{11}$$

where

$$f(w_0) = \frac{\int_{-\infty}^{\infty} dt e^{-\sqrt[2]{4w_0}t}\partial_t x_h(t)\int_{-\infty}^{t} dt' e^{\sqrt[2]{4w_0}t'} x_h^2(t')}{\int_{-\infty}^{\infty} dt e^{-\sqrt[2]{4w_0}t}\partial_t x_h(t)\int_{-\infty}^{t} dt' e^{\sqrt[2]{4w_0}t'} x_h(t')}.$$

In Fig. 8 we draw the function $f(w_0)$. A numerical simulation of the model (4) using the above condition (11) is depicted in Fig. 6.

5. Mechanical Example of Shilnikov Bifurcation: Shilnikov Particle

5.1. *Reversible 0^3-bifurcation*

Due to the universal nature of the instability under study, we can make up a simple mechanical system that exhibits the quasi-reversal 0^3-instability. We

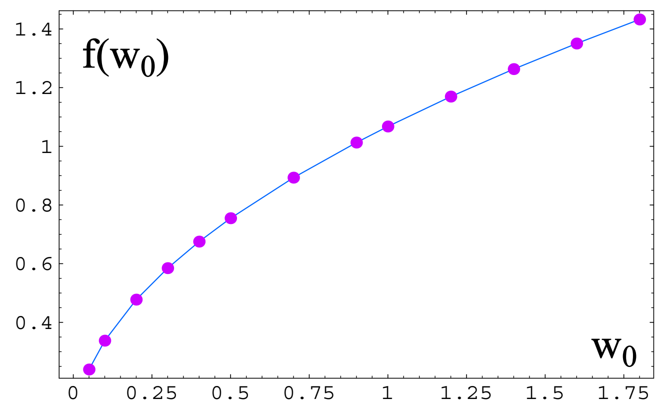


Fig. 8. Function f versus w_0.

consider a system composed of a ring of mass m, which slides over the rod OA without friction. This rod together with the other vertical rod OB are a rigid solid with a fixed angle α between the two rods (see Fig. 9) and this system can rotate in the vertical direction under the influence of the gravitational field. The Lagrangian $\mathcal{L}$ that characterizes this system is

$$\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2 \sin^2\alpha\ \dot{\phi}^2) + \frac{I}{2}\dot{\phi}^2 - mgr\cos\alpha,$$

where r stands for the distance between the ring and the contact point between the rods (point O, cf. Fig. 9), $\dot{\phi}$ is the azimuthal angular velocity of the system, I is the vertical moment of inertia of the two rods and g stands for the gravity. Note that the azimuthal angle ϕ is a cyclic variable, then the equation of motion reads

$$\ddot{r} = \frac{r\sin^2\alpha P_\phi^2}{(I + mr^2\sin^2\alpha)^2} - g\cos\alpha, \tag{12}$$

where $P_\phi \equiv (I + mr^2\sin^2\alpha)\dot{\phi}$ is the azimuthal angular momentum, which is a conserved quantity. The terms on the right-hand side of Eq. (12) stand for the centrifugal and gravitational forces, respectively. The centrifugal force has only one maximum as function of r at

$$r^* = \sqrt{\frac{I}{3m\sin^2\alpha}}.$$

Hence, for small centrifugal force, the system does not have equilibria, and the ring moves to point O. Contrary, for large centrifugal forces the system has two equilibria, one stable and another unstable. Therefore, there is a critical value of the azimuthal

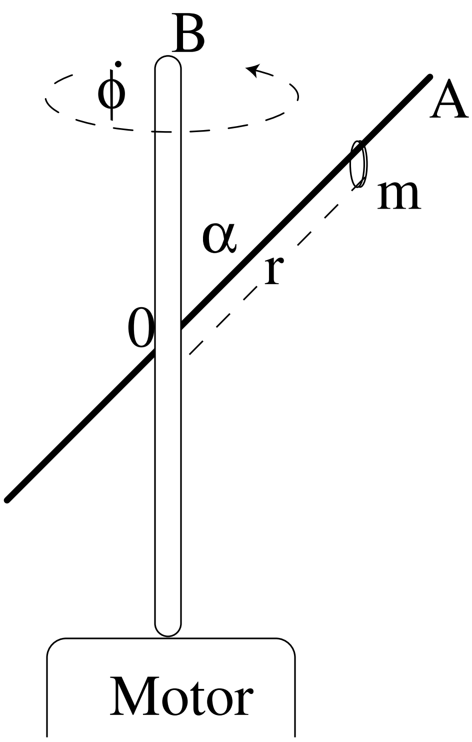


Fig. 9. **Shilnikov particle:** the system is composed of a ring of mass m, which slides over the OB-rod without friction. This rod is soldered to another vertical OA-rod, between these two rods there is an angle α and this system can rotate in the vertical direction.

