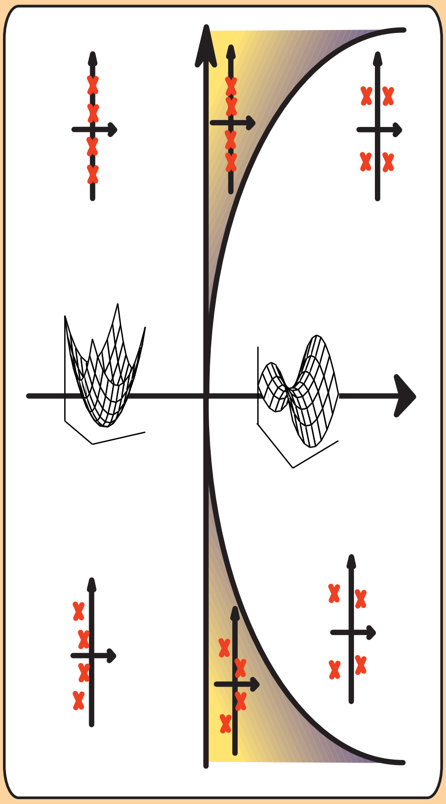


Latent Bifurcation

"Latent Bifurcation, Dissipation induce instability and applications".

M.G. Clerc and J.E. Marsden

CDS, Caltech

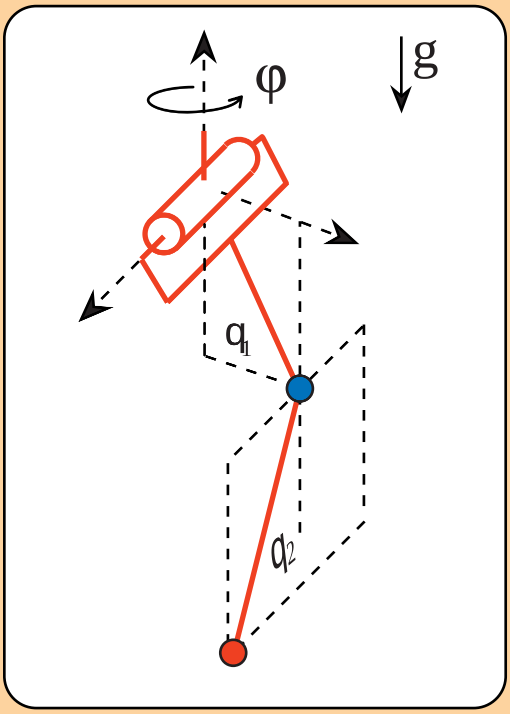
DFI, Universidad de Chile

DFI

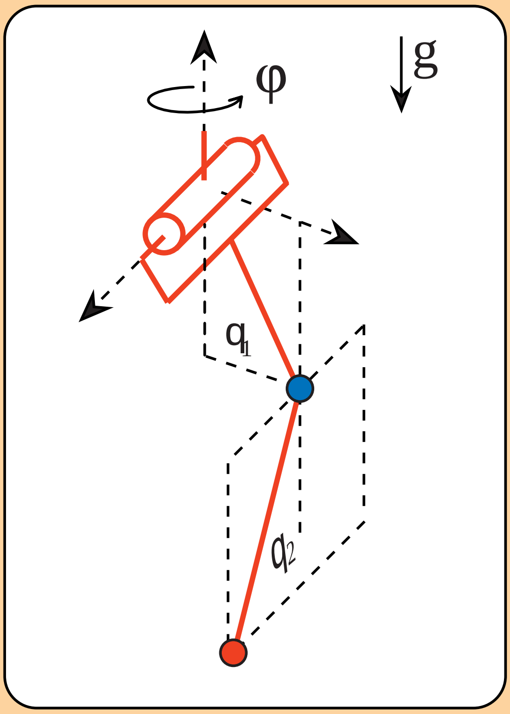
Outline

- Examples.
- Definition of linearlly and spectrally instability.
- Dynamics around the 1:1 resonance.
- Latent bifurcation.
- Dissipation induced instability.
- Applications.

Mechanical Laser



Two coupled spherical pendula in a gravitational field, with a support, which can rotate around the vertical axis. The lower pendulum is constrained to move in a plane that is orthogonal to the plane of the upper pendulum.



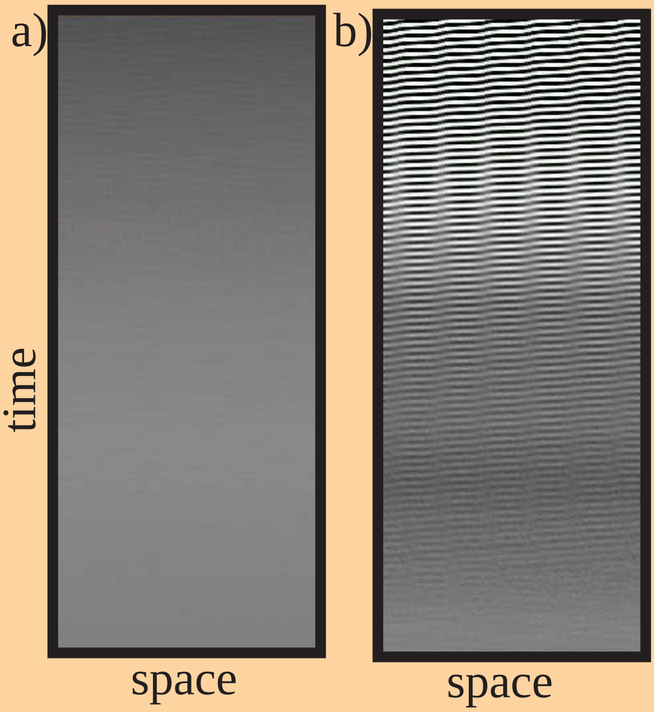
$$\begin{aligned}
\ddot{\theta}_1 = &-\sigma^2 \sin\theta_1 \sin\theta_2 \ddot{\theta}_2 - \sigma^2 \sin\theta_1 \cos\theta_2 \dot{\theta}_2^2 \\
&-2\sigma^2 \cos\theta_1 \cos\theta_2 \dot{\varphi}\dot{\theta}_2 + \sin\theta_1 \cos\theta_1 \dot{\varphi}^2 \\
&-\sigma^2 \cos\theta_1 \sin\theta_2 \ddot{\varphi} - \frac{g}{l}\sin\theta_1 - \nu_1\dot{\theta}_1, \\
\ddot{\theta}_2 = &-\sin\theta_1 \sin\theta_2 \ddot{\theta}_1 - \cos\theta_1 \sin\theta_2 \dot{\theta}_1^2 \\
&+2\cos\theta_1 \cos\theta_2 \dot{\varphi}\dot{\theta}_1 + \sin\theta_2 \cos\theta_2 \dot{\varphi}^2 \\
&+\sin\theta_1 \cos\theta_2 \ddot{\varphi} - \frac{g}{l}\sin\theta_2 - \nu_2\dot{\theta}_2,
\end{aligned}$$

$$\frac{d}{dt}\left\{ \begin{array}{c} \left(\sin^2\theta_1 + \sigma^2\sin^2\theta_2\right)\dot{\varphi} + \sigma^2\cos\theta_1\sin\theta_2\dot{\theta}_1 \\ -\sigma^2\sin\theta_1\cos\theta_2\dot{\theta}_2 + I\dot{\varphi} \end{array} \right\} =$$

