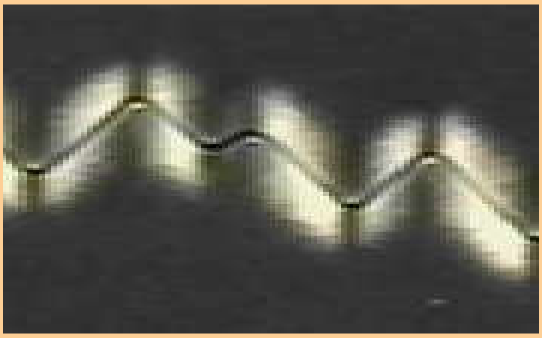
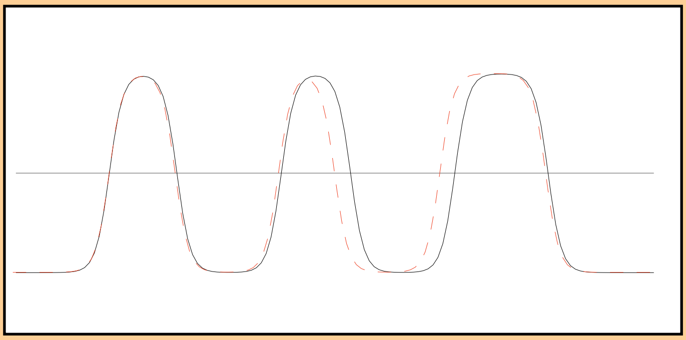
Bubbles interaction in the Cahn-Hilliard equation





H. Calisto, M. Clerc, R. Rojas, E. Tirapegui

INLN, CFNS

Motivation (Zigzag Instability)