angular momentum for which the system exhibits a saddle-node bifurcation

$$P_\phi^* = \left[\frac{16\sqrt{3mI^3}}{9} g \cot\alpha\right]^{1/2}$$

In Fig. 10 we show the centrifugal and gravitational forces as function of r, the stable and unstable equilibrium points are represented by the full and empty circles, respectively. Note that the unstable fixed point is closer to point O than the stable one.

Close to the saddle-node bifurcation, we can introduce the following ansatz

$$\begin{aligned} r &= r^* + \rho_0 x, \\ P_\phi &= P_\phi^* + z_0 z, \end{aligned} \tag{13}$$

where $\rho_0 \equiv 1/(6P_\phi^* \sin^3\alpha/(I + mr^{*2}\sin^2\alpha)^3 - 12m^2 r^{*2}\sin^4\alpha/(I + mr^{*2}\sin^2\alpha)^4)$ and $z_0 \equiv \rho_0 (I + mr^{*2}\sin^2\alpha)^4/r^{*2}\sin\alpha$, in Eq. (12). The system is described to dominant order by (reversible 0^3-normal form)

$$\begin{aligned} \ddot{x} &= z - x^2, \\ \dot{z} &= 0. \end{aligned}$$

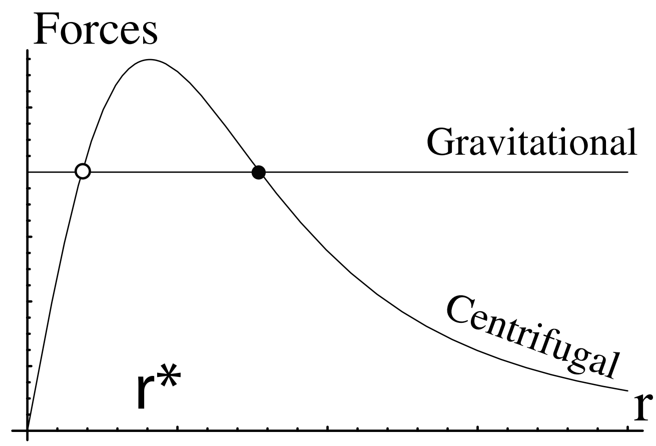


Fig. 10. Schematic representation of gravitational and centrifugal forces as a function of r. The horizontal and curved lines represent the gravitational and centrifugal forces, respectively. The full and empty circles give account of the stable and unstable equilibria.

Hence, the mechanical system exhibits a saddle-node bifurcation driven by the conservative quantity, that is, the model presents a 0^3 instability.

5.2. *Quasi-reversible instability*

In order to realize physically the previous system, we must consider the dissipative effects related to the macroscopic nature of the system and consider injection of energy in order to balance the dissipation. The system is then described by

$$\begin{aligned} \ddot{r} &= \frac{r\sin^2\alpha P_\phi^2}{(I + mr^2\sin^2\alpha)^2} - g\cos\alpha - \mu\dot{r}, \\ \dot{P}_\phi &= -\gamma\left(\frac{P_\phi}{(I + mr^2\sin^2\alpha)} - \omega_0 \right. \\ &\quad \left. + \delta' \frac{P_\phi^2}{(I + mr^2\sin^2\alpha)^2}\right) - \frac{(\nu' + \beta' r^2 \sin\alpha) P_\phi}{(I + mr^2\sin^2\alpha)}, \end{aligned} \tag{14}$$

where the injection of energy is through a motor with constant angular velocity — ω_0 is the angular velocity imposed by the motor — applied in the support of the vertical rod OB. The term proportional to γ accounts for the motor, and now P_ϕ is promoted to a dynamical variable. We have modeled the torque in the azimuthal direction as a polynomial function of the angular velocity with the attractor in ω_0, that is, $\gamma(\delta'\dot{\phi}^2 + \dot{\phi} - \omega_0)$ is the torque generated by the motor. Dissipation will occur through: (a) friction in the rotation of the rod

