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# Cosmology course summary

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# 1. Introduction

The present document pretend to be a quick review of the cosmology course dictated by Gonzalo Palma and Domenico Sapone on the spring-semester of this year. It's important to remark that this is not a replacement of the lectures inspiring this work, but it will be useful to have a notion of the most relevant subjects mentioned during classes. Also, we use as guiding books the ones written by Weinberg, Dodelson and others very similar.

On the other hand, I want to deeply thank Sebastián Vargas, José Fuentealba and Daniela Grandón for their participation in the correction and construction of many of the ideas deposited along this document.

## 1.1. About the notation

We will use natural units (c $= \hbar = 1$) and mostly the plus signature for the spacetime metric, i.e., (-,+,+,+). We will suppress the indices for the Boltzmann and Newton's gravitational constants, so they will be denoted just as $k$ and $G$, respectively.

Integrals in momentum will be in Fourier space of momentum, then we will have the convention:

$$\int_{\mathbf{p}} = \int \frac{d^3p}{(2\pi)^3} \tag{1}$$

However, when we deal with the Boltzmann equation and distribution of particles we will use:

$$\int_{\bar{\mathbf{p}}} = \int \frac{d^3p}{(2\pi)^3\sqrt{-detg}} \tag{2}$$

On the other hand, given a fluctuation $\phi(\mathbf{x}, t)$, we will define its Fourier counterpart as:

$$\tilde{\phi}(\mathbf{k}, t) = \int_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{x}, t) \tag{3}$$

## 1.2. Contact us

In case you have any correction/observation about this summary, please send me an e-mail to felipe.cubillos@ug.uchile.cl

# 2. Main results of General Relativity

One of the main uses of the metric is to compute lengths. We define the proper length $ds$ between two infinitesimally separated points located at $x^\mu$ and $x^\mu + dx^\mu$ as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

With the intention of defining a derivative operator consistent with our notion of the universe as a manifold, we introduce the Christoffel conections, given by:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} \left( g_{\lambda\nu,\mu} + g_{\lambda\mu,\nu} - g_{\mu\nu,\lambda} \right) \tag{2}$$

Then, we can define the covariant derivative of a tensor $A^{\mu\ldots\nu}{}_{\alpha\ldots\beta}$ as:

$$\begin{aligned} \nabla_\lambda A^{\mu\ldots\nu}{}_{\alpha\ldots\beta} = & A^{\mu\ldots\nu}{}_{\alpha\ldots\beta,\lambda} + \Gamma^\mu_{\lambda\sigma} A^{\sigma\ldots\nu}{}_{\alpha\ldots\beta} + \ldots + \Gamma^\nu_{\lambda\sigma} A^{\mu\ldots\sigma}{}_{\alpha\ldots\beta} \\ & - \Gamma^\sigma_{\lambda\alpha} A^{\mu\ldots\nu}{}_{\sigma\ldots\beta} - \ldots - \Gamma^\sigma_{\lambda\beta} A^{\mu\ldots\nu}{}_{\alpha\ldots\sigma} \end{aligned} \tag{3}$$

The equation of motion determining a geodesic followed by a point particle is:

$$\frac{du^\rho}{d\tau} + \Gamma^\rho_{\mu\nu} u^\mu u^\nu = 0 \tag{4}$$

where $u^\mu = \frac{dx^\mu}{d\tau}$ is the four velocity and $\tau = \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$ is the proper time of the particle.

The energy-momentum tensor describes the energy content in the spacetime under study and consequently, it determine the geometry of the spacetime. Also, the energy-momentum tensor respects the continuity equation $\nabla_\mu T_{\mu\nu} = 0$. The energy content bends spacetime in a way described by Einstein's equations, given by:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G T_{\mu\nu} \tag{5}$$

where $R_{\mu\nu}$ the Ricci tensor, defined as the contraction $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ of the Riemann tensor, and $\mathcal{R} = R^\rho{}_\rho$ is the Ricci scalar. The Riemann tensor can be computed as:

$$R^\rho{}_{\mu\lambda\nu} = \Gamma^\rho_{\mu\nu,\lambda} - \Gamma^\rho_{\mu\lambda,\nu} + \Gamma^\rho_{\lambda\sigma} \Gamma^\sigma_{\mu\nu} - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda} \tag{6}$$