C. Chevallard, M. Clerc, P. Coullet, J.-M. Gilli

Nematic liquid crystal cell

5 CB

d ~100mm

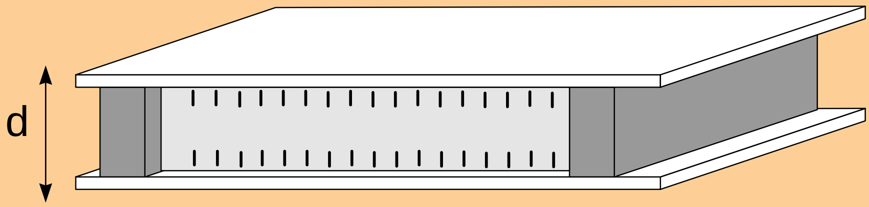
c_a =1.142

e_a =11.3

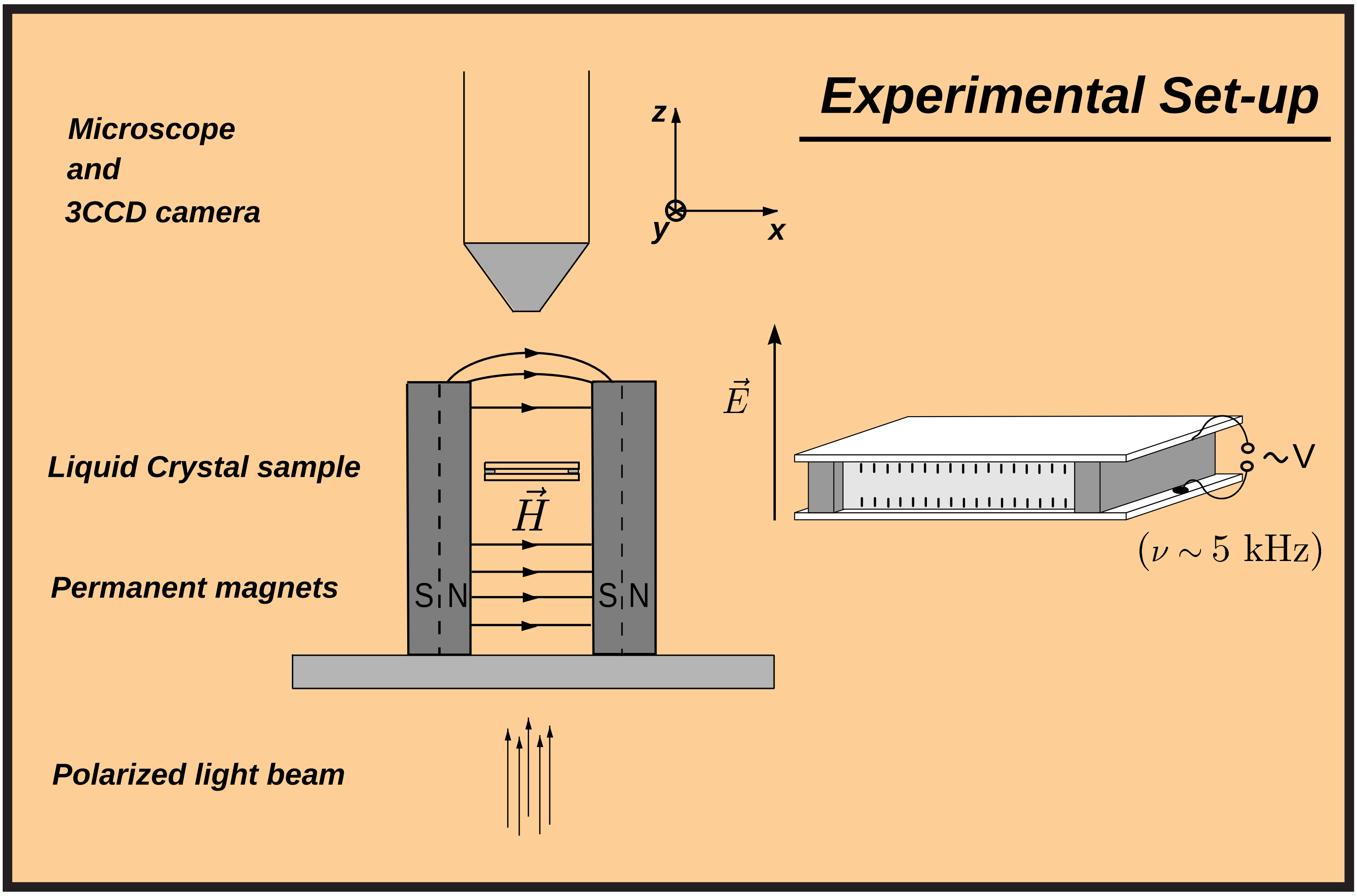
$K_1 = 6.3$

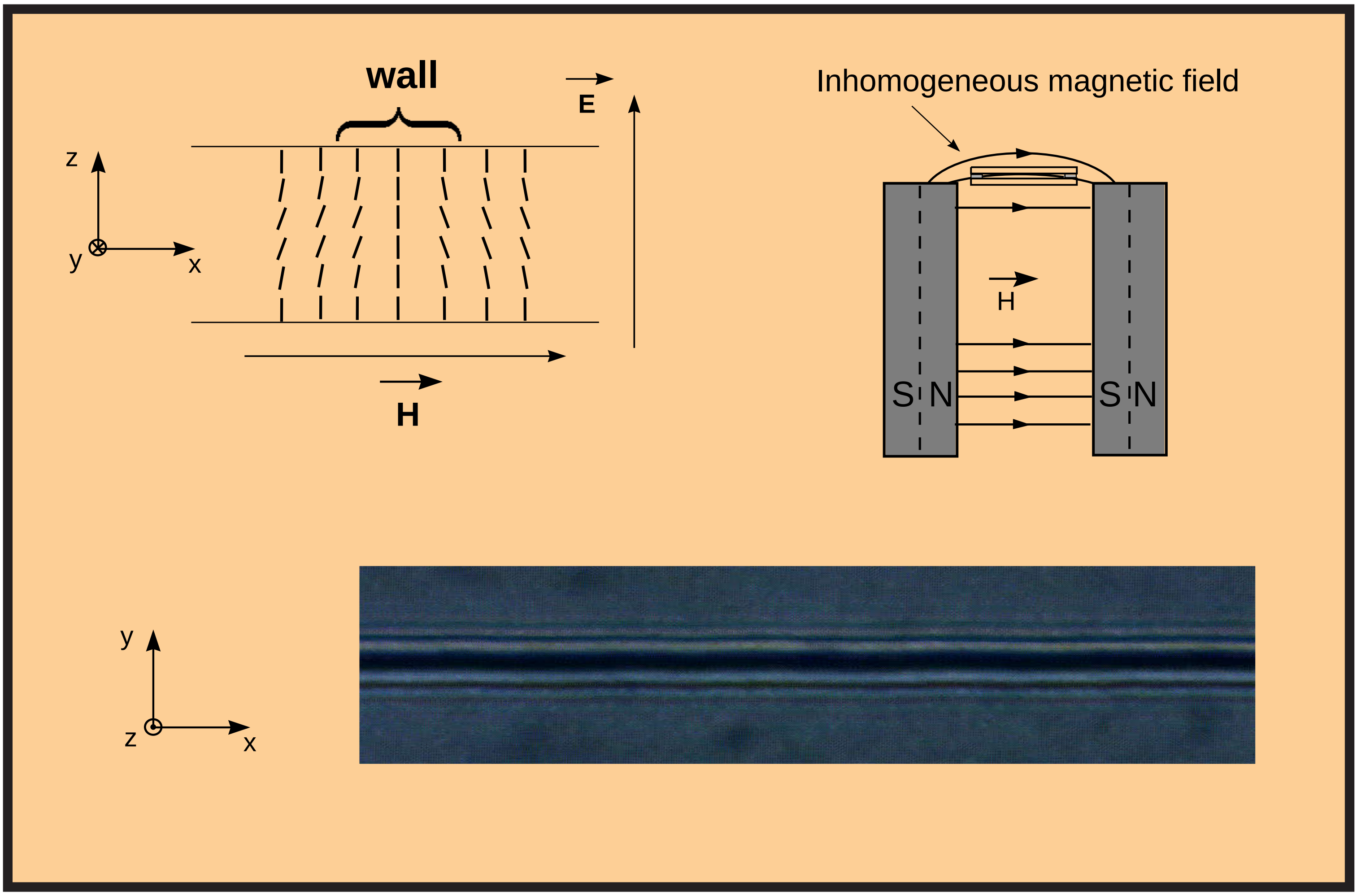
$K_2 = 4.1$ $(10^{-12}\mathrm{N})$

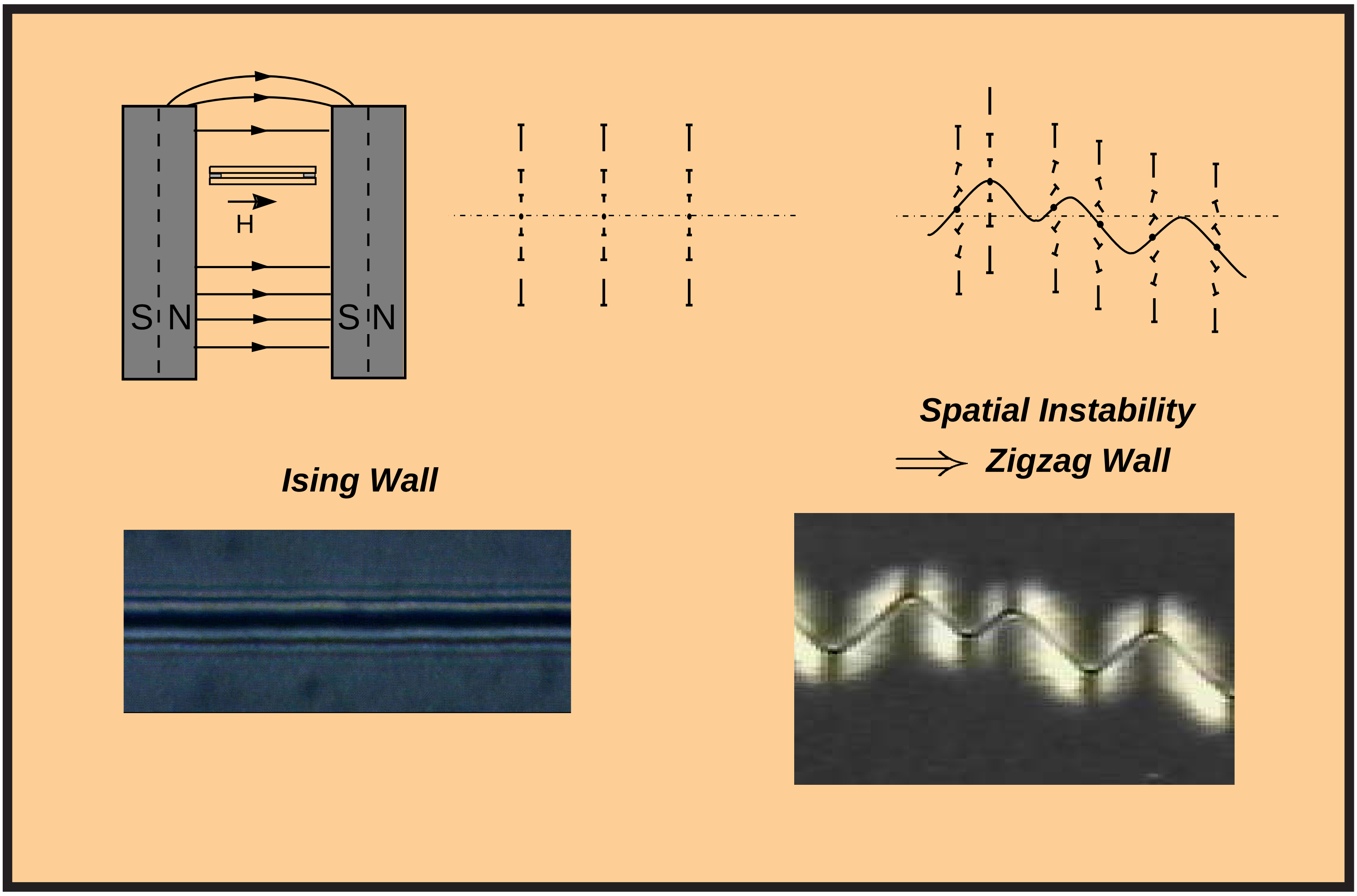
$K_3 = 8.4$

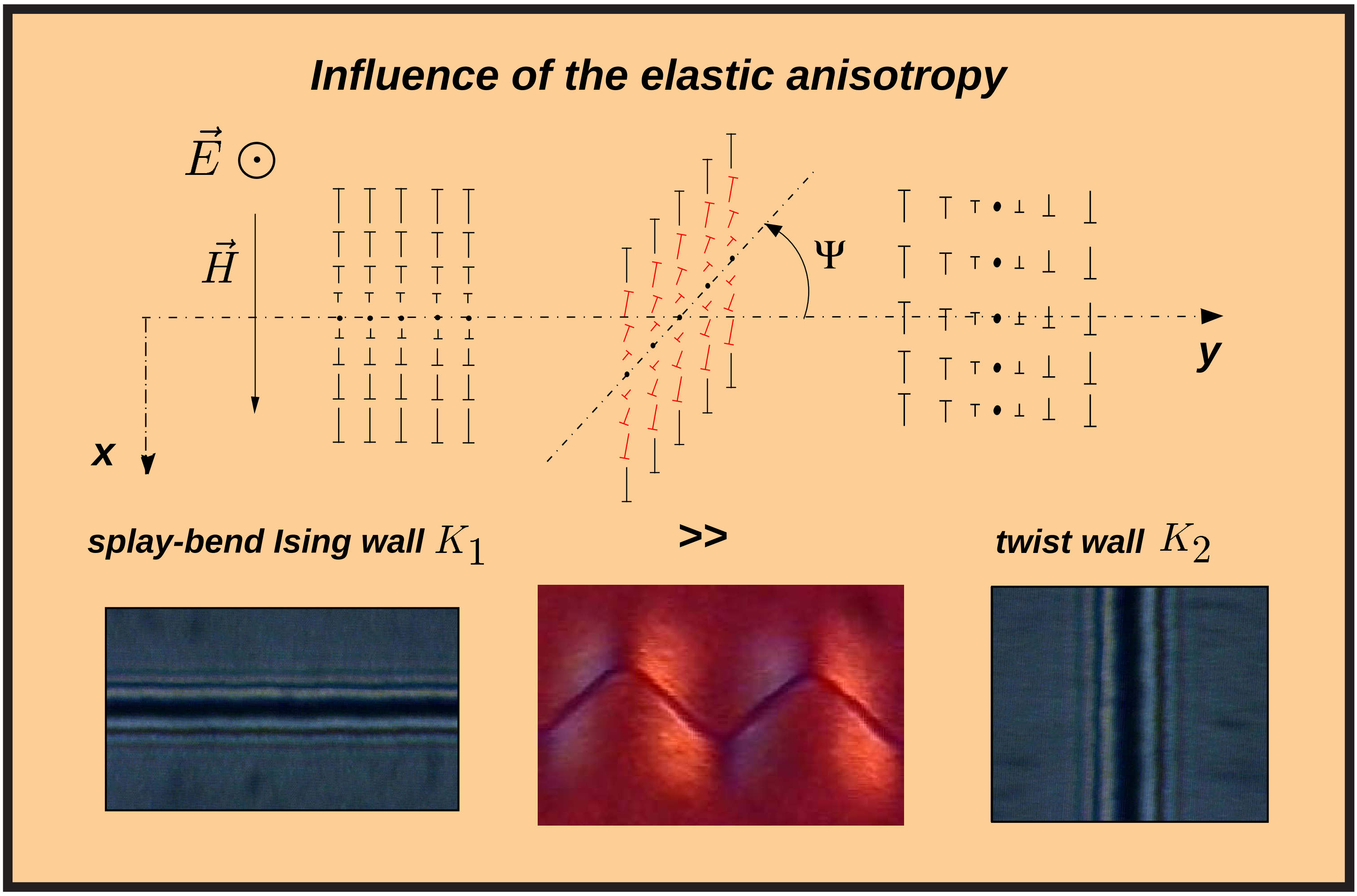


Homeotropic anchoring

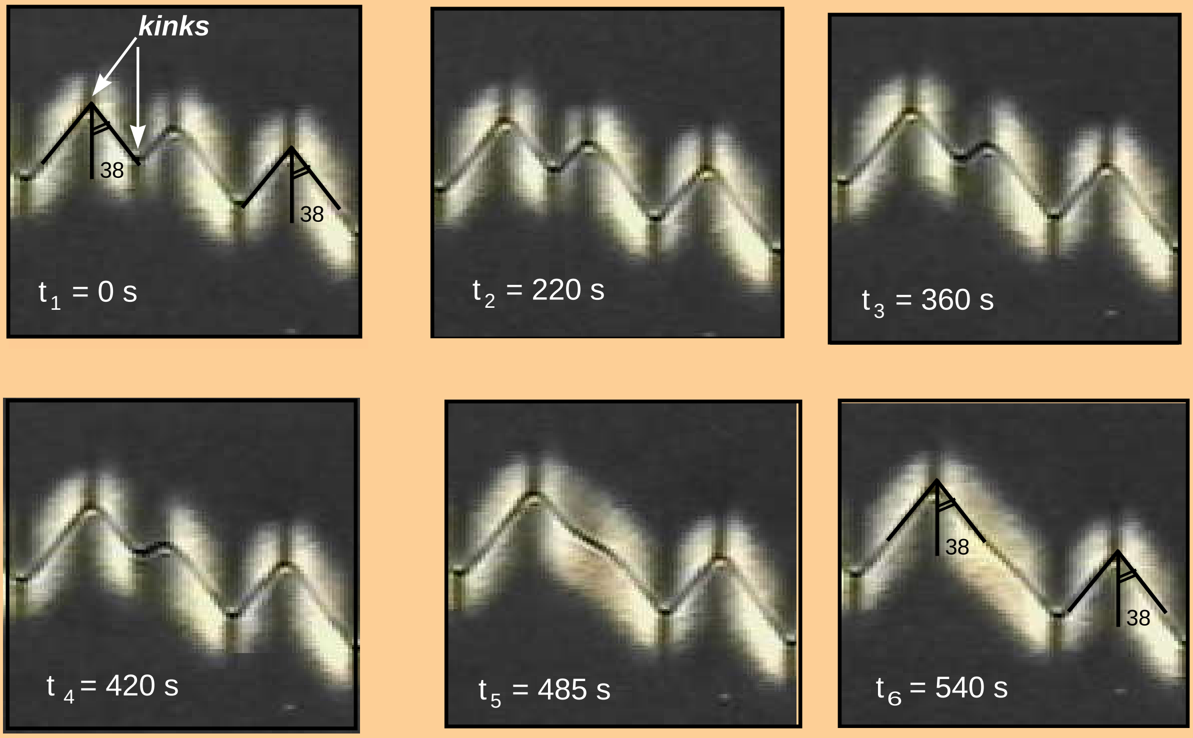