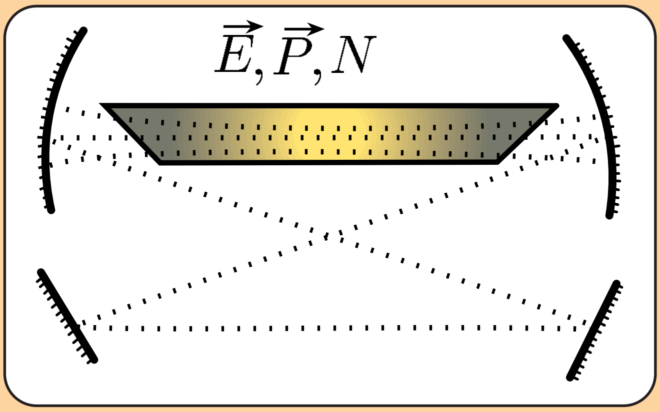
$$-\nu_\varphi\left(\dot{\varphi} - \Omega\right) - \mu_1\sin^2\theta_1\dot{\varphi} - \mu_2\left(\sin^2\theta_1 + \sin^2\theta_2\right)\dot{\varphi}.$$

where $l_1 = l_2 = l$ and $\sigma = \sqrt{m_2/\left(m_1 + m_2\right)}$

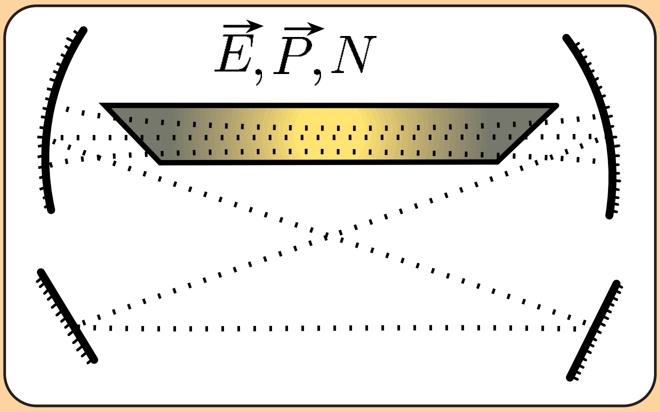
Laser



Nonlinear optical cavity : Light Amplification by the Stimulated Emission of Radiation



Semiclassical Laser Model



$$\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial x^2} - \frac{\partial^2 P}{\partial t^2} - \kappa \frac{\partial E}{\partial t},$$

$$\frac{\partial^2 P}{\partial t^2} = -\gamma_\perp \frac{\partial P}{\partial t} - (\gamma_\perp^2 + \Omega^2) P - \mu^2 N E,$$

$$\frac{\partial N}{\partial t} = -\gamma_\parallel (N - N_0) + E \left(\frac{\partial P}{\partial t} + \gamma_\perp P \right),$$

where the terms proportional to $\kappa, \gamma_\perp$ and $\gamma_\parallel$ are dissipatives.

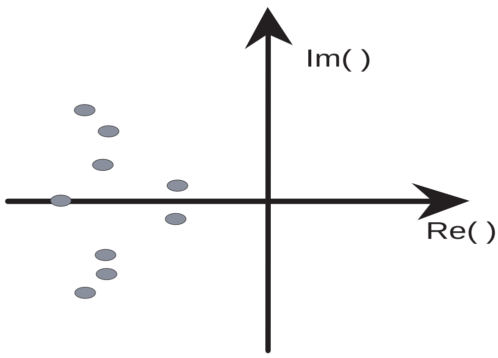
The nonlasing solution (equilibrium) is

$$E = P = 0 \text{ and } N = D_0$$

Stability of Equilibria

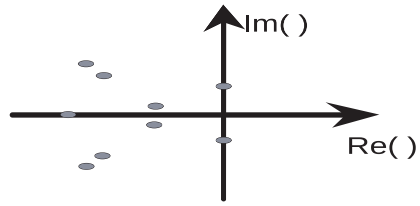
When all solutions of linearized equation at equilibrium are stable (Lyapunov), the equilibrium is said to be linearlly stable or linearized stable.

Stable equilibrium

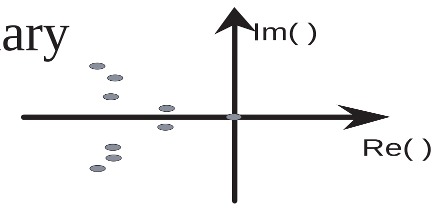


Codimension-One bifurcations

- Hopf



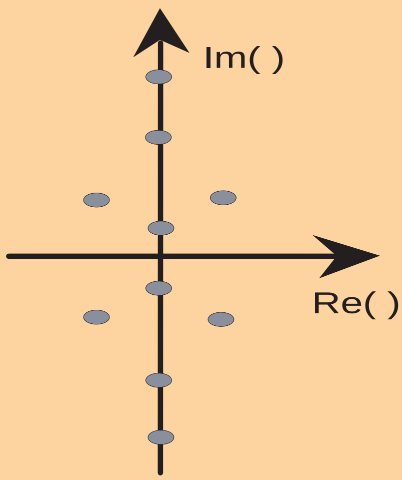
- Stationary



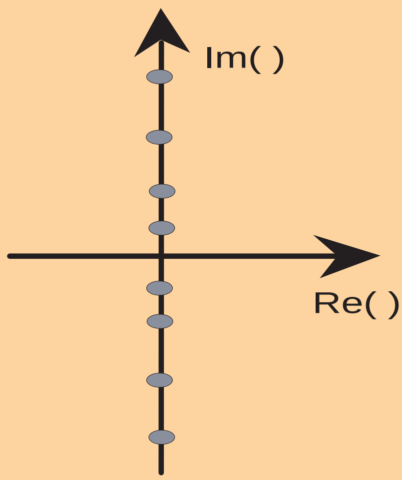
Hamiltonian and time reversible systems

- If l is an eigenvalue, then so is -l .

characteristic spectrum

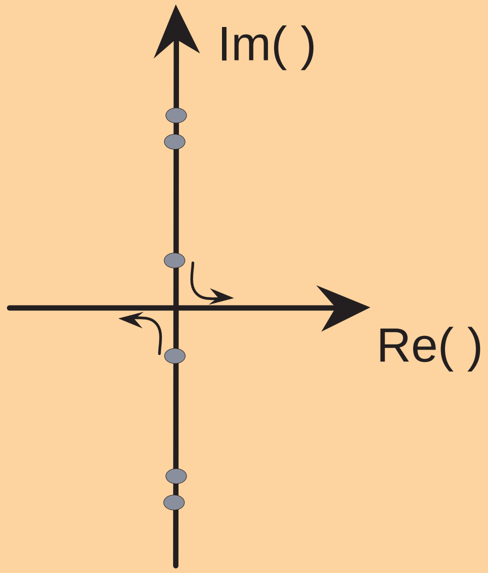


spectral stability

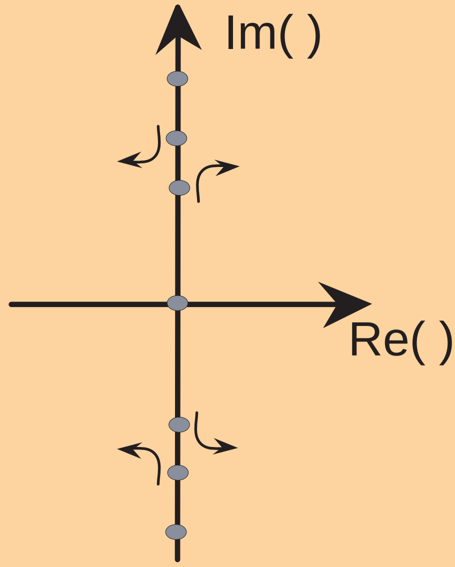


- There are two codimension-one spectral instabilities

i) Stationary instability

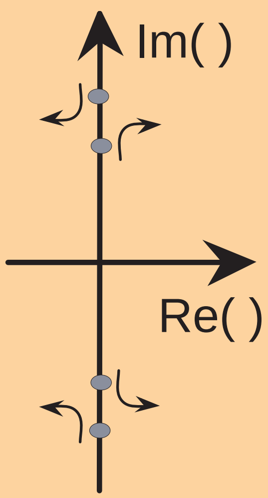


ii) 1:1 resonance



The spectral instability can predict instability, but not stability!