The relation that connect the representation of a given tensor (the metric, for example) in two different coordinate systems $x^\mu$ and $x^{\mu'}$ is given by:

$$A_{\mu'\ldots\nu'}(x') = \frac{\partial x^\mu}{\partial x^{\mu'}} \ldots \frac{\partial x^\nu}{\partial x^{\nu'}} A_{\mu\ldots\nu}(x) \tag{7}$$

If we now consider a "small" change of the coordinate system $x^\mu \to x'^\mu = x^\mu + \xi^\mu$, it follows that:

$$A'_{\mu'\nu'\ldots\rho'}(x) = A_{\mu\nu\ldots\rho}(x) - \partial_\mu \xi^\lambda A_{\lambda\nu\ldots\rho}(x) - \partial_\nu \xi^\lambda A_{\mu\lambda\ldots\rho}(x) - \ldots - \partial_\rho \xi^\lambda A_{\mu\nu\ldots\lambda}(x) \tag{8}$$

This is known as a gauge transformation. For the particular case of the metric and the energy-momentum tensor we have, respectively:

$$g'_{\mu\nu} = g_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu \qquad T'_{\mu\nu} = T_{\mu\nu} - \xi_\lambda \nabla^\lambda T_{\mu\nu} - T^\lambda_\mu \nabla_\nu \xi_\lambda - T^\lambda_\nu \nabla_\mu \xi_\lambda \tag{9}$$

# 3. Background cosmological dynamics

## 3.1. Observable universe and co-moving distance

On sclaes larger than about 100 Mpc's our universe is both homogeneous and isotropic. Then we can use the FLRW metric to describe it. Also, it can be shown that the Riemann tensor of maximally symmetric manifolds satisfy $R^{(3)}_{ijkl} = K(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk})$, where for a vanishing curvature we have a flat spacial 3D manifold, for a positive one the spatial manifold has a closed geometry (3D-sphere), whereas if the curvature is negative the spatial manifold has an open geometry (anti-de Sitter3D space). If $K \neq 0$ it is convenient o choose radial coordinates, the the line element reads as follow:

$$ds^2 = -dt^2 + a^2 \left( \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right) \qquad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{1}$$

Before continue, it is important to mention that only the combination $x^i_P = a(t)x^i$ can be identified as a proper distance, where $x^i$ are the co-moving coordinates.

Recall that:

$$E = \gamma m = \sqrt{m^2 + \mathbf{p}^2} \qquad \mathbf{p}^2 \equiv g^{ij} p_i p_j \tag{2}$$

So, even though the proper time used in eq. (4) becomes ill defined in the massless limit $m \to 0$, we can rewrite the geodesic equation for a massless particle using the past relation:

$$E \frac{dp^\lambda}{dt} + \Gamma^\lambda_{\mu\nu} p^\mu p^\nu = 0 \tag{3}$$

With this equation, we can also find for a particle moving along a radial direction ($\theta$ and $\phi$ are fixed, i.e., $p^\theta = p^\phi = 0$ ), introducing the definition $p^2 \equiv \gamma^{ij} p_i p_j$ (so that $|\mathbf{p}| = p/a$), that $\dot{p} = 0$. This means that the proper momentum redshifts as the universe expands, in other words, $|\mathbf{p}| \propto a^{-1}$, this is consistent with the fact that wavelengths should stretch as the universe expands, as a consequence of the de Broglie relation $|\mathbf{p}| = 2\pi/\lambda$. Because of these statements, the wavelength of a photon at a time $t$ is being defined by the emitted wavelength $\lambda_e$ as:

$$\lambda(t) = \frac{\lambda_e}{a(t)} \tag{4}$$

With this expression we can measure how small was the universe when the photon was emitted. On the other hand, measuring the emission/absorption lines of a source, it can appear to be shifted by a common ratio $\delta\lambda/\lambda$; this shift motivate us to define the redshift parameter as:

$$z \equiv \frac{\lambda - \lambda_e}{\lambda_e} \tag{5}$$

Using the two previous relations, we can find:

$$a = \frac{1}{1 + z} \tag{6}$$

Then, the four-momentum of a photon is redshifted as:

$$p^{\mu} = \frac{p^{\mu}_{(e)}}{1+z} \tag{7}$$

Recall that the photons moving in a radial path have vanishing line element $ds = 0$. We are going to start call $r$, the co-moving distance. Then, integrating the relation coming from $ds^2 = 0$, we define:

$$S(r) \equiv \int_0^r \frac{dr}{\sqrt{1-Kr^2}} = \int_{t_e}^{t_0} \frac{dt}{a(t)} \tag{8}$$

If we now consider two consecutive emitted photons from a source at $r$ at times $t_e$ and $t_e + \delta t_e$, those particles will be observed on Earth at times $t_0$ and $t_0 + \delta_0$. Then, using that $S(r)$ is independent of time, we can find the following relation:

$$\delta t_0 = (1+z)\delta t_e \tag{9}$$

We can conclude that the rate of photons per unit of time arriving to our instruments differs from the rate at which they were emitted by a factor $(1+z)^{-1}$.

## 3.2. Luminosity distance

Luminosity distance is defined by the absolute luminosity $L$ and the apparent luminosity or flux $f$ as:

$$d_L \equiv \sqrt{\frac{L}{4\pi f}} \tag{10}$$

Then, by measuring the apparent luminosity of a source of well known absolute luminosity, we can infer the luminosity distance. The flux of a source of absolute luminosity $L$ at a co-moving distance $r$ is given by:

$$f = \frac{L}{(1+z)^2 A(r_e)} \tag{11}$$

where the factor $(1+z)^2$ is due to both redshift of the energy per photon, eq. (7) with $\mu = 0$, and the redshift of the rate at which photons arrive to our instruments, eq. (9). We can also find the relation:

$$d_L = r_e(1+z) \tag{12}$$

By virtue of eq. (6), we can Taylor expand z as a function of the emission time $t_e$ of an observed photon, valid as long as $H_0(t_0 - t_e) << 1$ and then find an expression for the deceleration parameter $q_0$ given by:

$$q_0 \equiv -\frac{\ddot{a}(t_0)}{H_0^2} \tag{13}$$

If we also expand the both sides of eq. (8) and we mix it with the expansion mentioned before, we can find the useful relation:

$$d_L = \frac{1}{H_0}\left[z + \frac{1}{2}(1-q_0)z^2 + ...\right] \tag{14}$$

Notice that this last relation it does not depend on the content of the universe.

Sometimes is also useful to define the reduced Hubble constant as $h \equiv \frac{H_0}{100km/s/Mpc}$.

## 3.3. Angular diameter distance

Consider a distant object of known proper extension $\delta s$ perpendicular to the line of sight, that is seen in the sky subtending an angle $\delta\theta$. We define the angular diameter distant to this object, and its relation with the distances defined before, as:

$$d_A \equiv \frac{\delta s}{\delta\theta} = \frac{r_e}{1+z} = \frac{d_L}{(1+z)^2} \tag{15}$$

The diameter angular distance corresponds to the distance of the object from us if the universe was flat and static. Given that it is hard to determine the proper extension of a distant object, in practice, the last equality in this expression is not so useful with objects at low redshifts. Nevertheless, $d_A$ will become useful in the study of the CMB.

## 3.4. Cosmological evolution

Recall that $T_{\mu\nu}$ must describe an homogeneous and isotropic fluid, i.e., a perfect fluid, in the same rest frame where the universe is experienced to be homogeneous and isotropic. So, we can write the energy-momentum tensor of a perfect fluid as follow:

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu} \tag{16}$$

where the four velocity in the frame mentioned before has componentes $(u^0, u^i) = (1, 0)$; $\rho(t)$ is the energy density and $p(t)$ is the pressure of the fluid. The continuity equation $\nabla_\mu T_{\mu\nu} = 0$ implies:

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{17}$$

If a particular type of fluid $a$ do not interact with the rest of the fluids composing the universe, then it will respect the continuity equation independently.

