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Characterization of Faraday patterns and spatiotemporal chaos in parametrically driven dissipative systems

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ABSTRACT

In this work, we have studied numerically the dynamics of the parametrically driven damped nonlinear Schrödinger equation (PDDNLS). The PDDNLS is a universal model to describe parametrically driven systems. In particular, we have characterized stationary Faraday patterns, periodic, quasi-periodic, and spatiotemporal chaos as a function of the amplitude and the frequency of the parametric driving force. We have computed the Lyapunov spectra, the Fourier spectra, the amplitude norm, and the Kaplan–Yorke dimensions as valuable indicators for the identification of several dynamical regimes. We show that in the Faraday regime, close to the bifurcation of the trivial state, the pattern amplitude scales with power one-fourth (1/4) of the bifurcation parameter. Furthermore, we have found that the pattern wavelength decreases when the detuning parameter increases. In the case of the high dimensional spatiotemporal chaotic states, we have found that the Kaplan–Yorke dimension increases linearly with the length of the system, showing its extensive character in this dynamical regime. We have also found a transition from low to high dimensional chaos when the forcing amplitude is increased.

1. Introduction

Pattern formation and dissipative structures have been widely studied in science and technology [1]. Multiple physical and chemical systems exhibit a transition from a homogeneous state toward a patterned state. Many examples have been found in hydrodynamics, such as Taylor–Couette and Rayleigh–Bénard convection [2]; in solidification processes, nonlinear optics, oscillatory chemical reactions, biological systems, and granular media [3–6].

Spontaneous pattern formation has fascinated scientists for ages. These systems are examples of how a system out of thermodynamic equilibrium may organize itself into an ordered pattern. The great scientist Michael Faraday made a seminal contribution to the field of pattern formation [7]. He realized that a liquid layer vertically vibrated with a specific frequency and amplitude undergoes an instability to a patterned state. A critical driving force defines the onset of what is now called the Faraday instability [8]. After this first instability, the observed pattern consists of nonlinear surface waves (ripples) that organize themselves as stripes, squares, hexagons, and even quasi-crystals, depending on the system parameters and the excitation regimes [9]. Note that the Faraday instability is a parametric resonance where the created pattern oscillates at a frequency that is half of the excitation frequency.

Recently, one has observed a revival of interest in the study of the Faraday instability in the literature. Several studies have explored the Faraday instability in unexpected fields, for example, in Bose–Einstein condensates (BEC), where experimental and theoretical contributions have been proposed [10–12]. Another example is the non-linear optics domain, where parametric resonances are commonly observed in the

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interactions between laser and matter [13,14]. In particular, due to their characteristics of very short length and time scales, the optical and BEC systems have allowed the researchers to explore dynamic regimes much further from the first Faraday instability that was conventionally observed in fluid systems. These studies have been pivotal in unveiling the secondary transitions that may lead to chaos and turbulence in those systems.

From a technological standpoint, the applications of the Faraday instability are becoming increasingly more relevant. One can cite, non-exhaustively, the crafting of hydrodynamic crystals at the fluid interface where the surface rigidity is modified thanks to the addition of surfactants [15]. Another technological application concerns using Faraday waves as frequency combs to perform spectroscopy in the acoustic range, with massive potential in biomedical imaging applications [16]. Finally, and complementary to the two previously mentioned applications, one can cite industrial ultrasonic atomization, which consists of the ejection of fine droplets from a liquid film formed on an ultrasonically vibrating surface [17]. The study of combined effects adds some interesting complexity to the fundamental Faraday wave instability problem. One can cite a few situations that have been studied recently. The non-homogeneity of the excitation unavoidable in many experimental systems leads to interesting localized patterns [18]. The competition of these localized structures has been investigated recently and has shown a variety of complex situations that result from this competition [19]. Finally, the interaction of the magnetic field and the forcing frequency in a Fermi–Fermi mixture has also led to new interesting dynamical situations in the context of the Faraday wave instability [20].

The classical techniques to deal with Faraday's pattern above the first instability threshold are well-known: these are based on the *amplitude equations* [21–24]. The amplitude equations describe the slow modulations in space and time of the periodic structure envelope. In the case of the Faraday instability, the description results in a parametrically driven and damped nonlinear Schrödinger equation (PDDNLS) [25]. This equation is the standard normal form for 2:1 parametric resonance, and has been deeply studied from theoretical and experimental point of view in optics, fluids or even in magnetism showing several dynamical behaviors [26–49]. Generalization and other applications of this equation can be found in Refs. [50–63].