1:1 Resonance



Nearby this instability the system is decribed by the normal form

$$
\begin{aligned}
\partial_t A &= i\Omega A + B \\
\partial_t B &= i\Omega B + f\left(|A|^2, i\left(AB^* - BA^*\right), \{\lambda\}\right) A + \\
&\quad ig\left(|A|^2, i\left(AB^* - BA^*\right), \{\lambda\}\right) B,
\end{aligned}
$$

where f and g are real functions, and {l } is a set of parameters.

The linear evolution around the zero solution($\mathcal{A} = \mathcal{B} = 0$)

$$\partial_t \mathcal{A} = i\Omega \mathcal{A} + \mathcal{B},$$
$$\partial_t \mathcal{B} = i\left(\Omega + \delta\right)\mathcal{B} + \varepsilon \mathcal{A},$$

introducing the rotating variables $\mathcal{A} = Ae^{i\Omega t} = \left(x + iy\right)e^{i\Omega t}$ and $\mathcal{B} = \partial_t Ae^{i\Omega t} = \left(\partial_t x + i\partial_t y\right)e^{i\Omega t}$

$$\partial_{tt} x = \varepsilon x - \delta \partial_t y,$$
$$\partial_{tt} y = \varepsilon y + \delta \partial_t x.$$

- Gyroscopic system.

$$\partial_{tt} A = \varepsilon A + i\delta \partial_t A$$

The gyscopic system is hamiltonian

$$H = \partial_t A \partial_t A^* - \varepsilon |A|^2$$

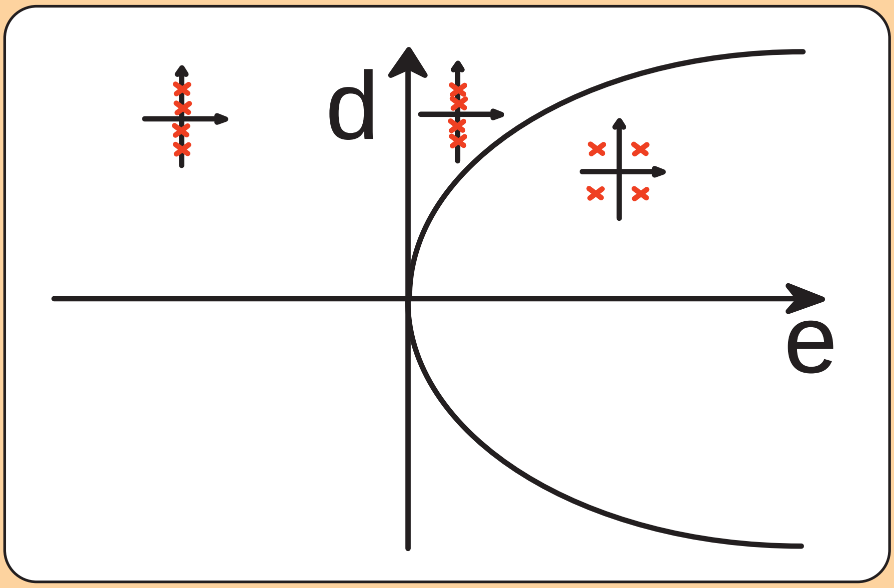
with the Poisson-bracket

$$\{F, G\} = \frac{\partial F}{\partial A}\frac{\partial G}{\partial A_t^*} - \frac{\partial G}{\partial A}\frac{\partial F}{\partial A_t^*} + i\delta \frac{\partial F}{\partial A_t}\frac{\partial G}{\partial A_t^*} + \text{c.c.}$$

The eigenvalues of this equilibrium are

$$\sigma = \pm 1/2\sqrt{4\varepsilon - 2\delta^2 \pm 2\delta\sqrt{\delta^2 - 4\varepsilon}}$$

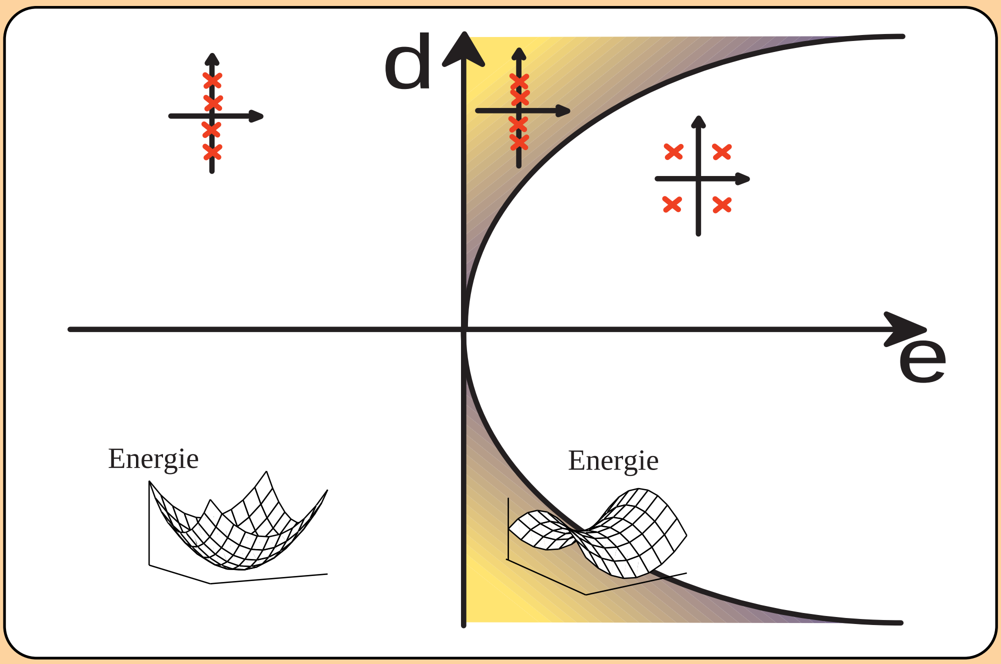
Bifurcation Diagram



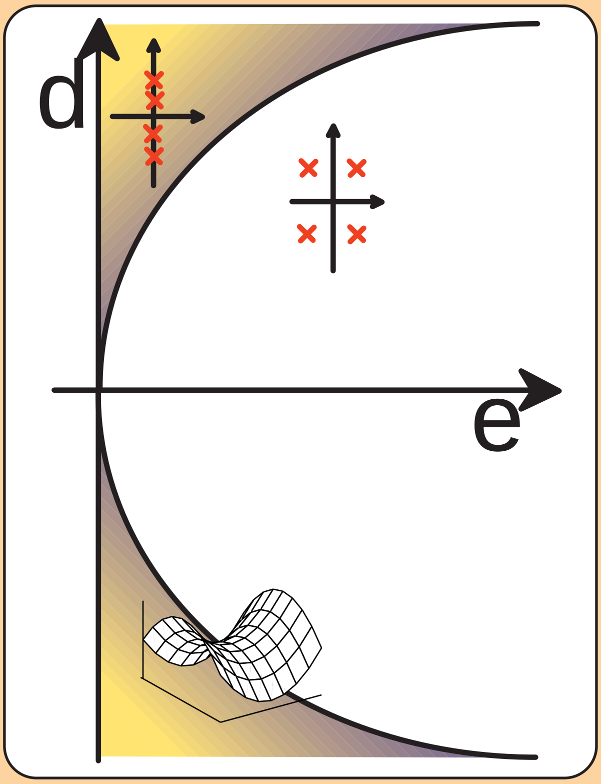
The conservatitive quantities

$$E = \frac{(\partial_t x)^2 + (\partial_t y)^2}{2} - \frac{\varepsilon}{2}\left(x^2 + y^2\right), \quad J = \partial_t x y - x \partial_t y + \frac{\delta}{2}\left(x^2 + y^2\right),$$

