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Solitary waves in van der Waals-like transition in fluidized granular matter

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Abstract

Solitary wave solutions exhibited at the onset of the phase transition in fluidized granular matter are perused. In the quasi-sonic limit the system is modeled by two Korteweg de Vries equations. We study the solitary wave interactions in order to understand the rich dynamics exhibited by the fluidized granular system at the onset of the gas–liquid phase transition.

Keywords: Granular matter; Soliton; Phase transition

Granular matter, when fluidized by continuous energy injection exhibits a variety of phenomena that resemble those of molecular fluid-like waves propagation, pattern formation, and phase transition, to mention a few. The main difference with molecular fluids is that, at collisions, grains dissipate kinetic energy into the internal degrees of freedom of the grains. Hence, energy must be supplied continuously to sustain a fluidized regime. Experimentally, energy is usually injected through vibrating walls or by the gravitational field. Recently, it has been shown that a fluidized granular system in two spatial dimensions with a vibrating wall and without gravity exhibits a phase separation [1–4], analogous to the spinodal decomposition of the gas–liquid transition in the van der Waals (VdW) model [5]. Molecular dynamics simulations of a granular system at the onset of phase transition reveal a rich dynamical behavior characterized by appearance, coalescence, and disappearance of bubbles (or clusters). The mechanism for this phase separation is triggered by a negative compressibility [1,2].

A continuous or macroscopic description of granular flows is still an open question. There are several models with different approximation schemes that produce different hydrodynamic models. Nevertheless, using simple generic arguments, independent of the specific macroscopic model, in Refs. [1,2] it is shown that a fluidized granular system that exhibits phase separation can be described, close to the critical point, in good detail by *the VdW normal form*. This model shows that the appearance, coalescence, interaction, and disappearance of bubbles is mediated by nonlinear waves.

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Our aim will be to identify and characterize the solitary waves of the VdW model, and to study their interactions in order to understand the rich dynamics that the fluidized granular system exhibits at the onset of the phase transition. In the quasi-sonic limit the system can be well described by two Korteweg de Vries equations.

The VdW normal form is [1,2]

$$\partial_{tt}u = \partial_{xx}(\varepsilon u + u^3 - \partial_{xx}u + \nu\partial_t u), \tag{1}$$

where u is the field that describes the correction to the critical average vertical density, x the coordinate that describes the horizontal direction of the granular system, ε the bifurcation parameter which is proportional to the compressibility coefficient, and ν the effective viscosity. The two first terms in the right-hand side give account of the pressure around the critical average vertical density. The term with high spatial derivative depicts the interface tension [2].

Solitary wave solutions: The inviscid VdW model has the form $\partial_{tt}u = \partial_{xx}(\varepsilon u + u^3 - \partial_{xx}u)$. In the moving framework, $z = x - ct$, the previous model reduces to a Newton type equation

$$\frac{d^2u}{dz^2} = (\varepsilon - c^2)u + u^3 - \lambda,$$

where λ is an integration constant related to the total mass, compatible with periodical or zero flux boundary conditions. The equilibrium fixed points (u_0) of this system satisfy $\lambda = (\varepsilon - c^2)u_0 + u_0^3$. It is easy to show that, when $|\lambda| < 2(c^2 - \varepsilon)/3\sqrt{3}$ and $\varepsilon < c^2$, this cubic equation has three real solutions, otherwise it just has one solution. In the first case, two of them are hyperbolic fixed points, while the other is a center fixed point. Then, Newton type equations in general, have a homoclinic loop, which corresponds to a traveling solitary wave of the inviscid VdW model. To have a solitary wave solution, we must impose

$$u_0^2 + \varepsilon < c^2 < v_s^2 \equiv 3u_0^2 + \varepsilon. \tag{2}$$

Hence, the wave speed is bounded. v_s is the sound speed about the homogeneous state u_0, therefore the *solitary waves are subsonic*. Due to the symmetry of $\lambda \to -\lambda$ and $u \to -u$, we will suppose without loss of generality $\lambda > 0$. In that case, the homoclinic orbit lies below the negative state $u_0 < 0$ (the lowest fixed point). And we have *bright solitary wave* solutions

$$u = u_0 + \frac{2(3u_0^2 + \varepsilon - c^2)}{\sqrt{2(c^2 - u_0^2 - \varepsilon)}\,\mathrm{Cosh}\left[\sqrt{(3u_0^2 + \varepsilon - c^2)}((x - x_0) - ct)\right] - 2u_0}. \tag{3}$$

In the opposite case, $\lambda < 0$, we have *dark solitary wave solutions*, which hold up the upper fixed point. In the limiting case, $\lambda = 0$, there are two heteroclinic connections. Hence, we have kink or anti-kink solutions. Fig. 1 illustrates the solitary wave solution.