Careful studies of the experimental characterization of Faraday's pattern have been reported in fluids [64]. Some experiments have reported the co-existence of two competing states in localized regions [65]. Faraday's instability has also been evidenced in multimode laser dynamics [66]. From the theory side, the competition between different patterns (squares, stripes, and hexagons) has been elucidated through asymptotic analysis of the amplitude equation near the threshold and also extensive numerical calculations [67,68]. A more involved experimental setup has investigated the possibility of Faraday's instability via parametric forcing with two commensurate frequencies [69], giving rise to superlattice and quasi-crystal patterns. These scaling laws associated with the spatiotemporal chaos in Faraday's surface waves were compared to scaling laws for turbulence and the differences were examined in [70]. The typical phenomenon of intermittency with a period of regular dynamics succeeding chaotic dynamics was also studied in great length in the context of Faraday's instability in [71–73] and compared with the classical theoretical predictions [74,75]. The connection between Faraday's waves on the surface of a vertically oscillated fluid layer and their relation with weak wave turbulence has been studied and shown to be generally disagreeing with each other [76]. Faraday's experiment with a fluid flowing down an inclined plane has been performed recently and compared satisfactorily with full numerical simulations in [77].

On the other hand, the characterization of spatiotemporal chaos has been studied in several contexts in recent years due to the access to improved computational power. Examples of complex spatiotemporal dynamics can be found in fluids, optics, coupled nonlinear oscillators, electric network devices, liquid crystals, chemical reactions, and cardiac dynamics, to mention a few. In the context of Faraday's instability, one can refer to the group of J. Gollub that performed a systematic experimental survey of both the primary patterns and the secondary instabilities of parametrically forced surface waves (Faraday waves) in the large system limit [78,79]. Universal scaling laws, a classical hallmark of chaos, have also been found in the Faraday instability in highly dissipative fluids [80,81]. Recently, a liquid crystal light valve experiment has led to the drawing of a complete bifurcation diagram of the observed complex spatiotemporal dynamics close to the spatial instability of waves [82].

The main characterization method used here is the computation of the Lyapunov exponents [83]. This paper aims to add numerical tools for the characterization of the secondary bifurcations of Faraday patterns in weakly dissipative parametrically driven systems described by PDDNLS equation. In particular, we will extensively compute the Lyapunov spectrum to characterize the level of the observed chaotic dynamics. We will also compute the associated Kaplan–Yorke dimension that indicates the dimension of the chaotic attractor [84]. The Fourier spectra are used to discriminate between periodic and quasi-periodic dynamics. In addition, the spatial average of the amplitude of the pattern is monitored in time as an additional dynamical indicator.

The manuscript is organized as follows: The model and some analytical results are presented in Section 2. Results of a systematic numerical exploration of the Faraday patterns and spatiotemporal chaos are given in Section 3. Finally, conclusions and future directions are drawn in Section 4.

2. The model and some analytical results

We consider an array of coupled nonlinear oscillators in the continuum limit under the action of dissipation as well as a parametrically driven forcing close to the parametric resonance. The system can be characterized by the parametrically driven dissipative nonlinear Schrödinger (PDDNLS) equation [26]:

$$\frac{\partial A}{\partial \tau} = -i\nu A - i|A|^2 A - i\frac{\partial^2 A}{\partial x^2} - \mu A + \gamma A^*, \tag{1}$$

where $A(\tau, x)$ is a complex-valued field; the asterisk stands for the complex conjugate of A; τ is a normalized time, and x is the normalized spatial coordinate. In Eq. (1), $\gamma > 0$ is the parametric-drive coefficient, ν measures the detuning of the drive, and $\mu > 0$ is the damping constant. Note that the nonlinearity coefficient is scaled to unity. Note also that the sign of the nonlinear term in Eq. (1) corresponds to the case of self-focusing onsite nonlinearity. Let us remind that we can also choose the scaling factor for A in Eq. (1) such that it allows fixing one of the equation parameters, which here will be the damping parameter, μ.

Let us mention some analytical results about the PDDNLS equation. First, Eq. (1) possesses the trivial solution $A = 0$ and two nontrivial homogeneous solutions, $A_{\pm,\pm} = \pm(1 \pm i\sqrt{(\gamma-\mu)/(\mu+\gamma)})x_0$ such that $x_0 \equiv \sqrt{(\gamma-\mu)(\phi-\nu)/2\gamma}$, with $\phi = \sqrt{\gamma^2-\mu^2}$. The two nontrivial solutions bifurcate from $A = 0$ at $\gamma^2 = \mu^2 + \nu^2$. This latter relation defines the first Arnold's tongue commonly used in the γ versus ν diagrams. The study of the stability of trivial state was analyzed in Ref. [25], whereas for the nontrivial ones in Ref. [57]. Let us remark that in the special case $\mu = \gamma = 0$, Eq. (1) is reduced to the nonlinear Schrödinger equation, which is a time-reversible equation [85]. Let us comment that the model (1) exhibits also soliton-like solutions that have been studied extensively in [26,28], two-soliton states [30,31,47,48], or soliton-radiation [32], just to mention a few.

Examples of patterns exhibited in the solutions of Eq. (1) are shown in Fig. 1 for different dynamical scenarios. In Fig. 1, we have fixed the detuning and damping parameters and varied the forcing amplitude γ in the range $[0.301, 0.8]$. We observe in Fig. 1 transitions from stationary to regular dynamical states to further complex states, from simple Faraday waves to high-dimensional chaos. In the time window in Fig. 1, we can observe the collective spatiotemporal dynamics with the maximum resolution in time and space available. The numerical details for the integration of Eq. (1) are given in Section 3.