Bifurcation diagram



Latent Bifurcation



The instability is present but non perceptible with the spectrum, and requieres a large time in come into view.

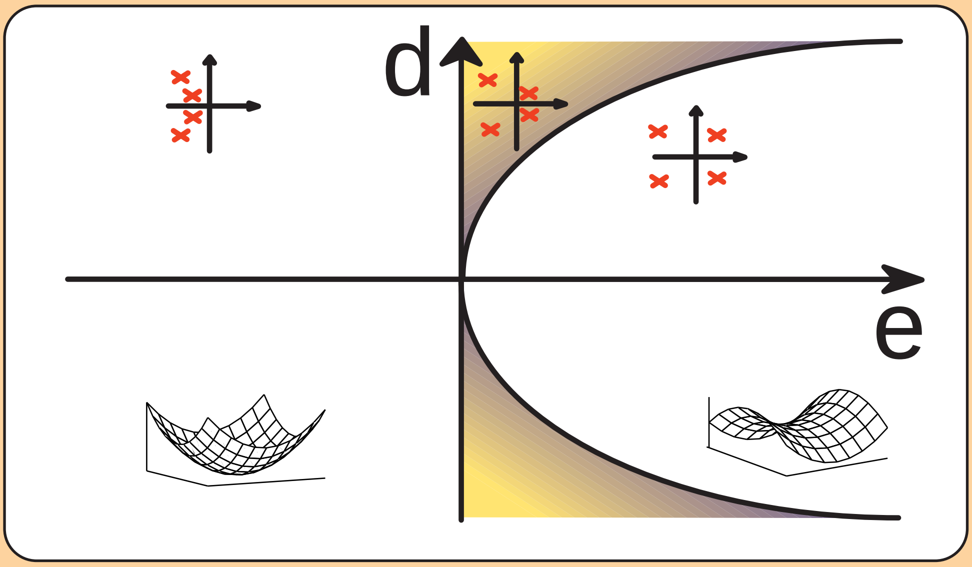
There is not resonance between the frequencies.

Dissipation induced instability

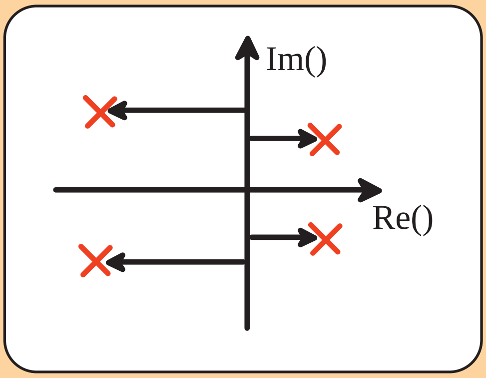
The equations under the presence of small dissipative terms read (m<<1)

$$\partial_{tt} x = \varepsilon x - \delta \partial_t y - \mu \partial_t x,$$
$$\partial_{tt} y = \varepsilon y + \delta \partial_t x - \mu \partial_t y,$$

- Bifurcation diagram



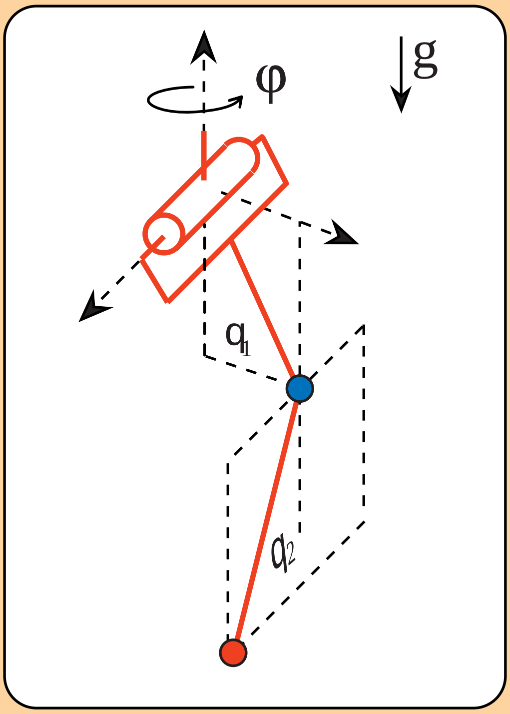
Observations



- The unperturbed system is marginal, the eigenvalues with larger frequency move to the left of the imaginary axis and these are the furthest from this axis.
- The destabilizing effects through positive or negative total dissipative perturbation was know a long time ago (Lord Kelvin, 1897).
- The asymptotic normal form of 1:1 resonance

$$\partial_{tt} A = \varepsilon A - (\mu - i\delta)\, \partial_t A - \alpha |A|^2 A,$$

Mechanical Laser

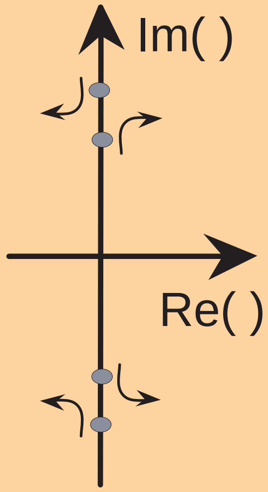


$$\begin{aligned}
\ddot{\theta}_1 = & -\sigma^2 \sin\theta_1 \sin\theta_2 \ddot{\theta}_2 - \sigma^2 \sin\theta_1 \cos\theta_2 \dot{\theta}_2^2 \\
& -2\sigma^2 \cos\theta_1 \cos\theta_2 \dot{\varphi}\dot{\theta}_2 + \sin\theta_1 \cos\theta_1 \dot{\varphi}^2 \\
& -\sigma^2 \cos\theta_1 \sin\theta_2 \ddot{\varphi} - \frac{g}{l}\sin\theta_1 - \nu_1 \dot{\theta}_1, \\
\ddot{\theta}_2 = & -\sin\theta_1 \sin\theta_2 \ddot{\theta}_1 - \cos\theta_1 \sin\theta_2 \dot{\theta}_1^2 \\
& +2\cos\theta_1 \cos\theta_2 \dot{\varphi}\dot{\theta}_1 + \sin\theta_2 \cos\theta_2 \dot{\varphi}^2 \\
& +\sin\theta_1 \cos\theta_2 \ddot{\varphi} - \frac{g}{l}\sin\theta_2 - \nu_2 \dot{\theta}_2,
\end{aligned}$$

