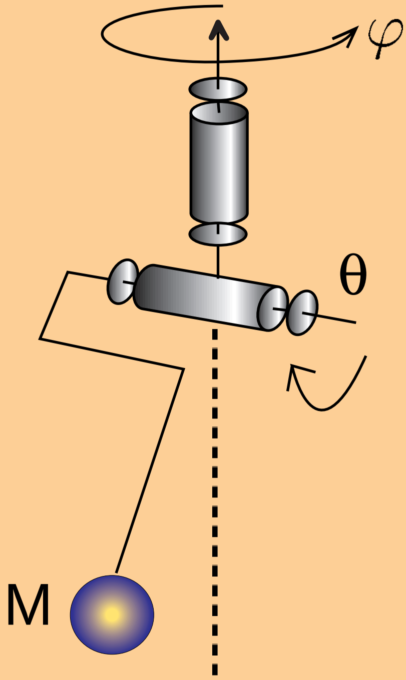

Lorenz Bifurcation and Quasi-reversible instabilities

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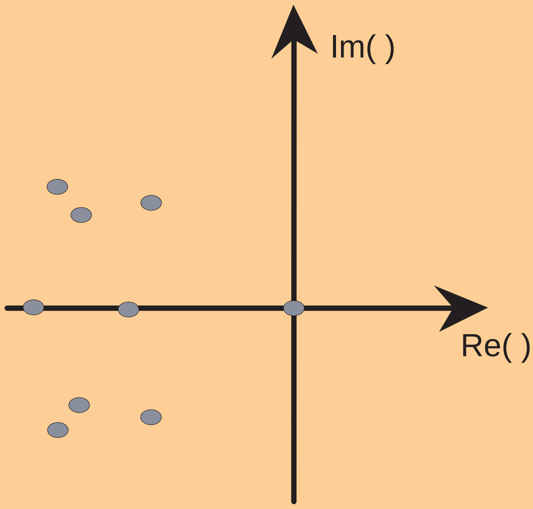
Outline

- Generical Instabilities of the dissipative systems.
- Generical Instabilities of the reversal systems.
- Quasi-reversal systems, Lorenz Bifurcation.
- Mechanical system which experimentally displays lorenz chaotic behavior.
- Several examples of Lorenz Bifurcation.
- Brief comments of the other Quasi-reversal instabilities.
- Conclusion.

Generical Instabilities

(Dissipative systems)

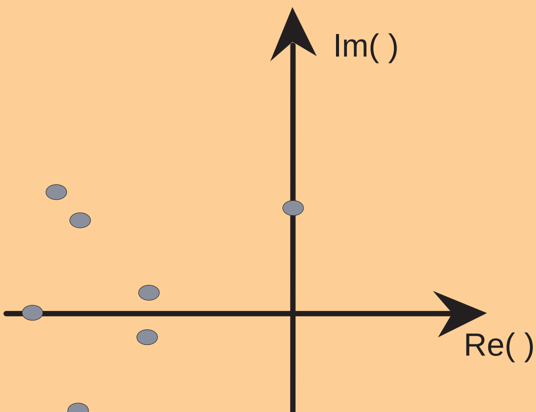
i)Stationary instability (Saddle-node)



example : dissipative rotating pendulum

(x ⟶ -x, Pitchfork)

ii) Andronov-Hopf bifurcation



example : Belousov Zhabotinski reaction, Laser

Reversible systems

$$\partial_t u = f(u), \; \partial_t Su = -f(Su), \; S^2 = 1$$

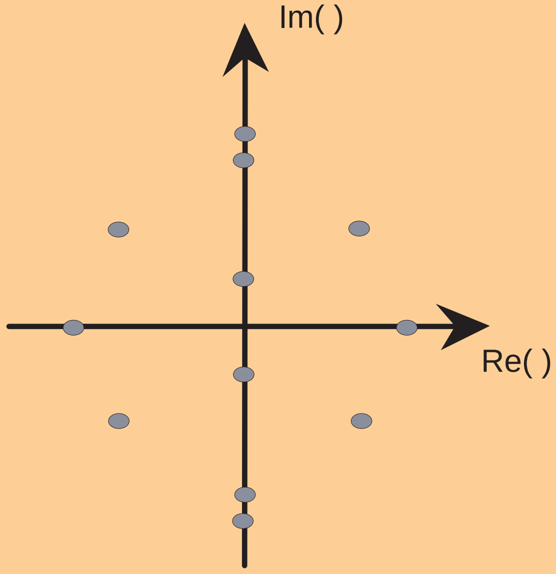
$$t \longrightarrow -t, \; u \longrightarrow Su$$