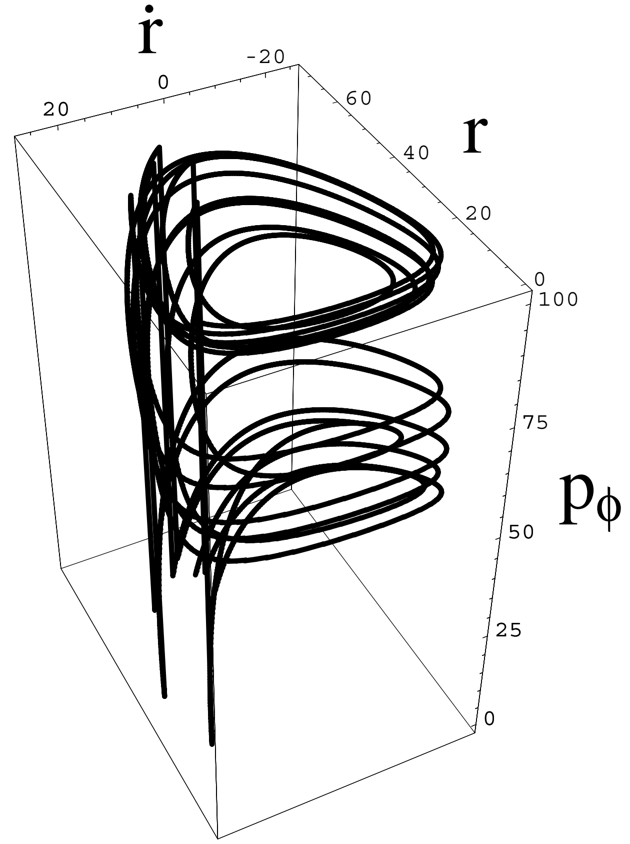


Fig. 11. Strange attractor exhibits by the mechanical system Shilnikov particle, model (14), by $g = 9.8$, $\alpha = 0.619$, $I = 2.6$, $m = 50$, $\omega_0 = 5.3$, $\mu = 2.92$, $\nu' = 4.3$, $\beta' = 0.6$, $\gamma = 350$.

OB around its axis, this dissipation is modeled by the term proportional to ν'; (b) wet friction between the ring and the rod OA, this loss of energy is modeled by the term proportional to μ, (c) motion of the ring in the fluid surrounding it (e.g. the atmosphere), this effect is described by the term proportional to β'. This parameter is related to the lateral section of the ring.

Using the ansatz (13), we obtain Eq. (4), with the parameters

$$\mu = \mu,$$

$$\delta = \frac{\gamma\omega_0}{z_0} - \frac{1}{z_0\sqrt[4]{3^3 I^5 m^3}}(g\gamma\sqrt[4]{3^5 I^3 m^5}\cot\alpha + I\sqrt{g\cot a}(3m(\gamma+\nu') + I\beta'\csc\alpha)),$$

$$\nu = \frac{\rho_0}{2z_0\sqrt[4]{mI^7}}(6g\gamma\sqrt[4]{I^3 m^5}\cos\alpha + 3^{3/4} I\sqrt{g\cot\alpha}(m(\gamma+\nu')\sin\alpha - I\beta')),$$

$$\beta = \frac{3\delta'\gamma g\rho_0^2\sin(2\alpha)}{8z_0}\sqrt{\frac{3m^3}{I^3}}.$$

Numerical simulation of the mechanical system exhibits a similar bifurcation diagram with (2). In Fig. 11 we present the typical limit cycle and chaotic dynamics exhibited by the model.

6. Discussion and Conclusion

The understanding of bifurcations is a cornerstone to develop a qualitative theory of differential equations. Depending on the symmetry, the systems exhibit different type of bifurcations. The time reversible system exhibits complex dynamics like chaos, however the inclusion of small dissipation and injection of energy drastically changes the dynamical behaviors. We have studied the 0^3 stationary instability which arises in quasi-reversible systems, which is characterized by the confluence of three eigenvalues at the origin of the complex plane with only one eigenfunction. By means of normal form theory, we have characterized this bifurcation and its dynamics. A simple mechanical system — Shilnikov particle — that exhibits this quasi-reversible instability has been proposed, and displays Shilnikov chaotic dynamics.