$$\frac{d}{dt}\left\{ \begin{array}{c} \left(\sin^2\theta_1 + \sigma^2 \sin^2\theta_2\right)\dot{\varphi} + \sigma^2 \cos\theta_1 \sin\theta_2 \dot{\theta}_1 \\ -\sigma^2 \sin\theta_1 \cos\theta_2 \dot{\theta}_2 + I\dot{\varphi} \end{array} \right\} =$$

$$-\nu_\varphi\left(\dot{\varphi} - \Omega\right) - \mu_1 \sin^2\theta_1 \dot{\varphi} - \mu_2\left(\sin^2\theta_1 + \sin^2\theta_2\right)\dot{\varphi}.$$

where $l_1 = l_2 = l$ and $\sigma = \sqrt{m_2/\left(m_1 + m_2\right)}$



The vertical solution

$$\theta_1 = \theta_2 = 0,\ \varphi_t = \Omega_0$$

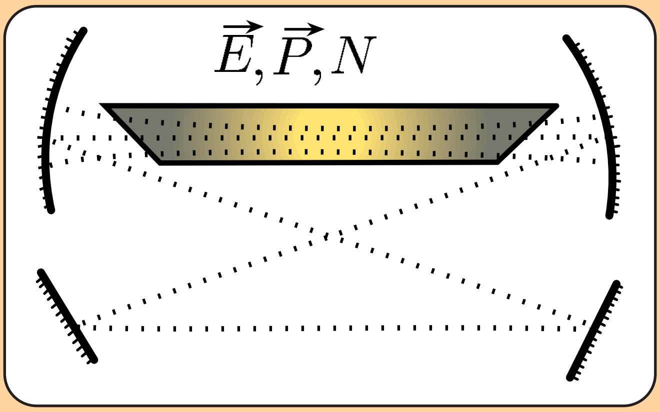
exhibits a 1:1 resonance when

$$\Omega_0 = \Omega_c = \sqrt{\frac{g}{l}\frac{(m_1+m_2)}{m_1}}$$

with $\omega_c = \pm\sqrt{gm_2/lm_1}$ frequencies. The system is described by

$$A_{tt} = \frac{2g\,(\Omega-\Omega_c)}{l\Omega}A + i\,(2\sigma\,(\Omega-\Omega_c))\,A_t$$

Semiclassical Laser Model



$$\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial x^2} - \frac{\partial^2 P}{\partial t^2} - \kappa\frac{\partial E}{\partial t},$$
$$\frac{\partial^2 P}{\partial t^2} = -\gamma_\perp\frac{\partial P}{\partial t} - (\gamma_\perp^2 + \Omega^2)P - \mu^2 N E,$$
$$\frac{\partial N}{\partial t} = -\gamma_\parallel(N - N_0) + E\left(\frac{\partial P}{\partial t} + \gamma_\perp P\right),$$

where the terms proportional to $\kappa, \gamma_\perp$ and $\gamma_\parallel$ are dissipatives.

The nonlasing solution (equilibrium) is

$$E = P = 0 \text{ and } N = D_0$$

Non dissipative semiclassical model

$$\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial x^2} - \left(\frac{\mu}{\Omega}\right)^2 \frac{\partial^2 P}{\partial t^2},$$
$$\frac{\partial P}{\partial t} = G, \quad \frac{\partial G}{\partial t} = -P - NE,$$
$$\frac{\partial N}{\partial t} = EG.$$

where $P \to \mu/\Omega P$, $E \to \Omega/\mu E$, $t \to t/\Omega$, $x \to x/\Omega$

This model has Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\left(D - \left(\frac{\mu}{\Omega}\right)^2 P\right)^2 + \frac{(\partial_x A)^2}{2} + \left(\frac{\mu}{\Omega}\right)^2 N,$$

with $E = \partial_t A = D - (\mu/\Omega)^2 P$.

and the Poisson-bracket

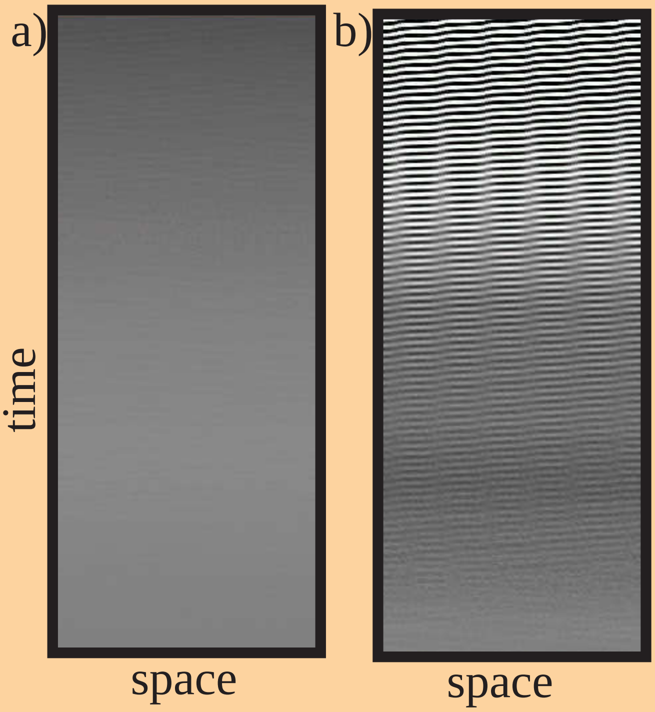
$$\{F, K\} = \int dx \left\{ \frac{\partial F}{\partial A}\frac{\partial K}{\partial D} - \frac{\partial F}{\partial D}\frac{\partial K}{\partial A} - \left(\frac{\mu}{\Omega}\right)^2 \vec{m} \cdot \left(\vec{\nabla}_m F \times \vec{\nabla}_m K\right) \right\},$$

where $\vec{m} = (N, P, G)$

Energie-Casimir Method

- $\{\Phi(N^2 + P^2 + G^2), F\} = 0$
- The effective energie

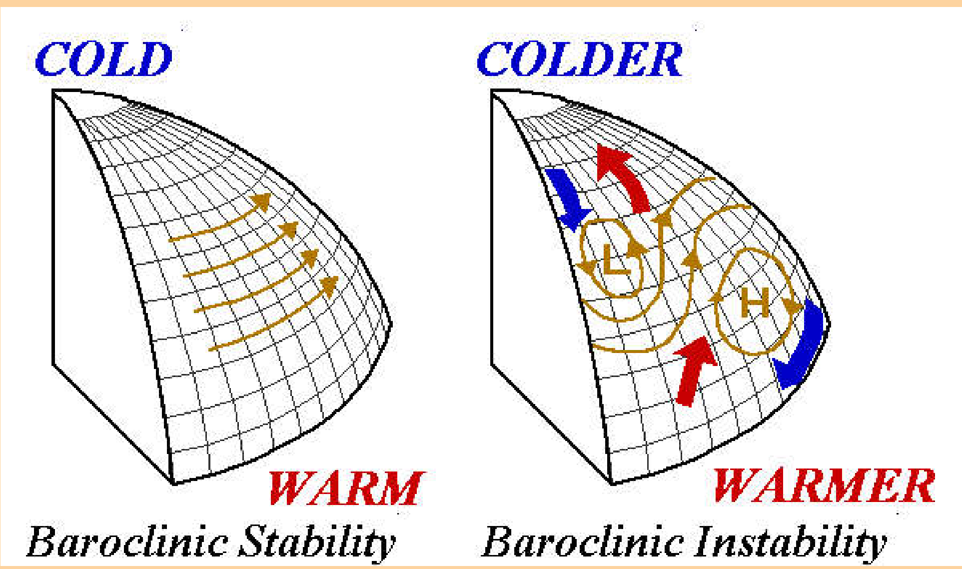
$$H_C = \int \left\{ \frac{1}{2}\left(D - \left(\frac{\mu}{\Omega}\right)^2 P\right)^2 + \frac{(\partial_x A)^2}{2} + \left(\frac{\mu}{\Omega}\right)^2 N + \left(\frac{\mu}{\Omega}\right)^2 \frac{(N^2 + P^2 + G^2)}{2D_o} + \alpha^2 (N^2 + P^2 + G^2)^2 \right\},$$