- Example : Hamiltonian Systems

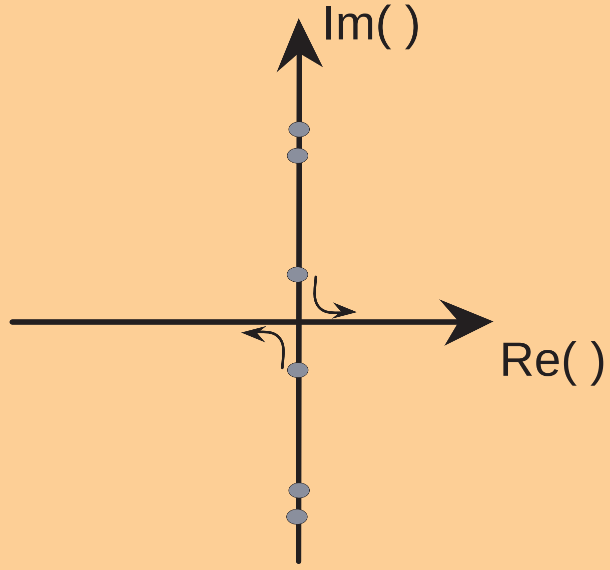
$$H(\vec{p}, \vec{q}) = \sum_{i=1}^{n} \frac{p_i p_i}{2} + V(\vec{q})$$

$$t \to -t, \; \vec{q}^{\,i} \to \vec{q}^{\,i}, \; \vec{p}_i \to -\vec{p}_i, \qquad i = 1, \cdots, n$$

- instabilities (l ⟶ - l , for reversal solution)

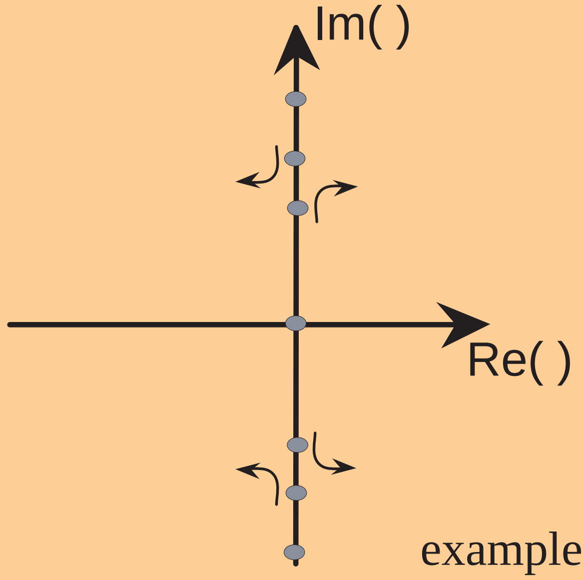


i)Stationary instability



example : rotating pendulum

ii)Confusion of frequencies (Rocard ('43))



example : wing's aircraft,
Laser

Reversible Normal Form

the stationary bifurcation in presence of a neutral mode, with the time reversal transformation ($t \longrightarrow -t, x \longrightarrow x$, $z \longrightarrow z$) and the reflection symmetry ($x \longrightarrow -x$), is described by

$$\ddot{x} = \epsilon x - x^3 - zx$$

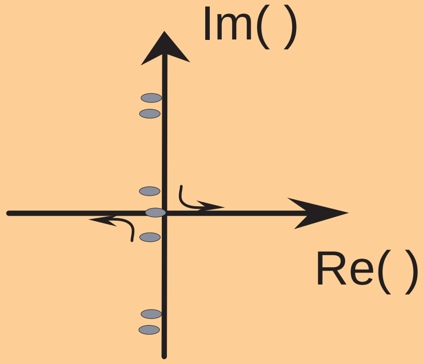
$$\dot{z} = 0$$

Quasi-reversible systems ($t \not\longrightarrow -t$)

"The systems in which the terms that break the time reversal symmetry are small and can be considered as perturbative terms near instabilities".

$$\ddot{x} = \epsilon x - x^3 - zx - \nu \dot{x}$$

$$\dot{z} = -\mu z + \eta x^2$$



where $\partial_t \sim \sqrt{\epsilon},\ x \sim \sqrt{\epsilon},\ z \sim \epsilon,\ \nu,\ \mu,\ \eta \sim \sqrt{\epsilon}$