Experimental Set-up
Microscope
and
3CCD camera
z
y
x
Liquid Crystal sample
$\vec{H}$
$\vec{E}$
~V
$(\nu \sim 5\ \mathrm{kHz})$
Permanent magnets
S N
S N
Polarized light beam


wall
E
z
y
x
H
Inhomogeneous magnetic field
H
S N
S N
y
z
x


H
S N
S N
Spatial Instability
$\Longrightarrow$ Zigzag Wall
Ising Wall

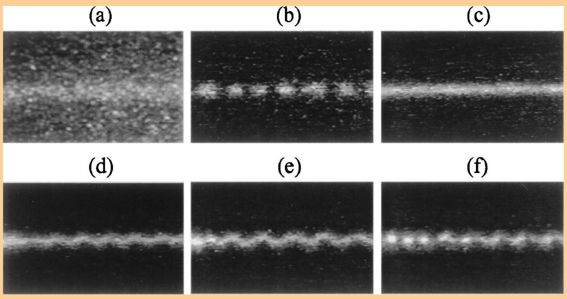

Influence of the elastic anisotropy
$\vec{E}$ $\odot$
$\vec{H}$
Ψ
y
x
splay-bend Ising wall K_1
>>
twist wall K_2

Domains dynamic (Facets dynamic, coarsening)?

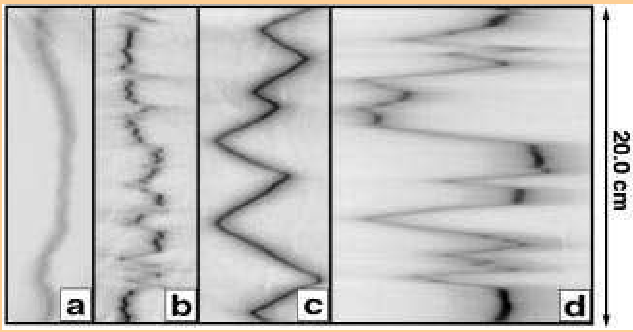


Universal behavior

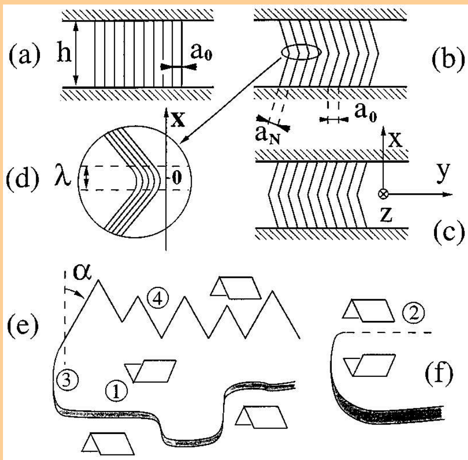
i)



ii)



iii)

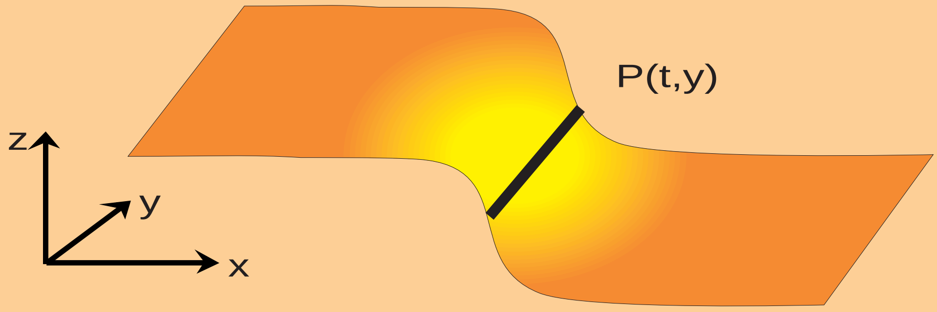


i) Gas discharge system, PRL, 78, 3129 (1997).

ii) Rifts in spreanding wax layer, PRL, 76, 3456 (1996).

iii) Zigzag walls in the chevron structure, Europhys. Lett., 44, 205 (1998).