The Einstein's equations (5) respected by the FLRW metric take the form:

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{18}$$

The equality on the left side is known as Friedmann's equation. It can be shown that (17) is a consequence of (18). Therefore, we can study the universe by only using the continuity and Friedmann's equation.

In cosmology, fluids are well described by a very simple equation of state, which is:

$$p(\rho) = w\rho; \qquad w_{RM,\gamma} = \frac{1}{3}, \quad w_{NRM} \to 0, \quad w_\Lambda = -1 \tag{19}$$

where RM stands for relativistic matter, $\gamma$ for radiation, NRM for non-relativistic matter and $\Lambda$ for dark energy (which negative value of $w$ consist of a fluid with negative pressure). If we use this relation in the continuity equation of a type of fluid $a$ and the integrating leads to the solution:

$$\rho_a = \rho_{a,*}\left(\frac{a}{a_*}\right)^{-3(1+w_a)} \tag{20}$$

Hence, we can conclude that $\rho_{RM,\gamma} \propto a^{-4}$ whereas $\rho_{NRM} \propto a^{-3}$. The * represent evaluation at an arbitrary time $t_*$. Recall that both $T^a_{\mu\nu}$ and $\rho_a$ are considered to be additive, i.e., the sum over all possible types $a$ results in the quantity for the complete universe.

We use to denote $t_0$ for the present time, so $a_0 = 1$. Hence, $\rho_a = \rho_{a,0}a^{-3(1+w_a)}$, where the constant of proportionality corresponds to the current energy density of $a$ species. Thus, by specifying the matter content of our universe at present time and expressing Friedmann's equation in function of the sum over types $a$, the solution is going to tell us how the universe evolved in the past.

## 3.5. Density parameter

It is useful to define the critical density $\rho_c(t)$ and the density parameter $\Omega_a$of a species $a$, respectively, as:

$$\rho_c(t) \equiv \frac{3H^2}{8\pi G}, \qquad \Omega_a \equiv \frac{\rho_a}{\rho_c} \tag{21}$$

We can also conveniently introduce $\Omega_K \equiv -\frac{K_*}{H_*^2}$ and, consequently, a density parameter that effectively describes a perfect fluid, given by $w_K = -\frac{1}{3}$. Notice that by introducing $\Omega_K$ we're making the total density parameter $\Omega = \sum_a \Omega_a = 1$ at any given time $t_*$. On the other hand, we have $\sum_{a\neq K} \Omega_a = 1 - \Omega_K$ independently of $\Omega_K$, then, if $\sum_{a\neq K} = 1$, the curvature necessarily vanishes ($K = 0$).

Finally, we can relate the deceleration parameter $q_0$ and the content of our universe by:

$$q_0 = \frac{1}{2}\sum_a (1 + 3w_a)\Omega_{a,0} \tag{22}$$

## 3.6. Universes dominated by a single fluid

The table below shows a comparison between two different observational methods for the measurement of some cosmological parameters.

Table 1: Values of some parameters and its errors.

| | Direct standard candles' observations | CMB anisotropies observations |
|---|---|---|
| Hubble constant | $H_0 = 75.35 \pm 1.68$ km/s/Mpc | $H_0 = 67.66 \pm 0.42$ km/s/Mpc |
| Deceleration parameter | $q_0 = -1.08 \pm 0.29$ | $q_0 = -0.55 \pm 0.01$ |

CMB observations indicate a value for the reduced Hubble parameter of $h = 0.67 \pm 0.004$ and then a critical density for the present time given by $\rho_{c,0} = (4.82 \pm 0.06) \times 10^9 [eV/m^3]$.

Now, let us focus in the content budget of our current universe, where we can find:

- **Photons:** these particles distributed according to a Planckian spectrum called Cosmic Microwave Background (CMB). The measure of the CMB offer us a picture of the very early universe, when they had a last chance to scatter with electrons (last scattering time). We can also find $\rho_{\gamma,0} = 2.6 \times 10^5 [eV/m^3]$ and $\Omega_{\gamma,0} = 4.48 \times 10^{-5}$.