Lorenz Bifurcation

Introducing the change of variables

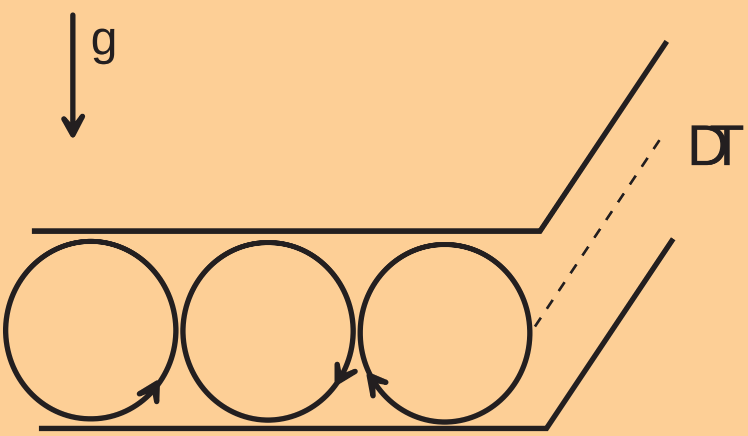
$$x = \frac{\epsilon}{\sqrt{\sigma}} x', \qquad \dot{x} = \frac{\epsilon^2}{\sqrt{2}} (y' - x'), \; z = \epsilon^2 \left(z' - \frac{x'^2}{2\sigma} \right),$$

$$\partial_t x' = \sigma(y' - x')$$
$$\partial_t y' = Rx' \pm y' - x'z'$$
$$\partial_t z' = -bz' + x'y'$$

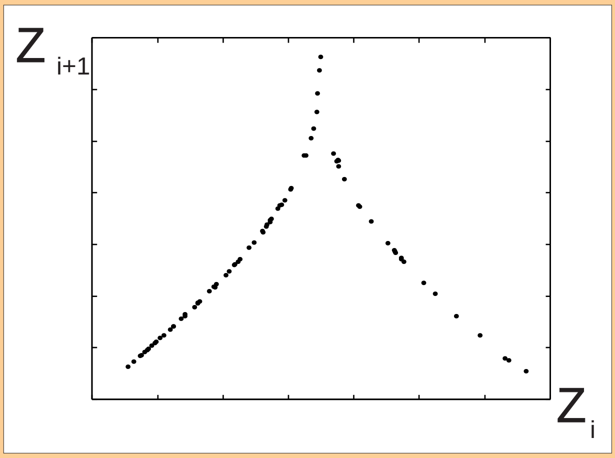
The sing $\pm$ is determined by $-\nu + \eta + \mu$

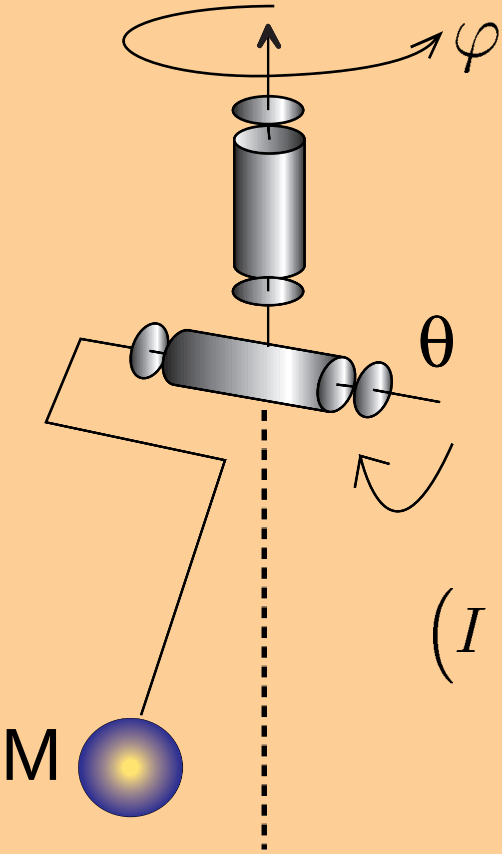
$$t \equiv \frac{\epsilon}{\sqrt{\sigma}} t', \; \nu = \epsilon \frac{\sigma+1}{\sigma}, \; \mu = \epsilon \frac{b}{\sqrt{\sigma}}, \qquad \eta = \epsilon \frac{2\sigma - b}{\sqrt{\sigma}}, \quad \epsilon = \sqrt{R-1}$$