Interface dynamics

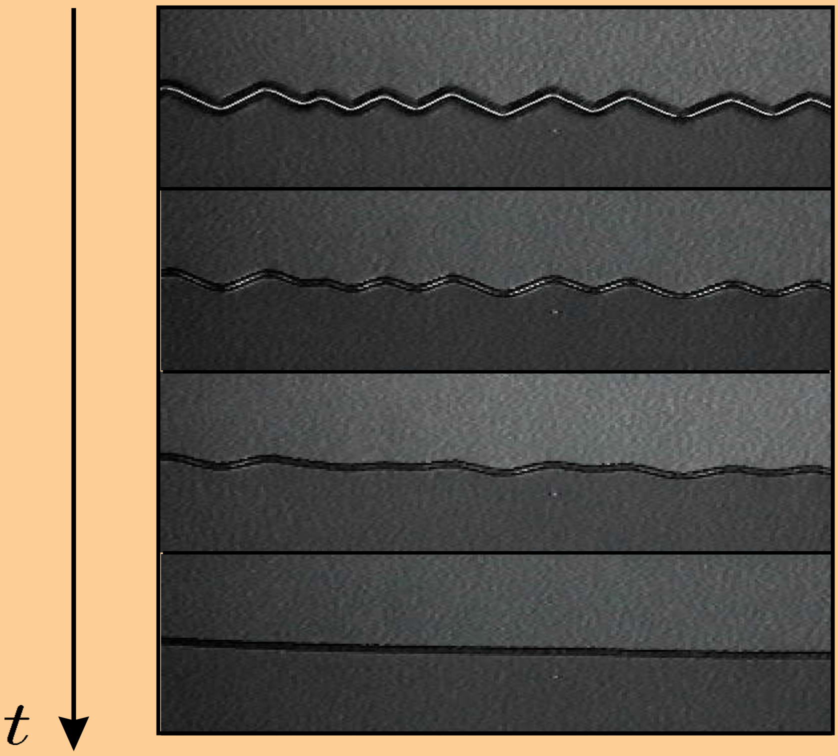


$$Z_o = \frac{\epsilon}{\sqrt{a}} \tanh\left(\frac{\epsilon}{\sqrt{2}}(x-P)\right)$$ ***is the Ising wall solution (splay-bend) close of Fréedericksz transition.***

In order to investigate the interface dynamics, one introduces the following ansatz :

$$Z(x,y,t) = \frac{\epsilon}{\sqrt{a}} \tanh\left(\frac{\epsilon}{\sqrt{2}}(x-P(y,t))\right) + \eta(x-P,P)$$

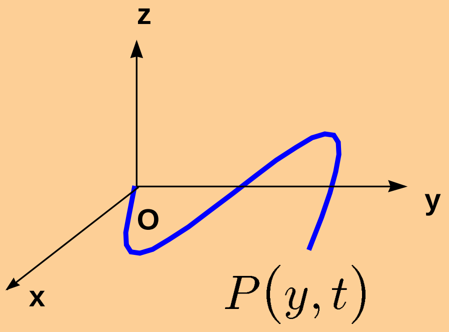
where $\eta\big(x-P,P\big)$ ***is a perturbation, generically*** P ***satisfies the diffusion equation.***



$$\partial_t P = \epsilon P_{yy}$$

- ***Experimental check of the diffusive interface equation.***

Symmetry analysis



Translational invariance along the x-axis

$$x \leftrightarrow x + x_o \quad , \quad P \leftrightarrow P + P_o$$

Reflection symmetry in the direction tangent to the interface

$$y \leftrightarrow -y \quad , \quad P \leftrightarrow P$$

Therefore the position of the interface satisfies $\partial_t P = f(\partial_y^n P)$

Reflection symmetry in the direction normal to the interface

$$x \leftrightarrow -x \quad , \quad P \leftrightarrow -P$$

For small perturbation the order parameter satisfies the diffusion equation :

$$\partial_t P = \epsilon P_{yy}$$

When ϵ is small (positive or negative), the order parameter is described by the asymptotic equation

$$\partial_t P = \epsilon P_{yy} + P_y^2 P_{yy} - P_{4y}$$

With ϵ is the diffusion (antidiffusion), $P_y^2 P_{yy}$ the nonlinear diffusion and the last term is the hyperdiffusion.

The latter equations are only valid for anisotropic systems.

Observation

- $$\partial_t P = -\frac{\delta \mathcal{F}[P]}{\delta P}$$ ***with*** $$\mathcal{F}[P] = \int \left[\frac{\epsilon}{2} P_y^2 + \frac{3}{12} P_y^4 + \frac{1}{2} P_{yy}^2 \right] dy$$

  $\Rightarrow$ ***relaxational dynamics***

- ***Continuity equation :***

$$\partial_t P = \partial_y (\epsilon P_y + P_y^3 - P_{3y})$$

for an infinite medium, the system conserves the following global quantity

$$\int P \; dy$$

- ***Introducing the new variable*** $\Lambda = P_y$ ***the equation reads***

$$\partial_t \Lambda = \partial_{yy} (\epsilon \Lambda + \Lambda^3 - \Lambda_{yy}) = \partial_{yy} \frac{\delta \mathcal{F}[\Lambda]}{\delta \Lambda}$$

where $$\mathcal{F}[\Lambda] = \int \left[\frac{\epsilon}{2} \Lambda^2 + \frac{3}{12} \Lambda^4 + \frac{1}{2} \Lambda_y^2 \right] dy$$

- ***This is the Cahn-Hilliard equation for a one-dimensional system.***

- ***Variational problem under constraint***

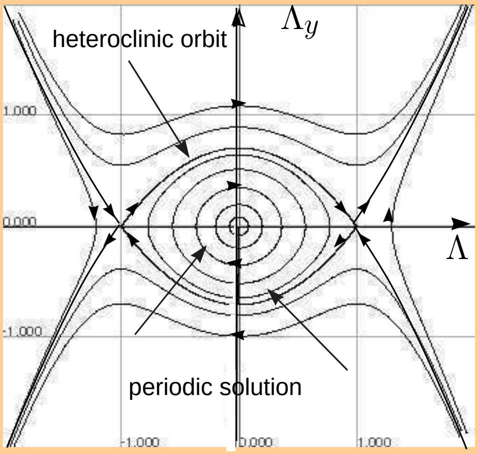
$$\left\{ \begin{array}{l} \partial_t \Lambda = \partial_{yy} \left[\frac{\delta \mathcal{F}}{\delta \Lambda} \right] \\ \mathcal{G}[\Lambda] = \int \Lambda \; dy = M \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \frac{\delta \mathcal{F}[\Lambda]}{\delta \Lambda} + \lambda \frac{\delta \mathcal{G}[\Lambda]}{\delta \Lambda} = 0 \\ \int \Lambda \; dy = M \end{array} \right.$$