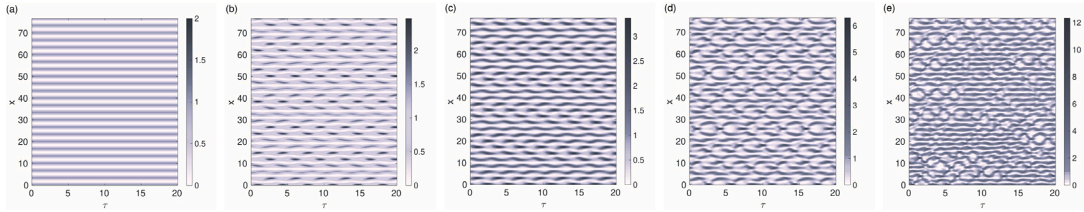

Fig. 1. Five panels showing the space–time dynamics of amplitude $|A|^2$ as color coded. Parameters in Eq. (1) are varied from regular to spatiotemporal chaotic dynamics (left to right panels). The chaotic level is quantified with the maximum Lyapunov exponent λ_{max}. From left to right: $\gamma = 0.390$ (Faraday waves, $\lambda_{max} = -0.0002$), $\gamma = 0.462$ (periodic state, $\lambda_{max} \approx 0$, up to the fourth decimal place), $\gamma = 0.534$ (quasi-periodic state, $\lambda_{max} \approx 0$, up to the fourth decimal place), $\gamma = 0.561$ (low dimensional chaos, $\lambda_{max} = 0.005$) and $\gamma = 0.687$ (high dimensional chaos, $\lambda_{max} = 0.117$). The other parameters, detuning $\nu = 0.4$ and damping $\mu = 0.3$ are held constant. For all the simulations, the transient time (not shown) was of 1.5×10^6 time units. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

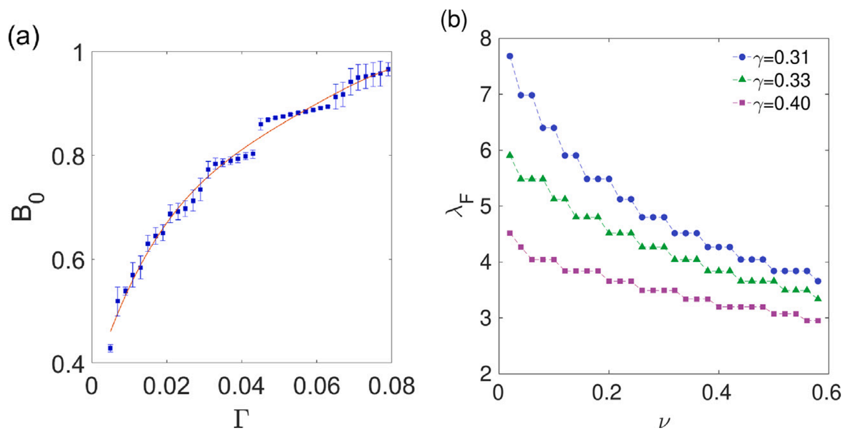

Fig. 2. (Color online) On the left panel (a), the squares indicate the pattern amplitudes extracted from the full simulation of Eq. (1) and the dashed line indicates the 1/4 predicted scaling law as a function of Γ for parameter $\nu = 0.3$ held constant. The right panel (b), displays the estimated Faraday wavelength λ_F (through the Fourier spectrum method; see text) in the case of stationary dynamics as a function of parameter $\nu > 0$. Squares correspond to $\gamma = 0.4$; triangles are for $\gamma = 0.33$, and circles are for $\gamma = 0.31$.

2.1. Instability of the trivial state: Faraday waves

To understand the mechanism of pattern formation in the model Eq. (1), let us briefly recall the results of the stability of the trivial solution $A = 0$ studied in Ref. [25]. Without the spatial term, the trivial state is stable outside of the Arnold's tongue. The spatial coupling, expressed by the Laplacian term in the PDDNLS Equation (dispersive term), modifies this scenario. The zero state exhibits a spatial instability at $\gamma = \mu$ for positive detuning, $\nu > 0$. This instability gives rise to the occurrence of a patterned state like the one shown in Fig. 1(a). Close to the bifurcation, the ansatz:

$$Re(A) = \left(B + \frac{1}{8\nu}B^3 e^{2ik_c Z}\right) e^{ik_c Z} + c.c. + \mathcal{O}(B^5)$$
$$Im(A) = -\frac{3}{2\gamma} B|B|^2 e^{ik_c Z} + c.c. + \mathcal{O}(B^5)$$

with $k_c \simeq \sqrt{\nu}$, results in an amplitude equation for the pattern amplitude B,

$$\frac{\partial B}{\partial \xi} = \Gamma B - \frac{9}{2\gamma}|B|^4 B - i\frac{\partial^2 B}{\partial Z^2}, \tag{2}$$

where $\Gamma = \gamma - \mu$ is the bifurcation parameter and $\xi = \Gamma\tau$ is the time scale. This equation has the form of a quintic Ginzburg–Landau equation for the complex amplitude B with real coefficients [25]. We remark that close to the bifurcation, the pattern amplitude B is constant and increases with Γ according to the power law, $|B_0| \sim \Gamma^{1/4}$. Let us remark that this law was experimentally observed on parametric mercury surface waves [86].