E. Lorenz (1963), Galerkin expansion of convection.



Lorenz mapping





Lorenz pendulum

PRL, 83, 3820 (1999).

$$\ddot{\theta} = \frac{\sin(2\theta)}{2}\dot{\varphi}^2 - \sin(\theta) - \nu\dot{\theta},$$

$$\left(I + \sin^2(\theta)\right)\ddot{\varphi} = -2\sin(\theta)\cos(\theta)\dot{\varphi}\dot{\theta} - \tilde{\mu}(\dot{\varphi} - \Omega) - \tilde{\nu}\dot{\varphi}\sin^2(\theta),$$

Onset of the bifurcation

$$\ddot{\theta}' = \epsilon\theta' - \nu\dot{\theta}' - \theta'\zeta' - \theta'^3,$$

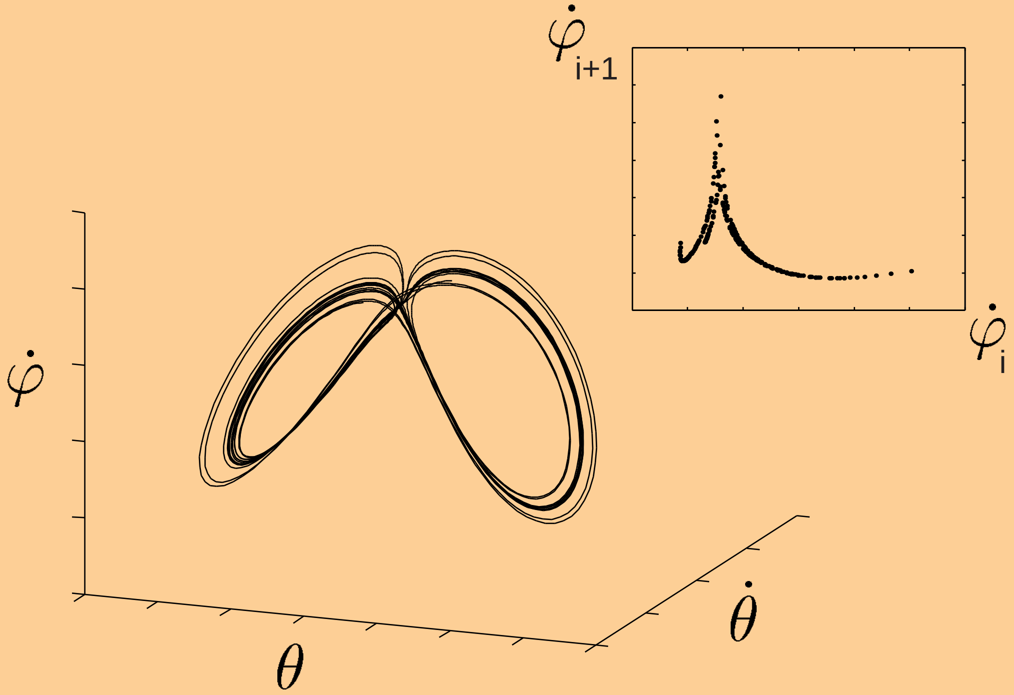
$$\dot{\zeta}' = -\mu\zeta' + \eta\theta'^2.$$

Where

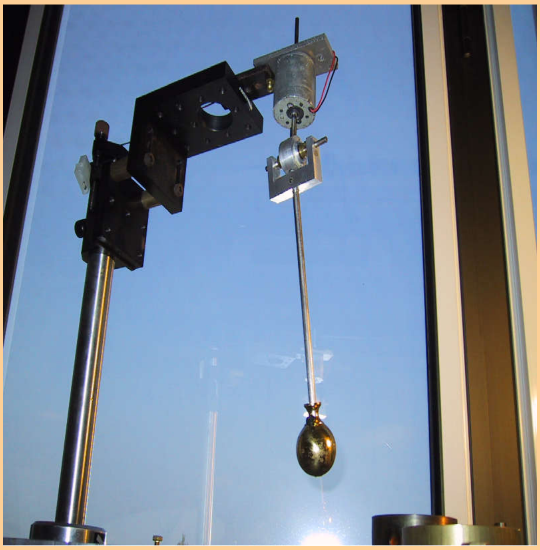
$$\epsilon = \Omega^2 - 1,\ \mu = \frac{\tilde{\mu}}{I},\quad \eta \equiv \frac{12\Omega^2(\nu-\mu)}{(4\Omega^2-1)I+12\Omega^2},$$

$$\dot{\varphi} = \Omega - \frac{\zeta'}{2\Omega} - \theta'^2\frac{6I\Omega}{I(4\Omega^2-1)+2\Omega^2},\ \theta = \frac{\theta'\sqrt{6I}}{\sqrt{4\Omega^2-1+12\Omega^2}}.$$