- **Neutrinos:** they decoupled from the hot bath much earlier than photons. In the massless limit we can relate their energy density with the one computed for photons by $\rho_\nu = 3 \cdot \frac{7}{8} \cdot \left(\frac{4}{11}\right)^{4/3} \rho_\gamma = 4.37 \times 10^5 [eV/m^3]$. We can also find $\Omega_{\nu,0} = 3.4 \times 10^{-5}$. We can see that, since the current value $\rho_{\nu,0}$ is negligible with respect to the energy density of other species, the true value is not very important to understand the evolution of our universe today.

- **Baryonic matter:** this type of matter corresponds to that we can observe them directly with our telescopes. CMB observations allow one to relate the density parameter of baryons and the reduced Hubble parameter via $\Omega_{B,0} h^2 = 0.0224 \pm 0.0001 \Rightarrow \Omega_{B,0} = 0.048 \pm 0.003$.

- **Dark matter:** this is a mysterious type of substance that behave like non-relativistic matter that only interact with baryons gravitationally. We can find by the same way we did it for baryonic matter that $\Omega_{DM,0} = 0.258 \pm 0.015$.

- **Curvature:** Observations of Baryon Acoustic Oscillations imprints in the structure of our universe, together with CMB observations, allow us to infer a value of the density parameter of curvature given by $\Omega_K = 0.001 \pm 0.002$, what is consistent with a flat universe.

- **Dark energy:** what we actually do to describe dark energy is to assumed to have the form of a cosmological constant, i.e., $T_{\mu\nu} \propto \Lambda g_{\mu\nu}$. Then, using the fact that $\sum_a \Omega_a = 1$, we have $\Omega_\Lambda = 0.6889 \pm 0.0056$ and $\rho_{\Lambda,0} = 3.32 \times 10^9 eV/m^3$. This last value result to be much smaller than the one expected from the Standard Model of particle physics.

## 3.7. Cosmic ages

If we group the photons and neutrinos into a single background fluid called radiation (R), and group baryonic matter and dark matter into a single fluid called matter (M), then the relevant density parameters today are:

$$\Omega_R = 8.4 \times 10^{-5}, \qquad \Omega_M = 0.31, \qquad \Omega_\Lambda = 0.69. \tag{23}$$

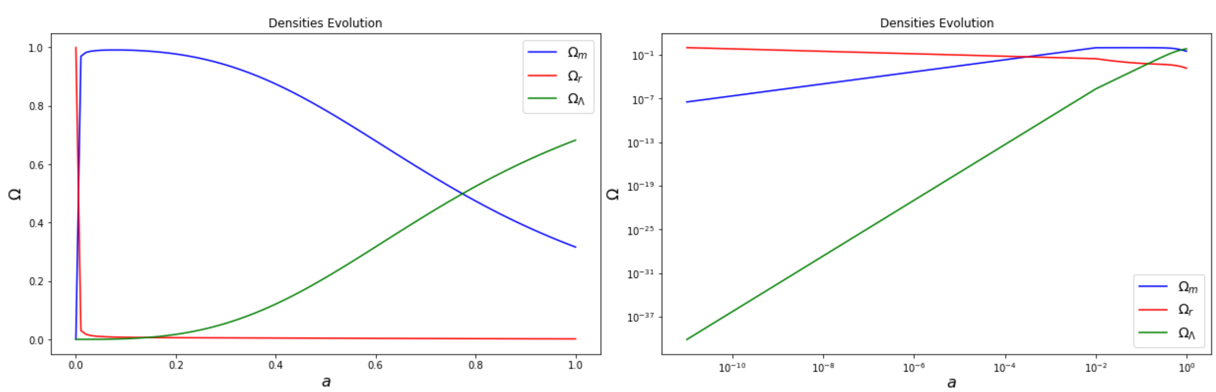


Figure 1: Densities evolution versus the scale factor plotted using CAMB.