with λ *Lagrangian multiplier*

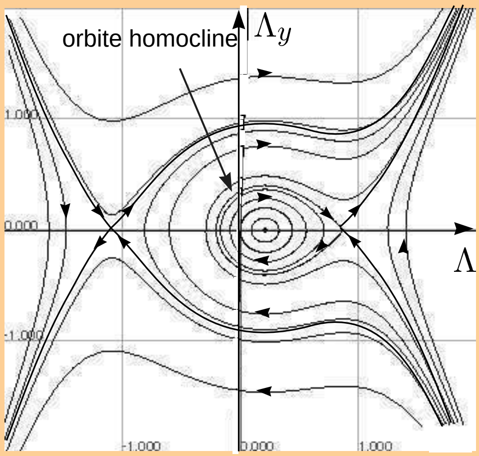
- ***Stationary solutions***

$$\partial_t P = 0 \Rightarrow -P_y + {P_y}^3 - P_{3y} = K$$
$$\Rightarrow -\Lambda + \Lambda^3 - \Lambda_{yy} = K$$

$K = 0$



$K \neq 0$



the global minimum are : heteroclinic, homoclinic, and uniform solutions.

Problems?

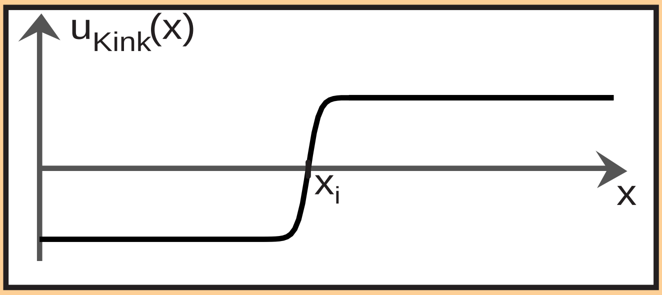
- ***Which are the particle like solutions of the Cahn-Hilliard equations?.***

- ***Which are the interaction between the particle like solution ("ulterior dynamics")?.***

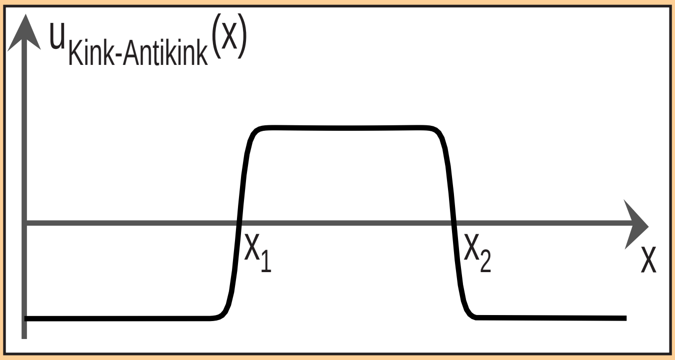
Kink solutions

This solutions are parametrized by the group of translation

$$u_i(x, x_i) = \sqrt{|\varepsilon|}\tanh\left(\sqrt{\frac{|\varepsilon|}{2}}(x - x_i)\right)$$



Interaction?, $u(x,t)=u_1(x,x_1(t))-u_2(x,x_2(t))+w$; $(w<<1)$.



As consequence of the area conservation, we have :

$$d_t(x_2-x_1) = 0$$

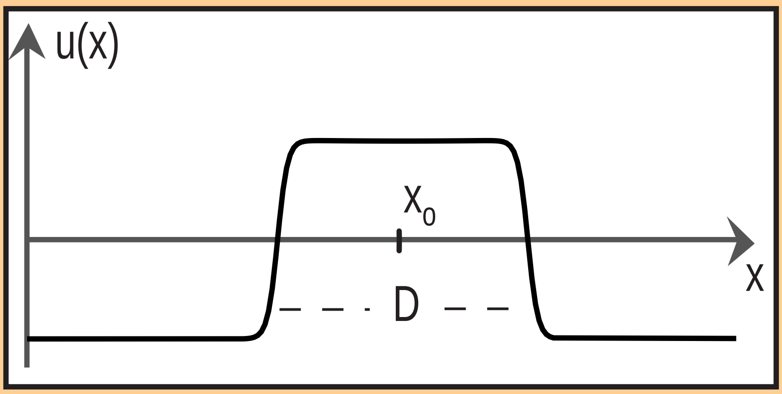
Bubble solutions

$$U(y \equiv x - x_o) = u_o + \frac{2\left(3u_o^2 + \varepsilon\right)}{-2u_o + \sqrt{2\left(|\varepsilon| - u_o^2\right)}\cosh\left(\sqrt{(3u_o^2+\varepsilon)}y\right)},$$

where

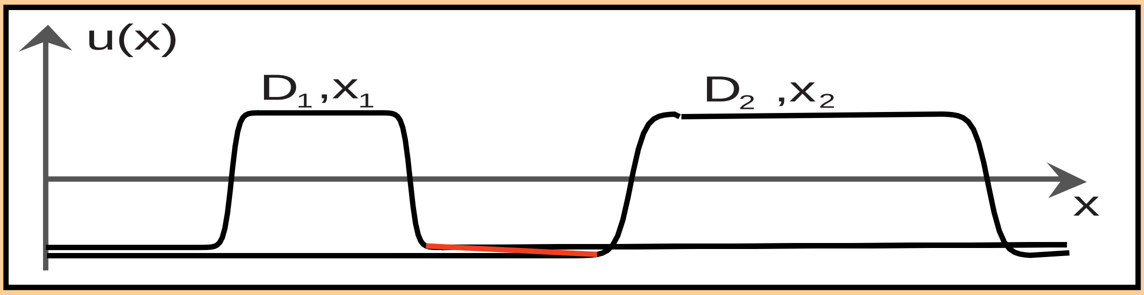
$$u_o = 2\sqrt{\frac{|\varepsilon|}{3}}\cos\left(\frac{1}{3}\arctan\left(\sqrt{\frac{4|\varepsilon|^3}{27\lambda^2}-1}\right)\right), \lambda < 0,$$

$$u_o = -2\sqrt{\frac{|\varepsilon|}{3}}\sin\left(\frac{1}{3}\left(\arctan\left(\sqrt{\frac{4|\varepsilon|^3}{27\lambda^2}-1}\right)+\frac{\pi}{2}\right)\right), \ \lambda \geq 0,$$