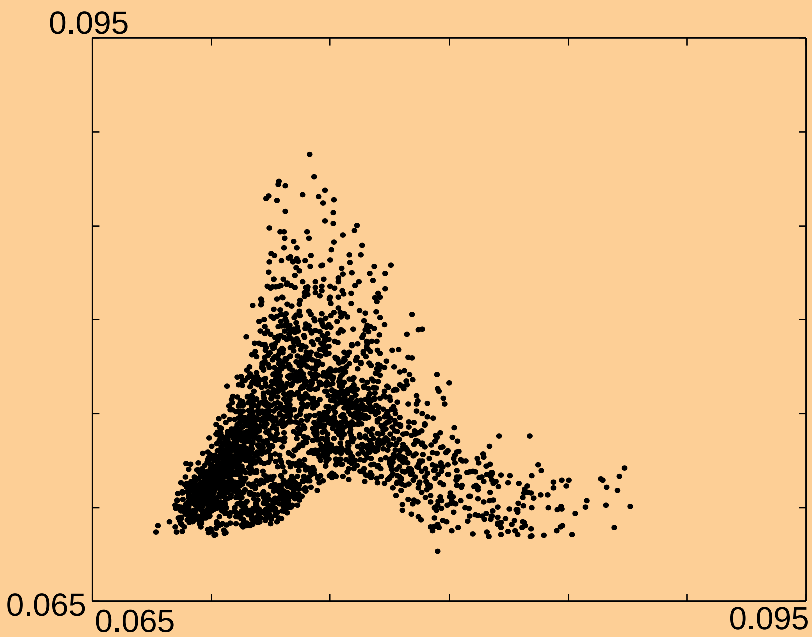
Numerical simulation



- Preliminary experimental observation

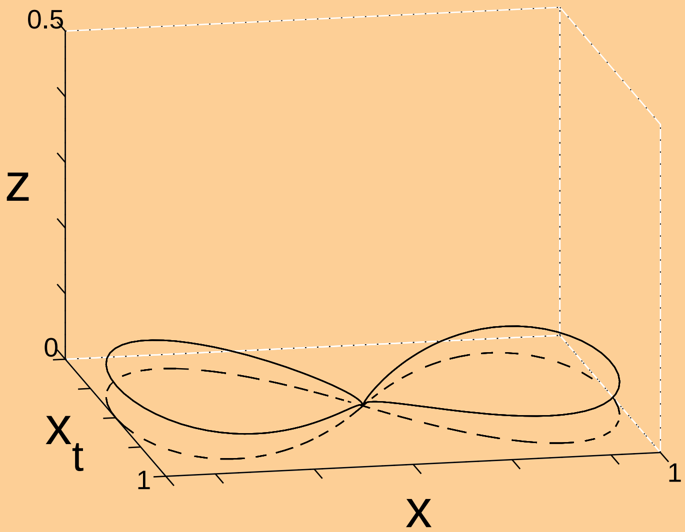


angular velocity mapping



Appearance of chaos

Homoclinic solutions



- $\nu = \eta = 0$

$$x_0 = \sec h\left(\frac{(t-t_0)}{\sqrt{2}}\right)$$

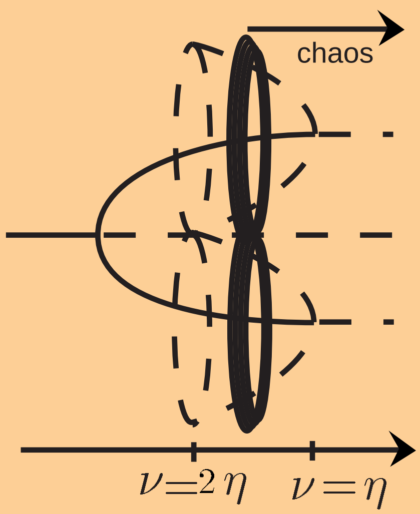
$$z_0 = 0$$

- Persistence condition

$$\boxed{\nu = 2\,\eta} \qquad \mu \ll \varepsilon$$

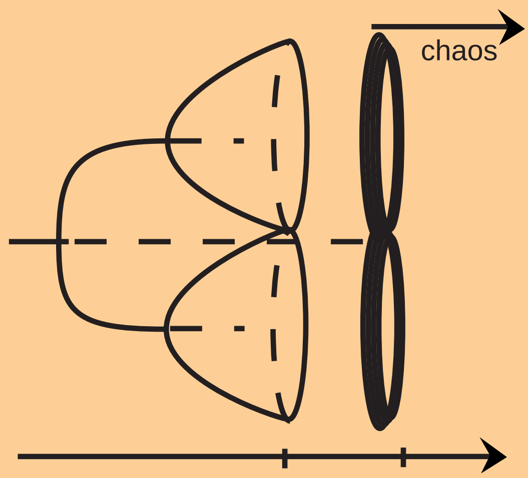
Bifurcation diagrams

i) "-"



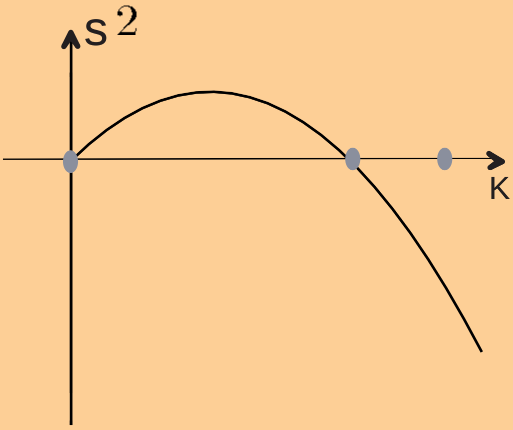
- Crisis scenario

ii) "+"



- Gluing scenario

Quasi-reversible Ginzburg Landau



$$\partial_t A = i|A|^2 A + i\partial_x^2 A + \varepsilon_1 A - \varepsilon_2 |A|^2 A + \varepsilon_3 \partial_x^2 A.$$

$$\varepsilon_1, \varepsilon_2, \varepsilon_3 \ll 1,$$

- $\partial_x A = 0 \qquad x = 0, L.$