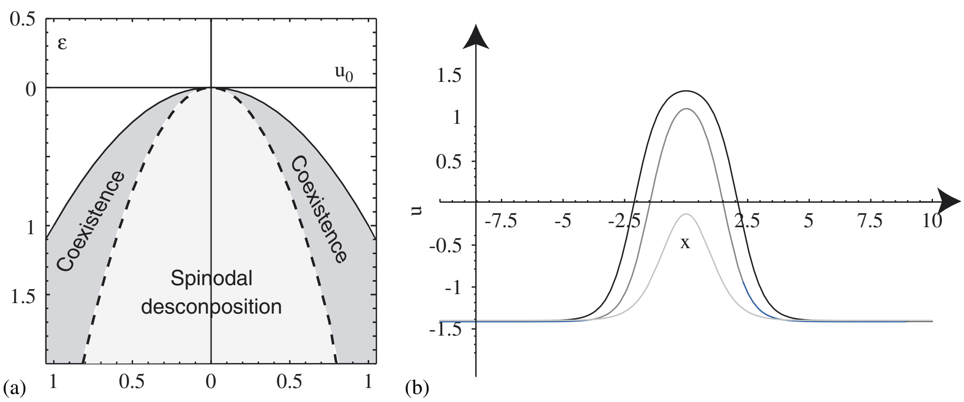

Fig. 1. Bifurcation diagram of VdW model and solitary waves: (a) bifurcation diagram, (b) three different solitary waves supported by the same homogeneous state.

The amplitude of the solitary wave decreases with the speed and has the analytical expression $2(3u_0^2+\varepsilon-c^2)/(-2u_0+\sqrt{2(c^2-u_0^2-\varepsilon)})$. When the speed gets near to the sound speed, the solitary wave tends to the homogeneous state $u=u_0$. On the other hand, when the wave speed decreases until $u_0^2+\varepsilon$, the solitary wave tends to the symmetric homogeneous state $u=-u_0$. The previous analysis is valid only in the parameter region where there is no phase separation ($\varepsilon>-u_0^2$, cf. Fig. 1). In the coexistence region, $-3u_0^2<\varepsilon<-u_0^2$ (see Fig. 1), the lower limit of speed exists no more. In this case, when the speed tends to zero the solitary wave tends to an unstable bubble. This solution is the nucleation barrier between the homogeneous state and the phase separation one. In the spinodal decomposition region, $\varepsilon<-3u_0^2$ (see Fig. 1), there is no solitary wave solution. However, in a state composed of one or more bubbles, there are regions in space where they are almost homogeneous. Then, we locally have a stable homogeneous state, which can support solitaire waves (cf. Fig. 2). These waves are well approximated by (3).

Quasi-sonic solitons: If we consider the solitary wave speed near the sound speed, $c=v_s-w$ ($w/v_s\ll 1$), and $u_0\sim\mathcal{O}(1)$, then we can approximate (3) by

$$u-u_0\simeq-\frac{v_s w}{u_0}\,\mathrm{Sech}^2\left[\sqrt{\frac{v_s w}{2}}\{x-(v_s-w)t-x_0\}\right]. \tag{4}$$

It is a typical soliton solution of Boussinesq or Korteweg de Vries equation, with the standard relation between its width (σ) and its amplitude (A): $\sigma\propto 1/\sqrt{A}$ where u_0 is fixed.