- $$U(x, x_o, \Delta) \approx -\sqrt{|\varepsilon|} + \sqrt{|\varepsilon|}\tanh\left(\sqrt{\frac{|\varepsilon|}{2}}\left(x - x_o - \frac{\Delta}{2}\right)\right) - \sqrt{|\varepsilon|}\tanh\left(\sqrt{\frac{|\varepsilon|}{2}}\left(x - x_o - \frac{\Delta}{2}\right)\right) + O(\sqrt{|\varepsilon|}e^{-\sqrt{2|\varepsilon|}\Delta})$$

Bubbles interaction $\left(L \gg \Delta \gg 1/\sqrt{|\varepsilon|}\right)$



In order to describe the interaction the parameters (D, x) are promoted to function. The first momentums of the Cahn-Hilliard are :

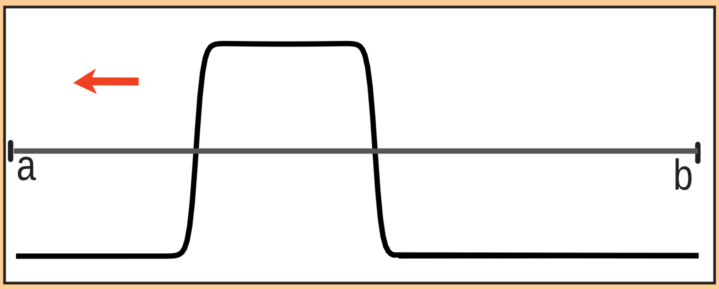
$$d_t \int_a^b u(x)dx = \partial_x \frac{\delta \mathcal{F}}{\delta u}\Big|_a^b ,$$
$$d_t \int_a^b x\ u(x)dx = x\partial_x \frac{\delta \mathcal{F}}{\delta u}\Big|_a^b + \frac{\delta \mathcal{F}}{\delta u}\Big|_b^a$$

- One bubble with periodic boundary conditions

$$\partial_t \int_a^b u(x)\, dx = 2\sqrt{\varepsilon} d_t \Delta = 0$$
$$\partial_t \int_a^b x\ u(x)\, dx = 2\sqrt{\varepsilon} d_t (x_o \Delta) = (b-a)\, \partial_x \frac{\delta \mathcal{F}}{\delta u}\Big|_b = 0$$

- with the boundary conditions $\partial_x u = \partial_{xx} u = 0$

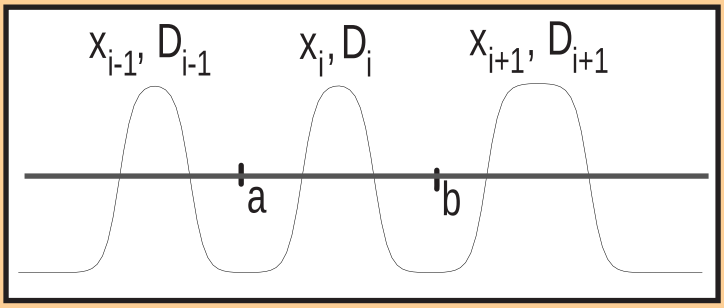
$$2\sqrt{\varepsilon} d_t \Delta = u_{xxx}|_b^a ,$$
$$2\sqrt{\varepsilon} d_t (x_o \Delta) = \varepsilon u + u^3 + x u_{xxx}\Big|_b^a$$



N. Alikakos, P. Bates, G. Fusco (1991).

Gas of diluted bubbles

Using the fact that the intermediated region between the bubbles is well approximated by straight lines, the equations for the position and width read



$$d_t\Delta_i = I_{i+1,i} - I_{i,i-1}, \qquad d_t x_i = \frac{I_{i+1,i} + I_{i,i-1}}{2},$$

where
$$I_{i+1,i} = \frac{8|\varepsilon|\sinh\left(\sqrt{\frac{|\varepsilon|}{2}}\,(\Delta_i - \Delta_{i-1})\right)}{\left(x_{i+1} - x_i - \frac{\Delta_i+\Delta_{i-1}}{2}\right)} e^{\left(-\sqrt{\frac{|\varepsilon|}{2}}(\Delta_i+\Delta_{i-1})\right)}$$

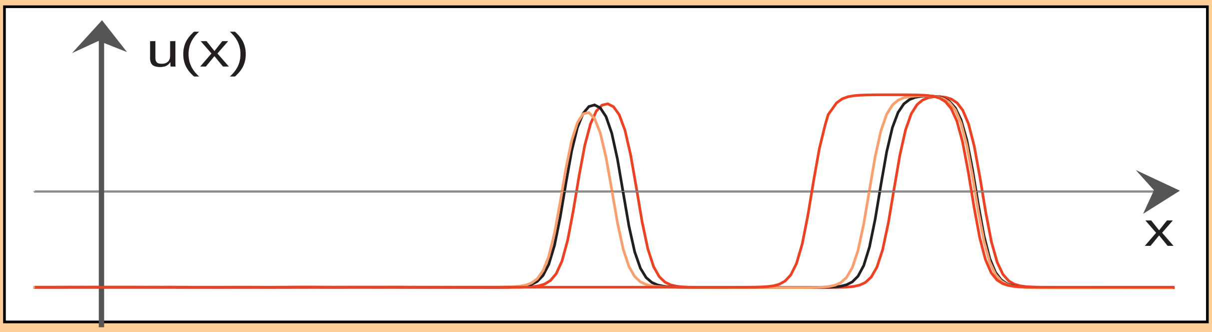
For n-bubbles with periodic boundary conditions we have the condition

$$x_{n+1} = x_1 + (b-a) \qquad x_0 = x_n - (b-a)$$

interaction of the two bubbles (periodic boundary conditions)

The dynamics will be given by

$$\begin{aligned} d_t\left(\Delta_1 + \Delta_2\right) &= 0,\\ d_t(x_1\Delta_1 + x_2\Delta_2) &= (b-a)I_{1,2},\\ d_t\Delta_1 &= I_{2,1} - I_{1,2},\\ d_t(x_2 - x_1) &= -|\varepsilon|^{-1/2}(I_{2,1} + I_{1,2})\frac{(\Delta_1 - \Delta_2)}{2\Delta_1\Delta_2}. \end{aligned}$$