Malomed et al (PRA 42, 6238 (90)).

Clerc et al (Progress T. Phys. Suppl. 139, 337 (2000)).

- Using the ansatz $\quad A = R\, e^{i\theta}$

$$R = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} + \rho_o + \rho_1 \cos(kx), \qquad \theta = \frac{\varepsilon_1}{\varepsilon_2} t + \psi_o + \psi_1 \cos(kx)$$

$$\partial_{tt} x = \varepsilon' x - zx + x^3 - \nu \partial_t x$$

$$\partial_t z = -\mu z + \eta x^2$$

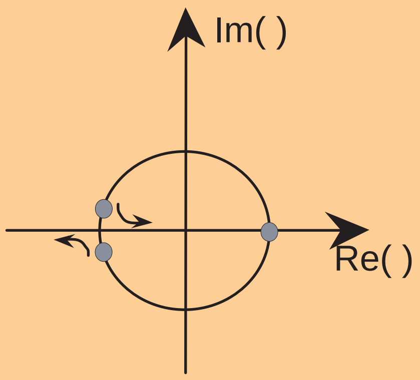
$$\partial_t \psi_o = \frac{z}{2k_c^2} - \frac{x^2}{7k_c^2} + \frac{6}{7k_c^2} {R_o}^2$$

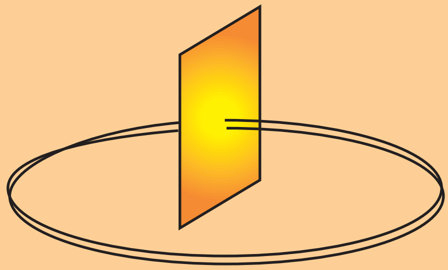
$$\rho_1 = x\sqrt{4/7k_c^2}, \quad \varepsilon' = \varepsilon k_c^2, \quad \nu = \varepsilon_3 k_c^2, \quad \mu = 2\varepsilon_1$$

$$\rho_o = -\left(z - \frac{x^2}{7k_c^2}\right)/4R_o k_c^2, \quad \eta = 4\varepsilon_1 / 3R_o k_c^2. \quad k = k_c - \varepsilon_1$$