Now, the main contribution today is that coming from the cosmological constant. Nevertheless, given that its value does not change together with expansion rate, both matter and radiation necessarily dominated in the past at different stages. Figure 1 shows the evolution of our three $\Omega_a$ in function of the scale factor in different scales.

The ratio between the energy density of matter and that of the cosmological constant is given by $\frac{\rho_M}{\rho_\Lambda} = \frac{\Omega_{M,0}}{\Omega_{\Lambda,0}} a^{-3}$. Then, both contributions become equally important to determine the expansion rate of the universe when the scale factor had a value $a_\Lambda = \left[\frac{\Omega_{M,0}}{\Omega_{\Lambda,0}}\right]^{1/3} \simeq 0.76$, which corresponds to a redshift $z_\Lambda \simeq 0.31$. On the other hand, $\frac{\rho_R}{\rho_M} = \frac{\Omega_{R,0}}{\Omega_{M,0}} a^{-1}$. So, the value of the scale factor at which both matter and radiation were equally important is deduced to be $a_{eq} = \frac{\Omega_{R,0}}{\Omega_{M,0}} \simeq 2.7 \times 10^{-4}$, which corresponds to a redshift $z_{eq} \simeq 3676$.

In other words, we can identify three main ages: (1) The radiation dominated era, (2) The matter dominated era and (3) The dark energy dominated era. These eras can be clearly seen in Figure 1.

### 3.7.1. Early time universe

Times much earlier than the $\Lambda$-dominated era ($a << a_\Lambda$) are commonly known as the early time universe. During this period, we can neglect the density parameter due to $\Lambda$, and then:

$$H^2 = H_0^2 \left[ \frac{\Omega_{R,0}}{a^4 + \frac{\Omega_{M,0}}{a^3}} \right] \tag{24}$$

Solving this equation we can see that at times $t << t_{eq} = 45248$ years the universe is radiation dominated.

### 3.7.2. Late time universe

Times where $a >> a_{eq}$ are usually known as the late time universe, at this regime we have the next Friedmann's equation:

$$H^2 = H_0^2 \left[ \frac{\Omega_{M,0}}{a^3} + \Omega_\Lambda \right] \tag{25}$$

Given that $\Omega_{M,0} = a_{eq R,0}$, we can compute the age of our universe to be $t_0 = 0.96 H_0^{-1} = 1.39 \times 10^{10}$ years.

After some algebra treatment of the Friedmann's equation solutions we find that in the limit where $a << a_\Lambda$ one obtains $a(t) \propto a_\Lambda exp \left( \sqrt{\frac{\Omega_{M,0}}{a_\Lambda^3}} H_0 t \right)^{2/3}$, whereas in the limit $a >> a_\Lambda$ one obtains $a(t) \propto \left(\sqrt{\Omega_{M,0}} H_0 t\right)^{2/3}$.

# 4. Kinectic theory in an expanding universe

A radiation dominated era is supposed to be hot, then we must take into account the kinematics of particles in a hot bath to have a better understanding of the early time universe.

We will also assume for the metric that $g_{00} = -1$ and $g_{0i} = 0$, nothing else. Recall that the only non vanishing Christoffel connections are $\Gamma^0_{ij}, \Gamma^i_{j0}$ and $\Gamma^i_{jk}$.

Then, taking into account the distribution function $f(\mathbf{x}, \mathbf{p}, t)$ describing the number of particles per units of spatial and momentum volume, we can define the number $dN$ of particles contained in a region of phase space whit co-moving coordinates between $x^i$ and $x^i + dx^i$, and co-moving momenta between $p_i$ and $p_i + dp_i$, as:

$$dN(\mathbf{x}, \mathbf{p}, t) \equiv \frac{d^3p d^3x}{(2\pi)^3} f(\mathbf{x}, \mathbf{p}, t) \tag{1}$$

So, we can define the number density of particles as:

$$n(\mathbf{x}, t) \equiv \frac{1}{\sqrt{-detg}} \int_{\mathbf{p}} \frac{d^3p}{(2\pi)^3} f(\mathbf{x}, \mathbf{p}, t) \tag{2}$$