Considering the change of variables: $T=v_s^2 t$, $X=v_s x$ and $u=u_0+(v_s^2/3|u_0|)v$; and defining $\beta=v_s^2/9u_0^2$, we can rewrite the inviscid VdW model as

$$\partial_{TT}v=\partial_{XX}(v-v^2+\beta v^3-\partial_{XX}v).$$

Taking as inspiration the KdV relation between the width and the amplitude, we can introduce the scaling $\xi=\chi(X-T)$, $\theta=\chi(X+T)$, $\tau=\chi^3 T$, $v=\sum_{n=1}^{\infty}\chi^{2n}v_n$, where $\chi\sim\mathcal{O}(\sqrt{w})$. The first order in χ gives us the wave equation

$$\partial_{\theta\xi}v_1=0\Longrightarrow v_1=f(\xi)+g(\theta),$$

that has the D'Alembert solution shown above. At second order we have

$$4\partial_{\theta\xi}v_2=\partial_\xi[-2\partial_\tau+2f\partial_\xi+\partial_{\xi\xi\xi}]f+\partial_\theta[2\partial_\tau+2g\partial_\theta+\partial_{\theta\theta\theta}]g+2(\partial_\theta+\partial_\xi)^2 fg,$$

which is a linear inhomogeneous equation for v_2. In order to have a bounded solution for v_2, we impose (solvability conditions)

$$-2\partial_\tau f+2f\partial_\xi f+\partial_{\xi\xi\xi}f=0,$$

$$2\partial_\tau g+2g\partial_\theta g+\partial_{\theta\theta\theta}g=0.$$

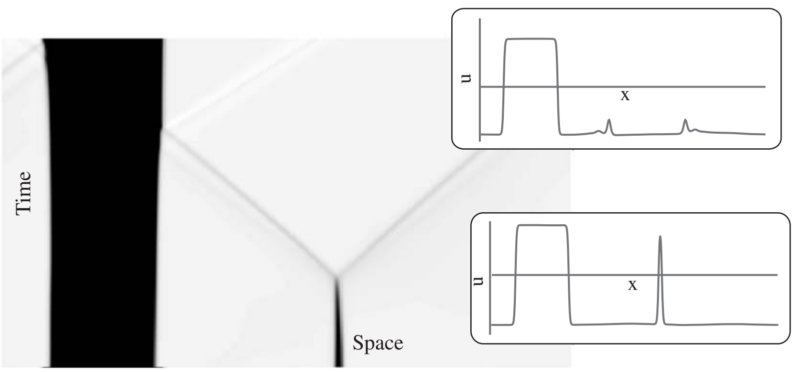

Fig. 2. Spatio temporal evolution of the VdW model at the onset of the phase transition, with time running up. The gray scale is proportional to the field u, with darker regions representing denser regions in the system. The inset figures illustrate two different snapshots before and after solitary wave emission.

Hence, we have two uncoupled KdV equations, which are associated to right and left moving frameworks with sound speed, respectively. The difference in sign guarantees that their solutions move against the sound velocity, that is, they are subsonic. The solitons (4) are exact solutions of these KdV equations. The advantage of having these KdV approximations consists in having a well-known integrable system [6]. For instance, when two solitons, at the same moving framework, collide, they just undergo a change in their phases. Then, they are genuine solitons, in the sense that they preserve their structure after a collision.

Left–right soliton interaction: Up to first order in χ, the left-waves do not interact with right-waves. To study the interaction between these waves, we must solve the next order. After imposing the solvability condition, we have

$$v_2 = \frac{1}{2}\left[(\partial_\theta g)\left(\int d\xi f\right) + 2fg + (\partial_\xi f)\left(\int d\theta\, g\right)\right].$$

Note that we have the freedom of an arbitrary integration constant. We can solve this ambiguity as in perturbation Quantum Mechanics theory, that is, imposing that the higher order corrections are orthogonal to the first one (i.e., do not contain any D'Alembert solution). Therefore, if we choose g and f as soliton profile (cf. Eq. (4)), then, as a consequence of the second-order correction v_2, we can infer that at soliton collision, the field u is higher and thinner than the superposition of both solitons. After the collision the solitons lose their reflection symmetry with respect to the maximum.

When the viscosity v is taken into consideration the solitary waves become unstable and they exhibit diffusive behavior. However, for small viscosity the decay time is small enough for the solitary waves to mediate the interaction between the bubble at the onset of phase transition. The experimental and numerical (molecular dynamics simulations) study of the solitary waves close to the phase transition is in progress.

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