Quasi-reversible period doubling

For Reversible system the Floquet multipliers $l \longrightarrow l^{-1}$





M={1,1,-1,-1}

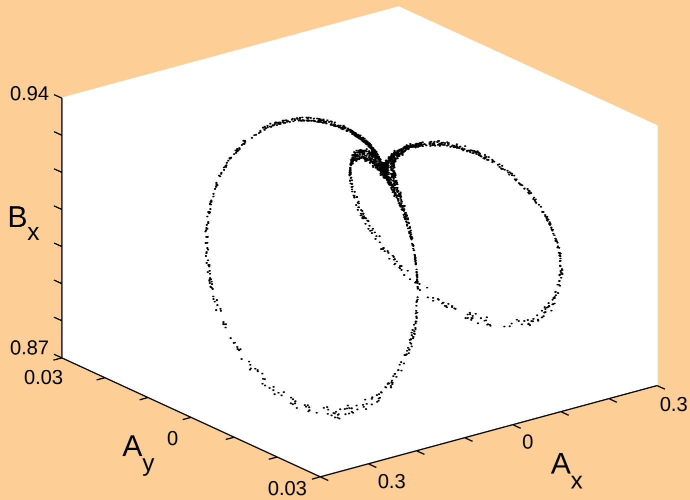
- The variables associated to M=-1 are invariant by reflection.

to appear in Phys. Lett. A 2001.

Example, the 1:2 resonance

$$i\dot{A} = iWA - aBA + a\,|A|^2A - imA$$

$$i\dot{B} = i2WB - bA^2 - |B|^2B + i\,d(R^2 - |B|^2)B$$



Poincaré section (Im(B)=0)