On the other hand, recall that $T^{\mu\nu}$ is defined to be the average flux of $p^\mu$ across a surface of constant $x^\nu$, what is leads to:

$$T^{\mu\nu}(\mathbf{x}, t) \equiv \frac{1}{\sqrt{-detg}} \int_{\mathbf{p}} f(\mathbf{p}, t) \frac{p^\mu p^\nu}{E} \tag{3}$$

## 4.1. Collisionless Boltzmann equation

If particles are collisionless, but subject to an external influence that produces changes in their momenta, then they will remain conserved along the trajectory in phasespace, i.e., $dN(\mathbf{x}, \mathbf{p}, t) = dN(\mathbf{x} + \dot{\mathbf{x}}dt, \mathbf{p} + \dot{\mathbf{p}}dt, t + dt)$. Thanks to the Liouville the volume $dV dV_p$ does not change, and so we infer that $f(\mathbf{x}, \mathbf{p}, t) = f(\mathbf{x} + \dot{\mathbf{x}}dt, \mathbf{p} + \dot{\mathbf{p}}dt, t + dt)$, from where it follows that:

$$\frac{d}{dt} f \equiv \frac{\partial}{\partial t} f + \dot{x}^i \partial_i f + \dot{p}_i \frac{\partial}{\partial p_i} f = 0, \qquad f(\mathbf{x}, \mathbf{p}, t) \equiv (2\pi)^3 \sum_a \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \delta^{(3)}(\mathbf{p} - \mathbf{p}_a(t)) \tag{4}$$

where the last expression corresponds to the distribution in terms of individual trajectories. Recall the next useful relations:

$$\dot{x}^i_a \partial_i f = \frac{p^i_a}{E_a} \partial_i f, \qquad \dot{p}^a_k - \frac{1}{2E_a} p^i_a p^j_a \partial_k g_{ij} = 0, \qquad \dot{p}^a_i \frac{\partial}{\partial p^a_i} = \frac{1}{2E_a} p^i_a p^j_a \partial_k g_{ij} \frac{\partial}{\partial p^a_i} f \tag{5}$$

The first expression is just the known relation between the velocity of a particle in terms of the energy and momentum, the second one corresponds to combine our assumption about the metric together with the geodesic equation (3) from section 3 and, the third one corresponds to combine the second relation of eq. (4) together with the second expression in (5).

With this, we finally find the collisionless Boltzmann equation:

$$\frac{\partial}{\partial t} f + \frac{p^i}{E} \partial_i f + \frac{1}{2E} p^i p^j \partial_k g_{ij} \frac{\partial}{\partial p_k} f = 0 \tag{6}$$

Applying eq (6) to the simple situation whereby the particles of a given species $a$ are homogeneously distributed in a flat FLRW background $g_{ij} = a^2\delta_{ij}$ and, therefore, $\partial_i f = 0$ and $\partial_k g_{ij} = 0$. This means that the distribution must respect $\frac{\partial}{\partial t} f = 0$. Using this together with $\sqrt{-detg} = a^3$ in eq (2) we can find that:

$$\dot{n} + 3Hn = \frac{1}{a^3}\frac{\partial}{\partial t}(a^3 n) = 0 \tag{7}$$

We can see that $n$ dilutes like $n \propto a^{-3}$ as the universe expands, what implies that the total number per co-moving volume stays constant.

## 4.2. Collision term for Boltzmann's equation

If we now take into account the presence of collision between several species $a$, then, the Boltzmann equation takes the form:

$$\frac{\partial}{\partial t} f_a + \frac{p^i}{E}\partial_i f_a + \frac{1}{2E} p^i p^j \partial_k g_{ij} \frac{\partial}{\partial p_k} f_a = C_a[f] \tag{8}$$

The collision term $C_a[f(\mathbf{p})]$ tells us how collisions contribute to the time variation of the distribution $f_a(\mathbf{x}, \mathbf{p}, t)$ due to collisions.