The pattern structure described by Eq. (2) not only exists and is stable for $\nu > 0$, for which it has been derived, but also for $\nu < 0$, if $\gamma^2 \geq \mu^2 + \nu^2$ (i.e. within the Arnold's tongue). By reducing γ (at negative ν), the pattern structure vanishes at the Arnold tongue border through a saddle–node bifurcation before reaching the $\gamma = \mu$ line.

The left panel of Fig. 2 compares the direct numerical simulations of Eq. (1) with the theoretical scaling prediction (as indicated by a dashed line). We observe an excellent agreement between the simulations and the theoretical predictions. The error bars in Fig. 2a depict the dispersion (plus and minus standard deviation) of the computed $|B_0|$ for a set of ten random different initials conditions per simulation point. We remark that we are focusing on the case of small forcing when $\Gamma \ll 1$. In addition, we can estimate the Faraday wavelength in the presence of stationary patterns. For this purpose, we first compute the square amplitude, $|A(\tau,x)|^2 = |A(x)|^2$, which is just a function of space, and then we compute its spatial Fourier transform:

$$\mathcal{F}(k) = \frac{1}{\sqrt{2\pi}} \int_0^L |A(x)|^2 \exp(-ikx)\, dx\,. \tag{3}$$

Later, we compute its power spectrum $|\mathcal{F}(k)|^2$, and from it, we extract its maximum peak located at k_F. The highest peak in the Fourier spectrum gives the approximate Faraday wavelength: $\lambda_F = 2\pi/k_F$. The right panel of Fig. 2 displays the Faraday wavelength λ_F as a function of the detuning parameter $\nu > 0$ for three selected values of γ. We observe from Fig. 2 that the Faraday wavelength decreases with ν but in a non-monotonous fashion. This stair-like decrease is due to the finite size effect in evaluating the dominant wavelength. Note that a similar wavelength decrease was recently recorded in an experiment of Faraday waves in a parametrically driven dissipative cold atom system [20]. Finally, it is interesting to note that closer to the bifurcation at $\gamma = \mu$, a wider range of values for λ_F can be reached.

In the previous situation, the dynamical regime leads to a stationary pattern. However, as we can observe from Fig. 1 if the parameter γ increases further, the pattern suffers multiple bifurcations and becomes chaotic. To the best of our knowledge, the full exploration of the spatiotemporal patterns of Eq. (1) has not yet been published. In the next section, we aim to numerically explore the regions of existence of the chaotic patterns and characterize their dynamics.

3. Spatiotemporal chaos

This section focuses on the spatiotemporal states observed after secondary bifurcations of the Faraday patterns take place. The first subsection describes the dynamical indicators that are used for their characterization. The second subsection describes the results of the numerical simulations through phase diagrams. The last subsection is devoted to quantifying the level of chaos in the spatiotemporal states.

Let us first briefly comment on the numerical integration scheme. To solve numerically the amplitude equation, Eq. (1), we use a classical fourth-order Runge–Kutta (RK) scheme for the time evolution and a second-order central finite difference method to discretize space. The system length L is discretized with $N = 512$ lattice points separated by $\Delta x = 0.15$, which implies a system size of $L = N\Delta x \approx 77$. At the edge of the spatial domain, we impose Neumann boundary conditions. Temporal discretization is set to $\Delta t = 0.001$ to ensure stability and high accuracy of the time integration scheme. All the simulations have been performed with a fixed damping coefficient value, $\mu = 0.3$. The initial conditions for the amplitude follow the homogeneous stationary pattern obtained for parameters $\nu = 0.6$ and $\gamma = 0.35$. The algorithm schemes were implemented in the C programming language [87], and the Fourier libraries needed for the subsequent analysis were borrowed from the open-access GSL libraries [88]. We have also checked the simulations using a Python code, with the library *Numpy* [89]. Let us emphasize that the spatial grid resolution Δx was chosen such that the Faraday wavelength λ_F (see Fig. 2) was notably larger than Δx for the parameters that we have explored in this work. Finally, let us remark that for all the simulations, we discard a transient time of 1.5×10^6 time units before measuring any of the dynamical indicators. This elapsed transient ensures that we are not measuring transitory regimes but stationary states. We have checked several values of the transient time to ensure that the considered states have converged into a steady one.