Acknowledgments

The authors thank M. Trejos, G. Kustrolovich and A. Corvalán for fruitful discussions and thank the support of *Anillo* grant ACT15 of programa bicentenario of Chilean Government and FONDAP grant 1020374.

References

Andronov, A. A., Vitt, A. A. & Khaikin, S. E. [1987] *Theory of Oscillators* (Dover, NY)

Arneodo, A., Coullet, P. H., Spiegel, E. A. & Tresser, C. [1985] "Asymptotic chaos," *Physica D* **14**, 327–347.

Arnold, V. [1980] *Chapitres Supplementaires de la Theorie des Equations Differentielles Ordinaires* (MIR, Moscou, 1980), Chap. 6, pp. 211–331.

Baker, N. H., Moore, D. W. & Spiegel, E. A. [1971] "Aperiodic behaviour of a non-linear oscillator," *Quart. J. Mech. Appl.* **24**, 394–422.

Clerc, M., Coullet, P. & Tirapegui, E. [1999a] "Lorenz bifurcation: instabilities in quasireversible system," *Phys. Rev. Lett.* **83**, 3820–3823.

Clerc, M., Coullet, P. & Tirapegui, E. [1999b] "The Maxwell–Bloch description of 1/1 resonances," *Opt. Commun.* **167**, 159–164.

Clerc, M., Coullet, P. & Tirapegui, E. [1999c] "The stationary instability in quasi-reversible systems and the Lorenz pendulum," *Int. J. Bifurcation and Chaos* **11**, 591–603.

Clerc, M., Coullet, P. & Tirapegui, E. [2000] "Reduced description of the confined quasi-reversible Ginzburg Landau equation," *Progr. Theoret. Phys. Suppl.* **139**, 337–343.

Clerc, M. & Marsden, J. E. [2001] "Dissipation-induced instabilities in an optical cavity laser: A mechanical analog near the 1:1 resonance," *Phys. Rev. E* **64**, 067603.

Coullet, P. & Tresser, C. [1978] "Iteration dendomorphismes et groupe de renormalisation," *CR Acad. Sci. Paris A* **287**, 577–580.

Coullet, P. [1985] "Nonlinear phenomena in dissipative systems," in *Nonlinear Phenomena in Physics*, ed. F. Claro (Springer).

Coullet, P. & Elphick, C. [1987] "Topological defects and Melnikov's theory," *Phys. Lett. A* **121**, 233–236.

Cross, M. & Hohenberg, P. [1993] "Pattern formation outside of equilibrium," *Rev. Mod. Phys.* **65**, 851–1112.

Elphick, C., Tirapegui, E., Brachet, M., Coullet, P. & Iooss, G. [1987] "A simple global characterization for normal forms of singular vector fields," *Physica D* **29**, 95–127.

Feigenbaum, M. J. [1978] "Quantitative universality for a class of non-linear transformations," *J. Stat. Phys.* **19**, 25–52.

Guckenheimer, J. & Holmes, P. [1983] *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Springer-Verlarg, NY) Chap. 3, pp. 117–156.

Marzec, C. J. & Spiegel, E. A. [1980] "Ordinary differential equations with strange attractor," *SIAM J. Appl. Math.* **38**, 403–420.

Moore, D. W. & Spiegel, E. A. [1966], "A thermally excited non-linear oscillator," *Astrophys. J.* **143**, 871–887.

Pismen, L. M. [1987] "Homoclinic explosion in the vicinity of bifurcation at the triple-zero eigenvalue," *Phys. Rev. A* **35**, 2709–2717.

Rocard, Y. [1943] *Dynamique Générale des Vibrations* (Masson et cie., Paris) Chap. 14, pp. 224–236.

Shilnikov, L. P. [1965] "A case of the existence of a countable number of periodic motion," *Sov. Math. Dokl.* **6**, 163–166.

Shilnikov, L. P. [1968] "A contribution of the problem of the structure of an extended neighborhood of a rough equilibrium state of saddle type," *Math. USSR Sbornik* **6**, 427–438.