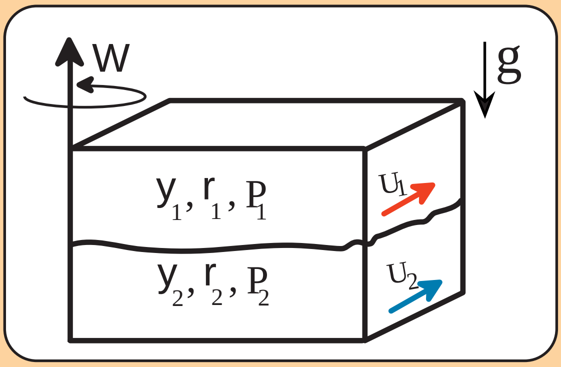
Thus, a slighty pumping optical cavity is unstable (D_0>0) when one take into account the dissipative effects and the nonlassing solution exhibits a latent bifurcation for D_0 equal to zero.

Baroclinic instability

This instability gives rise waves motion due to vertical shear of the basic current in the presence of Coriolis and buoyancy forces



Quasi-Geostrophic Two-layer Model



two layers of immiscible, incompressible, homogeneous fluid of slightly different densities ($r_2 > r_1$). The dimensionless quasi-geostrophic vorticity equations are

$$[\partial_t + \psi_{1,x}\partial_y - \psi_{1,y}\partial_x]\left[\vec{\nabla}^2\psi_1 + F(\psi_2 - \psi_1) + \beta y\right] = -r\vec{\nabla}^2\psi_1,$$

$$[\partial_t + \psi_{2,x}\partial_y - \psi_{2,y}\partial_x]\left[\vec{\nabla}^2\psi_2 + F(\psi_1 - \psi_2) + \beta y\right] = -r\vec{\nabla}^2\psi_2,$$

- This system is Lagrangean

$$\mathcal{L} = \int \frac{1}{2}\left(\vec{\nabla}\psi_1\right)^2 + \frac{1}{2}\left(\vec{\nabla}\psi_1\right)^2 + \frac{F}{2}(\psi_1 - \psi_2)^2 - \beta y(\psi_1 + \psi_2)\; dt dx dy,$$

constrained to Euler-Poincaré variations

$$\delta\psi_1 = \frac{\partial}{\partial t}\delta\phi_1 + \hat{z}\cdot\left(\vec{\nabla}\psi_1 \times \vec{\nabla}\delta\phi_1\right),$$

$$\delta\psi_2 = \frac{\partial}{\partial t}\delta\phi_2 + \hat{z}\cdot\left(\vec{\nabla}\psi_2 \times \vec{\nabla}\delta\phi_2\right).$$

Where the stream functions depent of the horizontal coordinate and time. The respective Hamiltonian is

$$\mathcal{H} = \int \frac{1}{2}\left(\vec{\nabla}\psi_1\right)^2 + \frac{1}{2}\left(\vec{\nabla}\psi_1\right)^2 + \frac{F}{2}(\psi_1^2 + \psi_2^2)\, dx dy,$$

If one considers perturbations of the geostrophic basic flow of the form

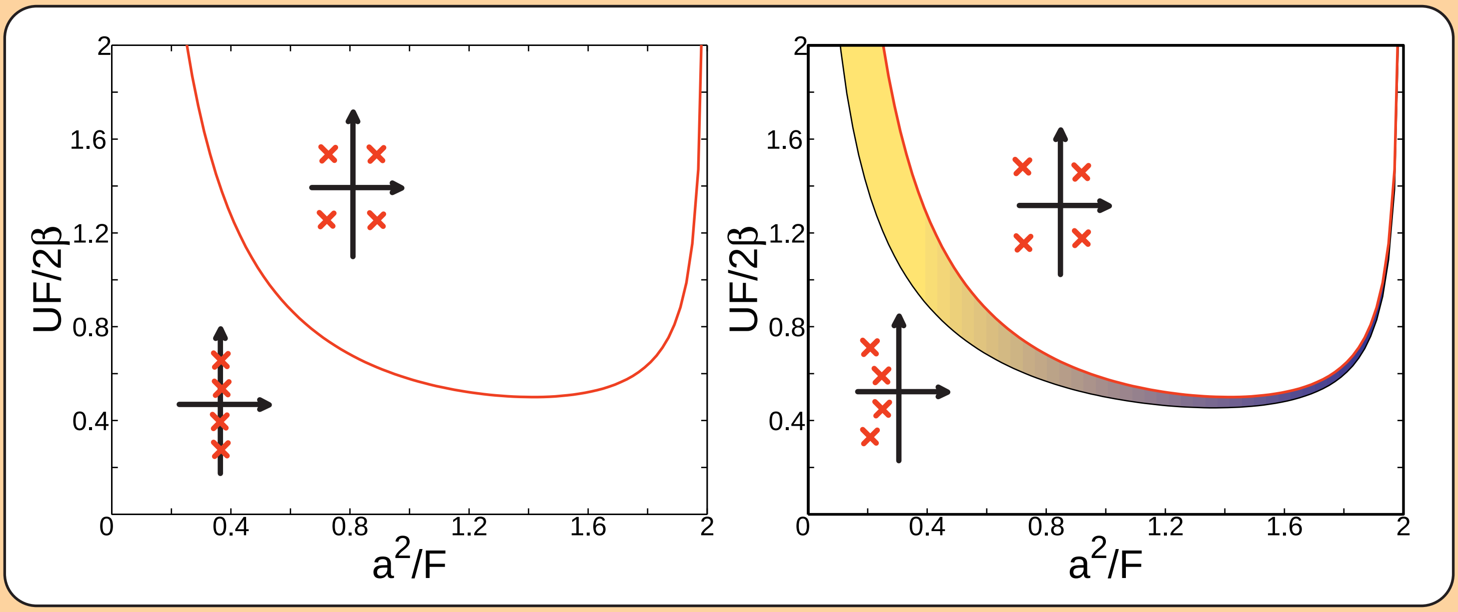
$$\psi_1 = -U_1 y + \mathrm{Re}\, A e^{i\alpha(x-ct)} \sin(m\pi y),$$
$$\psi_2 = -U_2 y + \mathrm{Re}\, \gamma A e^{i\alpha(x-ct)} \sin(m\pi y),$$

then dispertion relation is

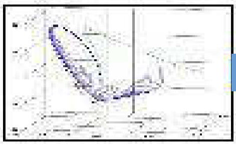
$$c = \frac{U_1+U_2}{2} - \frac{a^2+F}{a^2+2F}\left[\frac{\beta+i\alpha r}{\alpha^2}\right]$$
$$\pm \frac{\left[(\Delta U)^2 a^4 \left(a^4-4F^2\right) + 4F^2\left(\beta + ira^2\alpha^{-1}\right)\right]^{1/2}}{2a^2\left(a^2+2F\right)}$$

where $a^2 = \alpha^2 + m^2\pi^2$ and $\Delta U \equiv U_1 - U_2$.