3.1. Dynamical indicators

Presumably, the most relevant indicators for the characterization of the dynamics are given by the Lyapunov exponents [90,91]. The set of the Lyapunov exponents constitutes the Lyapunov spectrum. In principle, the Lyapunov spectrum can be continuous in the case of spatiotemporal dynamics. However, due to the discretization of the numerical scheme, we only get a discrete representation of the Lyapunov spectrum. Practically, it is obtained by computing the time evolution of the perturbation vectors δA_k of the linearized system according to:

$$\frac{\partial \delta A_k}{\partial \tau} = \bar{\mathbf{J}} \cdot \delta A_k, \tag{4}$$

where $\bar{\mathbf{J}}$ is the Jacobian matrix of Eq. (1) evaluated along the dynamical trajectory of Eq. (1). The vectors δA_k have N components and $k = 1, 2, \dots, N_{pert}$, with $N_{pert} \le N$. With the integration of Eqs. (4) over a time interval t_s, one can compute *instantaneous* Lyapunov exponents as:

$$\widetilde{\Lambda}_k = \frac{1}{t_s} \ln \frac{\|\delta A_k\|_{t_s}}{\|\delta A_k\|_0} \tag{5}$$

where $\|\delta A_k\|_0$ is the vector norm at the beginning of the time integration interval. After renormalization of the perturbation vectors δA_k, the process is repeated N_s times to compute an estimate of the finite time Lyapunov exponent:

$$\Lambda_k = \frac{1}{N_s} \sum_{i=1}^{N_s} \widetilde{\Lambda}_k, \tag{6}$$

where N_s denotes the number of renormalizations performed. Note that we also evaluated the standard error of each of the Lyapunov exponents by also computing the second moment of $\widetilde{\Lambda}_k$. The limit $N_s \to \infty$ yields the steady state Lyapunov exponent.

In the numerical scheme, the renormalization actually consists in an ortho-normalization of the different perturbation vectors δA_k through the Gram–Schmidt algorithm. This procedure is repeated every t_s time unit [90]. The set of Lyapunov exponents $k = 1, 2, \dots, N_{pert}$ forms the Lyapunov spectrum associated with the Eq. (1). From a practical point of view, here we have followed the numerical approach given in Ref. [91]. Let us remark that the Lyapunov exponents $\{\lambda_i\}$ have been used in the characterization of many dynamical systems [92–111].

In the second step, once the Lyapunov spectrum has been computed and ordered from largest to smallest, we compute the Kaplan–Yorke dimension D_{KY} following the formula [84]:

$$D_{KY} = p + \frac{1}{|\lambda_{p+1}|} \sum_{i=1}^{p} \lambda_i, \tag{7}$$

where p is the largest integer such that the sum of the first p Lyapunov exponents is non-negative. The Kaplan–Yorke dimension D_{KY} is conjectured to give the dimension of the chaotic attractor. For regular states, one gets that $D_{KY} = 0$, whereas $D_{KY} > 0$ means a chaotic state. We can also infer that when D_{KY} is small with respect to N, one has a dynamical state of low-dimensional chaos.

From the numerical point of view, we have set $N_{pert} = 128$ perturbation vectors δA_k. That means we must solve N_{pert} copies of the variational Eq. (4). We have set $t_s = \Delta t$, which means that we reorthonormalize the perturbation vectors δA_k at each time step [91]. We have checked that $N_{pert} = 128$ perturbation vectors are sufficient to ensure an accurate estimate of the Kaplan–Yorke dimension in the parameter space $\{\gamma - \nu\}$ considered here. The number of perturbation vectors N_{pert} drastically affects the computation time because each new perturbation vector δA_k implies the solution of an additional variational Eq. (4). Hence, the correct election of N_{pert} is essential to optimize the parameter space exploration.

Let us remark that transients are also important in evaluating the Lyapunov spectrum. Here, we solve Eq. (1) for a transient time of 1.5×10^6, and then we start to solve Eq. (1) and Eqs. (4) for another 3×10^4 time units. After this second transient time, we use only the last 5×10^3 time units to compute the average Lyapunov spectrum Λ_k. The Kaplan–Yorke dimension D_{KY} is evaluated once the Lyapunov spectrum Λ_k is obtained.

To validate our Lyapunov spectrum code, we have reproduced several results regarding the Kaplan–Yorke dimension that are available in the literature. In particular, we have successfully reproduced D_{KY} for the Lorenz-96 model [91] and the Lugiato–Lefever model [112–114].

Another relevant quantity to characterize different dynamical behaviors is the total norm of the amplitude:

$$Q_\tau = \frac{1}{L} \int_0^L |A(\tau, x)|^2 \, dx. \tag{8}$$

Note that Q_τ is a dynamical invariant for the NLS equation (when the damping and parametric force are vanishing), and it is time-independent when the dynamical state is steady. On the contrary, it is a function of time for non-stationary states. In the latter case, we can compute the temporal Fourier spectrum of Q_τ, $S(f) = |\mathcal{F}(f)|^2$, as a function of the time–frequency f. The temporal Fourier transform of the function Q_τ is evaluated through:

$$\mathcal{F}_Q(f) = \frac{1}{\sqrt{2\pi}} \int_0^{\tau_{\max}} Q_\tau \exp(-i f \tau) \, d\tau. \tag{9}$$

A general result is that $S(f)$ features a quasi-discrete spectrum with a set of narrow peaks that indicates regular solutions, which exhibit (quasi-) periodic evolution in time. On the other hand, the spectrum is expected to be continuous if the underlying time-dependent solution is chaotic. Practically, the Fourier transform was numerically computed using the open GSL library [88].