Approximate solution

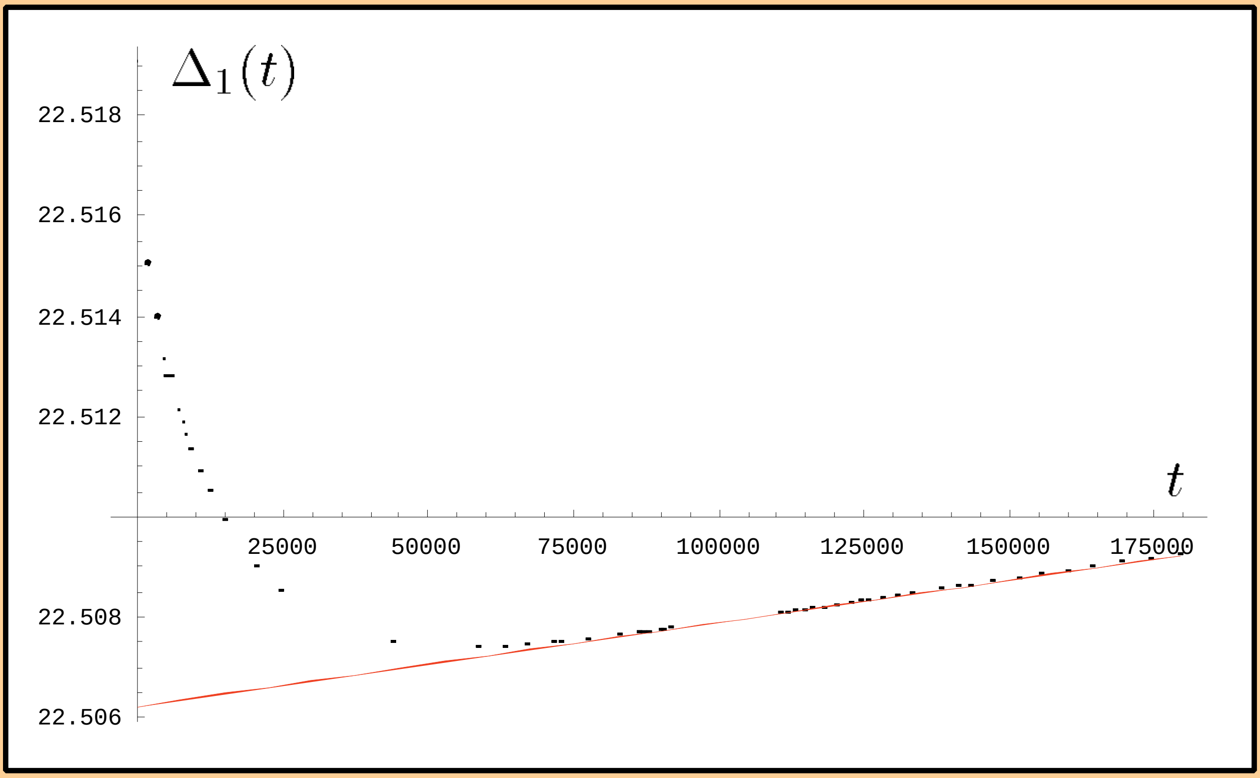
When it is considered the dominate terms in the latter equations, one obtain

$$
\begin{aligned}
\Delta_1(t) &= \frac{\Delta}{2} + \sqrt{\frac{2}{|\varepsilon|}}\,\mathrm{arctanh}\left(\tanh\left[\frac{1}{2}\sqrt{\frac{|\varepsilon|}{2}}\delta_o\right] \quad \sqrt{2|\varepsilon|}\,\mathcal{C}\,(t-t_o)\right)\\
\Delta_2(t) &= \frac{\Delta}{2} - \sqrt{\frac{2}{|\varepsilon|}}\,\mathrm{arctanh}\left(\tanh\left[\frac{1}{2}\sqrt{\frac{|\varepsilon|}{2}}\delta_o\right] \quad \sqrt{2|\varepsilon|}\,\mathcal{C}\,(t-t_o)\right)\\
x_1(t) &= x_1(t_o) - \frac{b-a-2R}{2(b-a-\Delta)}\left\{\frac{\delta_o}{2} - \sqrt{\frac{2}{|\varepsilon|}}\,\mathrm{arctanh}\left(\tanh\left[\frac{1}{2}\sqrt{\frac{|\varepsilon|}{2}}\delta_o\right] \quad \sqrt{2|\varepsilon|}\,\mathcal{C}\,(t-t_o)\right)\right\}\\
x_2(t) &= x_2(t_o) - \frac{b-a-2R}{2(b-a-\Delta)}\left\{\frac{\delta_o}{2} - \sqrt{\frac{2}{|\varepsilon|}}\,\mathrm{arctanh}\left(\tanh\left[\frac{1}{2}\sqrt{\frac{|\varepsilon|}{2}}\delta_o\right] \quad \sqrt{2|\varepsilon|}\,\mathcal{C}\,(t-t_o)\right)\right\}
\end{aligned}
$$

where $\Delta_1 + \Delta_2 \equiv \Delta, \quad x_2 - x_1 \equiv R, \quad \delta \equiv \Delta_1 - \Delta_2\,,$

$$\mathcal{C} \equiv \frac{8|\varepsilon|\,(b-a-\Delta)\,e^{-\sqrt{\frac{|\varepsilon|}{2}}\Delta}}{\left(R-\frac{\Delta}{2}\right)\left((b-a)-R-\frac{\Delta}{2}\right)}$$

Quantitative!



Gas of diluted bubbles

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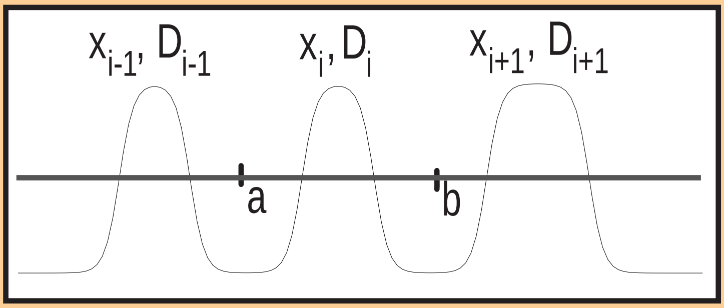

$$d_t\Delta_i = I_{i+1,i} - I_{i,i-1}, \qquad d_t x_i = \frac{I_{i+1,i} + I_{i,i-1}}{2},$$

where

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Summary

We have studied the dynamics of bubbles in the one dimensional Cahn-Hillierd equation. For a gas of diluted bubbles we have found ordinary differential equations describing their interaction which us to describe the ulterior dynamics of the system in very good agreement with numerical simulations.