- Bifurcation diagram for small viscosity



- references(E.O. Holopainen 1961, Romea 1977).



CR3BP - Circular Restricted Three-body Problem

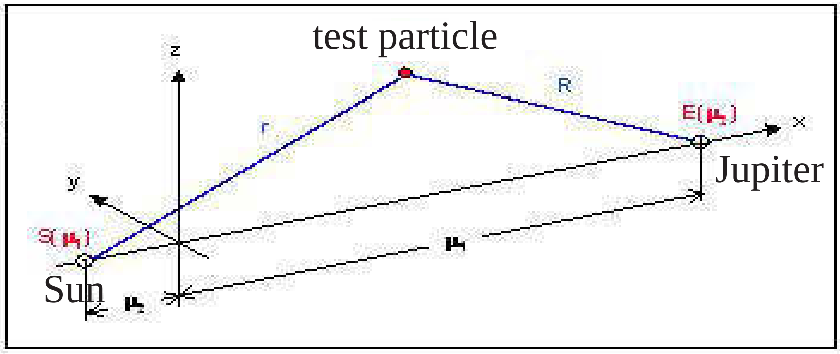
Equations of Motion
(nondimensional; rotating frame)

$$\mu_2 = \mu = 3.03591\cdot 10^{-6}$$

$$\mu_1 = 1-\mu$$

$$r = \sqrt{(x_1+\mu_2)^2 + x_2^2 + x_3^2}$$

$$R = \sqrt{(x_1-\mu_1)^2 + x_2^2 + x_3^2}$$



$$\dot{x}_1 = x_4$$

$$\dot{x}_2 = x_5$$

$$\dot{x}_3 = x_6$$

$$\dot{x}_4 = x_1 - \mu_1\frac{x_1+\mu_2}{r^3} - \mu_2\frac{x_1-\mu_1}{R^3} + 2x_5$$

$$\dot{x}_5 = x_2 - \mu_1\frac{x_2}{r^3} - \mu_2\frac{x_2}{R^3} - 2x_4$$

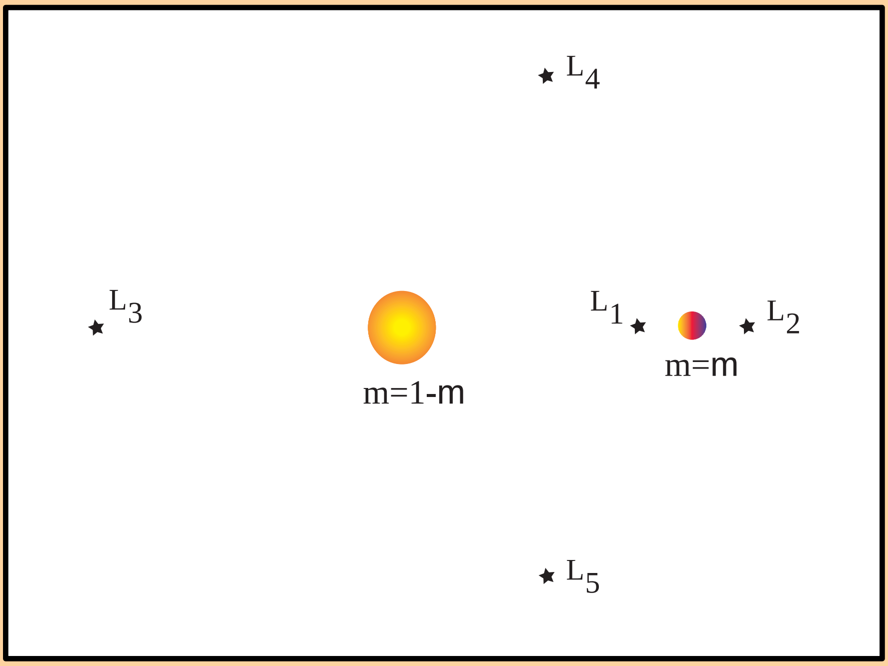
$$\dot{x}_6 = -\mu_1\frac{x_3}{r^3} - \mu_2\frac{x_3}{R^3}$$

- Using synodic coordinates, that is, we consider the rotating reference frame with the same frequency as primaries, one finds that the evolution of the test particle is described by

$$\boxed{\begin{aligned} \ddot{x} - 2\dot{y} &= \frac{\partial\Omega}{\partial x}, \\ \ddot{y} + 2\dot{x} &= \frac{\partial\Omega}{\partial y}, \end{aligned}}$$

where $\Omega(x,y) = \dfrac{x^2+y^2}{2} + \dfrac{1-\mu}{r_1} + \dfrac{\mu}{r_2}$,

$$r_1 = \sqrt{(x+\mu)^2 + y^2}, \text{ and } r_2 = \sqrt{(x-1+\mu)^2 + y^2},$$



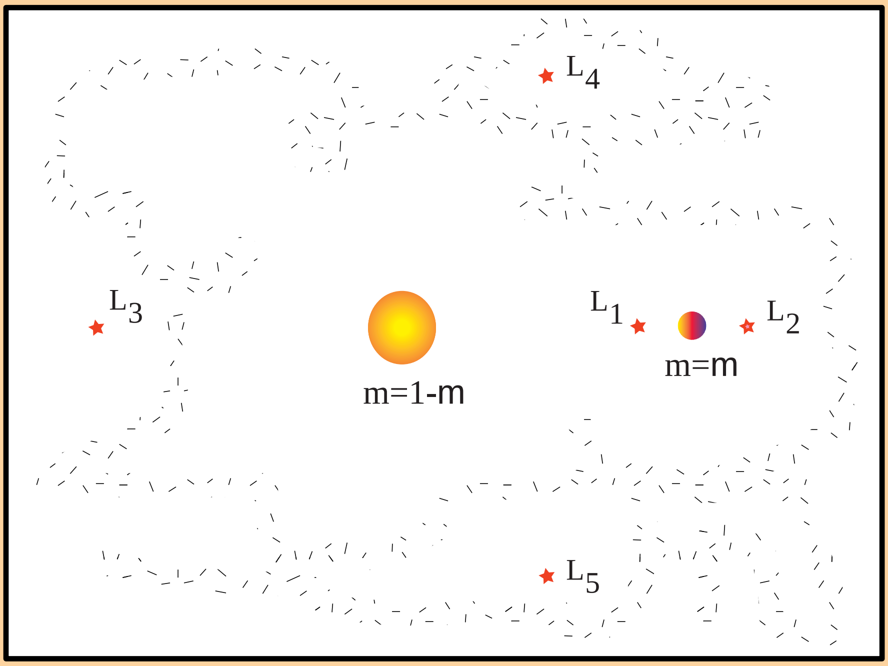
LAGRANGE POINTS

- Librations points in the planar circular restricted three-body problem. The stability of these points is classical problem in Celestial Mechanics.