3.2. Phase diagrams of the dynamical states

In this section, we analyze the results provided by the dynamical indicators. The maximum Lyapunov exponent and dynamical states in the $\gamma - \nu$ parameter space are displayed in Fig. 3.

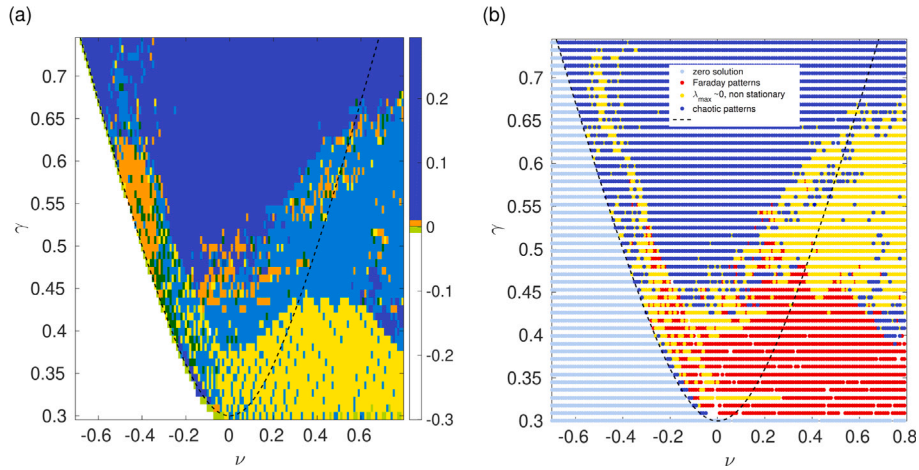

Fig. 3. Maximum Lyapunov exponent (left) and dynamical states (right, see text) in $\gamma - \nu$ parameter space for the PDDNLS equation. On the left, the colors indicate the following ranges: white, less than −0.01; light green, −0.01 to −0.001; yellow, −0.001 to −0.0001; light blue, −0.0001 to +0.0001; green, +0.0001 to 0.001; orange, 0.001 to 0.01; blue, greater than 0.01. On the right panel, the red dots correspond to Faraday waves. Yellow dots correspond to non-stationary states with small Lyapunov exponent (see text), while blue dots correspond to chaotic patterns. Dashed lines in both figures indicate the Arnold's tongue limits. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

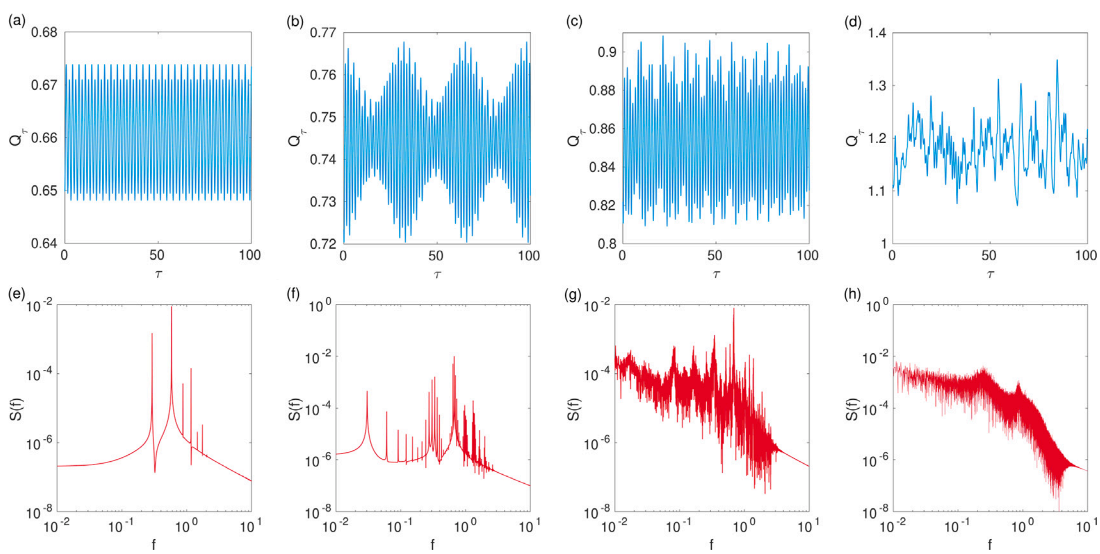

Fig. 4. (Color online) Upper panels show the time evolutions of the norm Q (Eq. (8)) and the lower panels their corresponding Fourier power spectra $S(f)$ (see text for details) for four characteristic dynamical scenarios. (a) and (e): $\gamma = 0.462$. (b) and (f) : $\gamma = 0.534$. (c) and (g) : $\gamma = 0.561$. (d) and (h) : $\gamma = 0.687$. The detuning parameter is hold to $\nu = 0.4$ in all cases. See also the corresponding patterns given in Fig. 1.