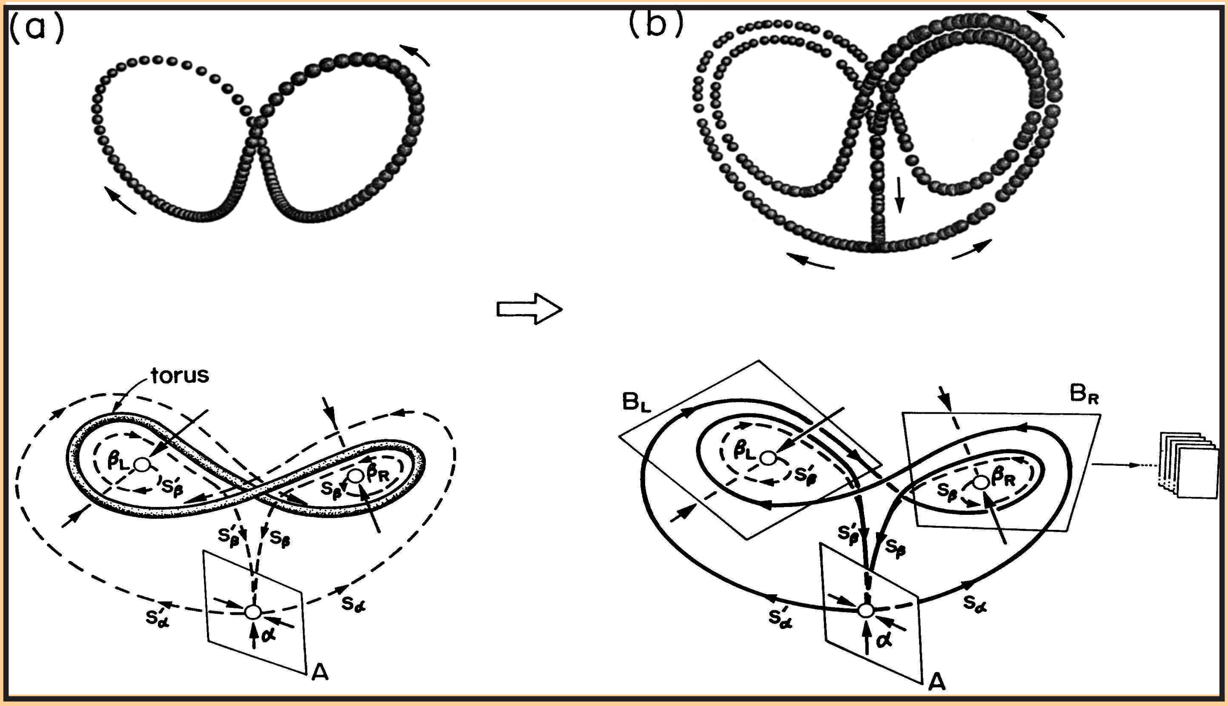
- Limit cycles

$A = 0, \; B = \exp\{iR\,t\}$

becomes unstable

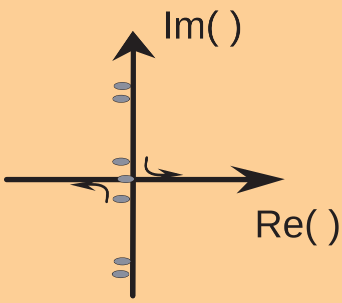
$R > 2a$

- Poincaré Section Im(a_1)=0.

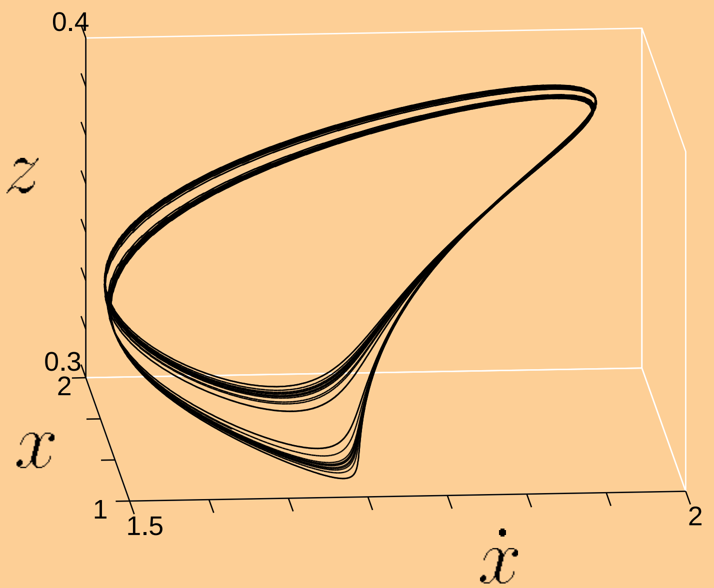


- without reflection symmetry the normal form is

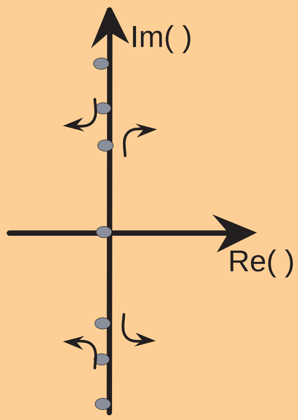
$$\ddot{x} = \epsilon \pm x^2 + azx \pm z^2 - \nu \dot{x}$$
$$\dot{z} = \delta - \mu z + \eta x$$



- Shilnikov type chaos



- In the quasi-reversible confusion of frequencies, the asymptotic normal form



$$A_{tt} = \epsilon A - (\nu + i\Delta) A_t - |A|^2 A$$
$$- \;\; zA$$
$$z_t = -\mu \; z + \eta \; |A|^2$$

- J. Gibbon et al ('80): the dispersive instability with small dissipation.

Introducing the change of variables

$$P = \kappa E + \partial_t E, \quad E = e^{-i\frac{\Delta\kappa}{\gamma+\kappa}t}\frac{A}{\sqrt{g}}, \qquad N = \frac{z - D_o}{g} + |E|^2,$$

the equations read (Maxwell-Bloch or complex Lorenz eqs.)

$$\partial_t E = -\kappa E + P$$

$$\partial_t P = -(\gamma_\perp + i\Delta) P - gNE$$

$$\partial_t N = -\gamma_{||}(N - N_0) + (E\bar{P} + \bar{E}P)$$

Summary

- The reversible dynamical systems present two generic instabilities : The stationary instability or resonance at zero frequency and the confusion of frequencies or resonance at finite frequency. We study the consequences when time reversal symmetry is weakly broken. We show that the resonance at zero frequency in the presence of reflection symmetry has as asymptotic normal form the well know Lorenz equations. We describe a simple mechanical system which displays Lorenz type chaotic behavior. In the case of confusion of frequencies we find that the asymptotic normal form is the Maxwell-Bloch equations, which describes the dynamic of two level atom gas in an optical cavity.

The end