- Equilateral libration points are marginal stable!, when mis small.

$$x = \frac{1}{2} - \mu, \quad y = \pm\sqrt{\frac{3}{2}},$$

Dissipation induced instabilities

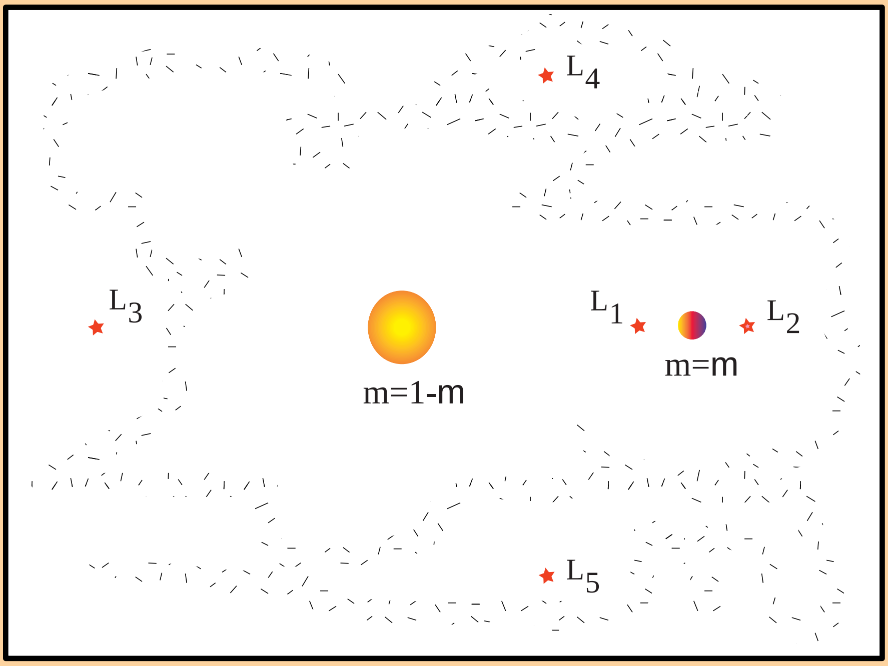


Inertial drag force

$$\begin{aligned}
\ddot{x} - 2\dot{y} &= \frac{\partial\Omega}{\partial x} - \nu\,(\dot{x} - y)\,, \\
\ddot{y} + 2\dot{x} &= \frac{\partial\Omega}{\partial y} - \nu\,(\dot{y} + x)\,.
\end{aligned}$$

- Equilateral libration points are stable! (Murray 1994).

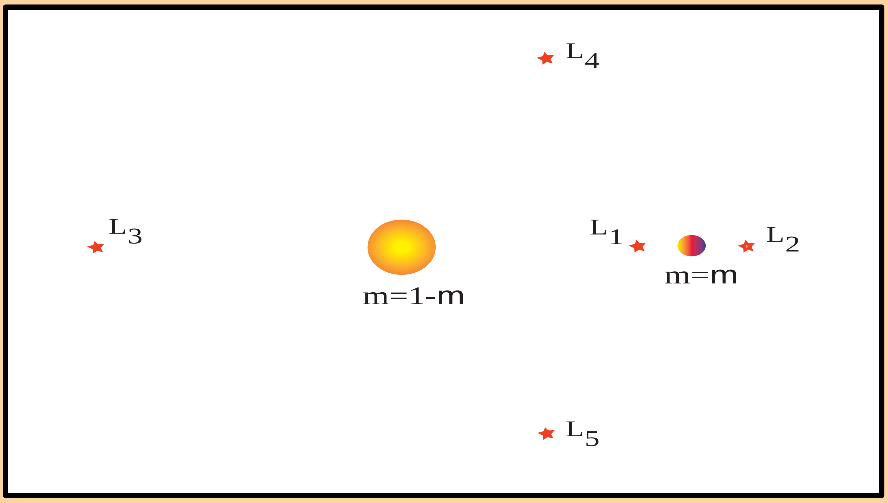
Dissipation induced instabilities



Nebular drag force

$$
\begin{aligned}
\ddot{x} - 2\dot{y} &= \frac{\partial \Omega}{\partial x} - \nu \dot{x}, \\
\ddot{y} + 2\dot{x} &= \frac{\partial \Omega}{\partial y} - \nu \dot{y}.
\end{aligned}
$$

- Equilateral libration points are unstable!



The Poyting-Roberson effect

$$
\begin{aligned}
\ddot{x} - 2\dot{y} &= \frac{\partial \Omega}{\partial x} - \frac{\nu}{r_1}\left(\dot{x} - y + \frac{x}{r_1^2}(x\dot{x} + y\dot{y})\right), \\
\ddot{y} + 2\dot{x} &= \frac{\partial \Omega}{\partial y} - \frac{\nu}{r_1}\left(\dot{y} + x + \frac{y}{r_1^2}(x\dot{x} + y\dot{y})\right).
\end{aligned}
$$

- Equilateral libration points are unstable!

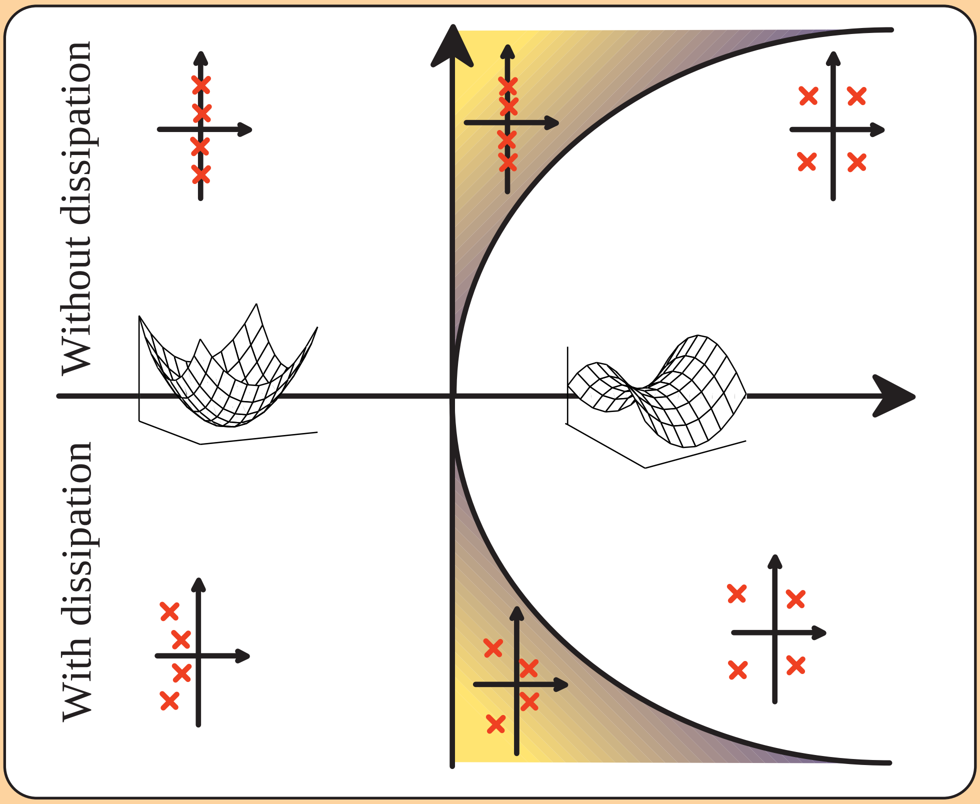
FOUCAULT PENDULUM

- As consequence of Earth rotation, the vertical solution exhibits a latent bifurcation!.

$$\ddot{x} = -\left(\frac{g}{l} - \Omega^2\right) x + 2\Omega \dot{y}$$

$$\ddot{y} = -\left(\frac{g}{l} - \Omega^2\right) y - 2\Omega \dot{x}$$

Conclusion



- Latent bifurcation.
- Dissipation induced instability.
- Applications : simple mechanical system, Laser, restricted three-body problem, Foucault pendulum, Chemical systems, nuclear physics and Baroclinic instability.