In Fig. 3, we observe that the left border given by the Arnold's tongue coincides with the limit between space–time chaos and zero solution (fixed point solution corresponding to $A = 0$). On the contrary, the right border of the Arnold's tongue does not discriminate dynamic states. Depending on the parameters $\gamma - \nu$, the dynamical state settles to a regular Faraday wave pattern, quasi-periodic pattern, or space–time chaos, as illustrated in Fig. 1.

Due to the numerical precision of the calculations of the Lyapunov exponents, we will consider as vanishing a Lyapunov exponent that satisfies the following condition of $|\lambda| < 10^{-4}$. Note that longer and more costly simulations would lower this limit for considering vanishing exponents.

Of particular relevance is the maximum Lyapunov exponent Λ_{max} as shown in Fig. 3(a). This dynamical indicator quantifies how fast the distance between two initially close trajectories δA of the vector field A either vanishes exponentially ($\Lambda_{max} < 0$) or diverges ($\Lambda_{max} > 0$). Fig. 3(b) displays that the parameters $\gamma - \nu$ selects the dynamical state of the system governed by Eq. (1).

3.3. Spatiotemporal chaos

In this last subsection, let us concentrate on the analysis of the spatiotemporal chaos exhibited in Eq. (1).

Fig. 4 displays four different typical dynamical behaviors exhibited by the PDDNLS Eq. (1) as detected by the norm of Q_τ and their corresponding temporal Fourier spectra. Note that the spatiotemporal patterns associated with the specific parameter values used in Fig. 4 are displayed in Fig. 1. They correspond with increasing values of the parameter γ of periodic, quasi-periodic, low dimensional, and high dimensional chaos.

The Lyapunov spectra and their cumulative sums $\sum \lambda_i$ are displayed in Fig. 5. From the calculations of the Lyapunov spectra, one can evaluate the Kaplan–Yorke dimension by using Eq. (7). In panel (b) of Fig. 5, we checked the expected extensivity [91] of D_{KY} with respect to the dimension of the system (spatial extension indicated by N the number of grid points). Note that the inset of panel (b) of Fig. 5 shows the collapse of the Lyapunov spectrum to a single curve within numerical fluctuations when the exponents are plotted against the spatial scaled

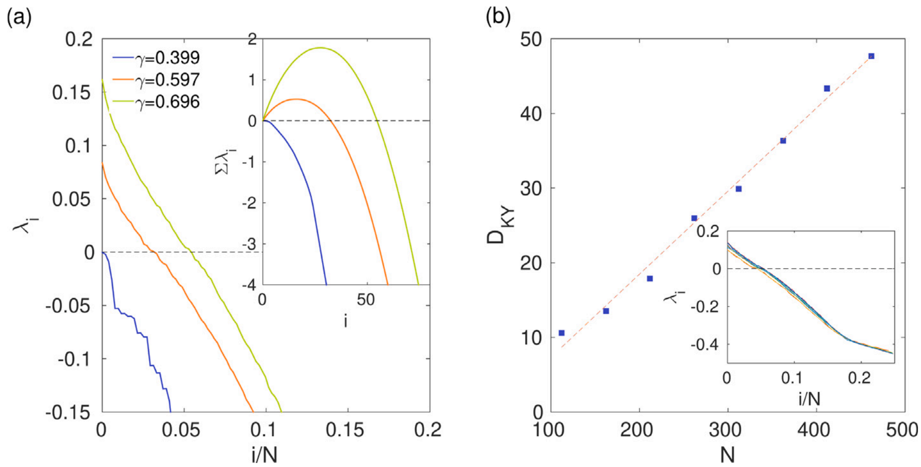

Fig. 5. (a) The Lyapunov spectra for three representative values of the parameter γ (γ = 0.399 blue curve; γ = 0.597 orange curve; γ = 0.696 green curve), with ν = 0.3 held constant. The inset in panel (a) shows the calculated cumulative sum of the ordered exponents $\sum \lambda_i$. The crossing with $\sum \lambda_i = 0$ gives the Kaplan–Yorke dimension D_{KY}. (b) Extensivity of D_{KY}, as a function of the system size (here indicated with the number of grid points N). The scaling of D_{KY} with N is obtained for parameters $\gamma = 0.7$ and $\nu = 0.2$. This linear scaling is expected in chaotic dynamical scenarios [91]. The inset in panel (a) shows the collapse, within numerical fluctuations, of the Lyapunov spectrum when we plot the Lyapunov exponents λ_i versus the space scaled index i/N. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

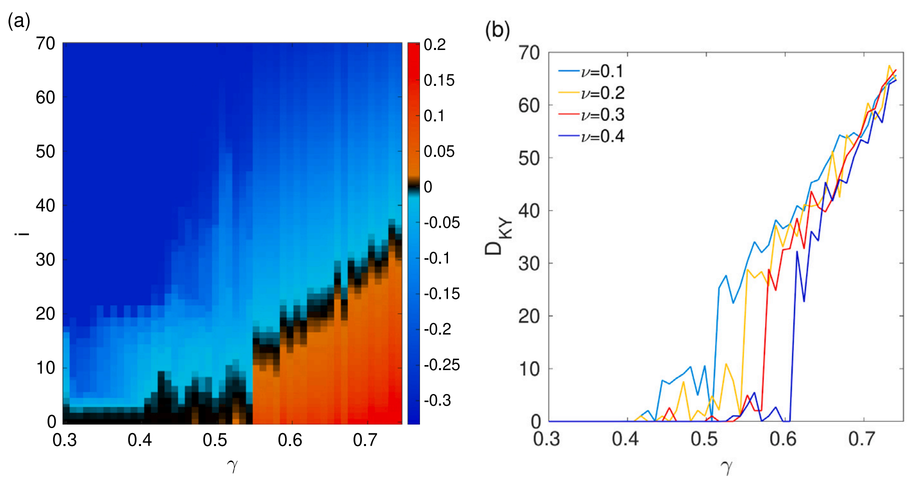

Fig. 6. A transition to spatio-temporal chaotic dynamics of increased dimensionality in parameter space. On the left it is shown the Lyapunov spectrum in color bar code as a function of oscillator's number and forcing parameter γ at $\nu = 0.2$. On the right the Kaplan–Yorke dimension D_{KY} as a function of γ for $\nu = 0.1$, $\nu = 0.2$, $\nu = 0.3$ and $\nu = 0.4$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

index i/N. From a numerical point of view, we can discern between a Faraday's wave regime and a high-dimensional chaotic regime using this dynamical indicator.

Let us discuss the distinction between low- and high-dimensional chaos. In the left panel of Fig. 6, the color code indicates the Lyapunov exponent values as a function of the forcing parameter γ and the spatial index i for fixed values of $\nu = 0.2$ and $N = 512$ lattice points. The region without chaos corresponding to the Faraday waves regime is clearly shown between $\gamma = 0.3$ and $\gamma = 0.4$ (approximately). An increase of the parameter γ in the range between about 0.4 and 0.55 shows a region of low-dimensional chaos. Finally, if γ is larger than 0.55, one observes that the number of positive Lyapunov exponents increases linearly with γ, showing a region of extensive spatiotemporal chaos.

In the right panel of Fig. 6, the computed Kaplan–Yorke dimension D_{KY}, Eq. (7), is displayed as a function of the parameter γ for several values of the detuning parameter ν. It is especially interesting to note that the approximations of the partial derivatives $\partial D_{KY}/\partial\gamma$ and $\partial D_{KY}/\partial\nu$ provide good indicators for identifying corresponding dynamical states. Indeed, the Faraday waves regime (here for small γ between 0.3 and 0.4) and the high dimensional chaotic regime (values of $\gamma > 0.6$) share the condition that both partial derivatives are constant, as it can be shown in the right panel of Fig. 6. For intermediate values of γ, the system is in the low dimensional chaotic regime, and $\partial D_{KY}/\partial\gamma$ fluctuates considerably, and large variations of D_{KY} are observed if the detuning parameter ν is varied.

4. Conclusions

In this paper, we have studied the dynamics associated with the PDDNLS equation numerically. We have shown that if the forcing amplitude is small, we observe the classical Faraday pattern. Increasing the forcing amplitude allows periodic, quasi-periodic, and chaotic dynamics to be observed. The dynamics were studied through several indicators, i.e., Lyapunov and Fourier spectra, Kaplan–Yorke dimension, and averaged amplitude norm. Using the information provided by these indicators, the phase diagrams of these dynamical states were built as a function of the amplitude and detuning of the parametrical forcing. Furthermore, the system size plays an important role in discriminating between low and high-dimensional chaos. In the latter case, the Kaplan–Yorke dimension scales linearly with the system size, ultimately proving its extensive property.

The present study's natural extensions will consider two-dimensional settings for the PDDNLS equation. In this case, one expects to encounter competing stationary patterns with spatiotemporal chaos that will further increase the complexity of the phase diagrams of the dynamic states.

CRediT authorship contribution statement

L.I. Reyes: Data curation, Investigation, Methodology, Software, Visualization, Writing – original draft. **L.M. Pérez:** Formal analysis, Investigation, Resources, Validation, Writing – review & editing. **L. Pedraja-Rejas:** Data curation, Software, Visualization. **P. Díaz:** Formal analysis, Investigation, Software, Visualization. **J. Mendoza:** Data curation, Software, Visualization. **J. Bragard:** Formal analysis, Investigation, Validation, Writing – review & editing. **M.G. Clerc:** Conceptualization, Formal analysis, Methodology, Supervision, Writing – review & editing. **D. Laroze:** Conceptualization, Funding acquisition, Investigation, Supervision, Validation, Writing – original draft, Project administration.

Declaration of competing interest

The authors declare that they have no competing interests.

Data availability

Data will be made available on request.

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