Let us restrict our analysis to a small region of volume $V$ whit some given characteristic length much smaller than the Hubble radius $H^{-1}$. To start with, let us say that the collision term have the form:

$$C_a[f(\mathbf{p})] = C_{a+}[f(\mathbf{p})] + C_{a-}[f(\mathbf{p})] \tag{9}$$

where the "+" stands for the collisions increasing the density distribution and the "-" stands for the ones that decrease it.

Let us now consider collisions of the form $a(\mathbf{p}) + b(\mathbf{q}) \to c(\mathbf{p}') + d(\mathbf{q}')$ that respect energy and momentum conservation conditions. The probability of the collision to occur, given that the reaction is taking place in a bath of particles, the probability may be enhanced or suppressed depending on whether the product particles are being injected in a bath of fermions or bosons, is found to be:

$$d\dot{P} = \frac{(2\pi)^4 \delta^{(4)}(p + q - p' - q')}{16 E_a(p) E_b(q) E_c(p') E_d(q') V^3} \left|\mathcal{T}_{a+b\to c+d}\right|^2 [1 \pm f_c(\mathbf{p}', t)][1 \pm f_d(\mathbf{q}', t)] \tag{10}$$

where we have used $d\dot{P} \equiv \frac{dP}{T} = \frac{V}{\langle i|i\rangle\langle f|f\rangle}(2\pi)^4\delta^{(4)}(p+q-p'-q')\left|\mathcal{T}_{a+b\to c+d}\right|^2$, where the numerator corresponds to the term $\langle f|i\rangle/T$, and that $\langle i|i\rangle = 4E_a(p)E_b(q)V^2$ and $\langle f|f\rangle = 4E_c(p')E_d(q')V^2$. Recall that our volume V corresponds to a box with periodic BC (boundary conditions), so their momenta is quantized as $\mathbf{q} = (2\pi/V^{1/3})\mathbf{n}$, with the components of $\mathbf{n}$ being integers. Then, integrals over momenta can be understood as sums over all possible integer values in $\mathbf{n} = (n_x, n_y, n_z)$. After some algebraic treatment we find that:

$$\begin{aligned} C[f(\mathbf{p})] =& \frac{1}{2E_a(p)} \sum_b \sum_c \sum_d \int_{\bar{\mathbf{q}}} \frac{1}{2E_b(q)} \int_{\bar{\mathbf{p}}'} \frac{1}{2E_c(p')} \int_{\bar{\mathbf{q}}'} \frac{1}{2E_d(q')} (2\pi)^4 \delta^{(4)}(p + q - p' - q') \\ & \times \left|\mathcal{T}_{a+b\to c+d}\right|^2 \{f_c(\mathbf{p}', t) f_d(\mathbf{q}', t)[1 \pm f_a(\mathbf{p}, t)][1 \pm f_b(\mathbf{q}, t)] \\ & - f_a(\mathbf{p}, t) f_b(\mathbf{q}, t)[1 \pm f_c(\mathbf{p}', t)][1 \pm f_d(\mathbf{q}', t)]\} \end{aligned} \tag{11}$$

where the positive term in the last general expression corresponds to $C_+[f(\mathbf{p})]$, whereas the negative one corresponds to $C_-[f(\mathbf{p})]$, and inside the squared brackets the "+" stands for bosons and the "-" stands for fermions.

## 4.3. Continuity equation

Now we are able to say that the energy-momentum tensors of each individual species does not necessarily respect the total energy-momentum tensor continuity equation $\nabla_\mu T^\mu{}_\nu = 0$. Provides we can find $T^\mu{}_\nu$ via eq. (3), it is possible to show, after some algebra, that each individual species obey the relation:

$$\nabla_\mu T_a{}^\mu{}_\nu = C^a_\nu \equiv \int_{\bar{\mathbf{p}}} C_a[f] p_\nu \tag{12}$$

Then, given that the total energy momentum tensor respects the continuity equation, it follows that $\sum_a C^a_\nu = 0$, which is due to conservation rules in microscopic reactions among species.

## 4.4. Interaction rates and average cross sections

Notice that if we multiply $d\dot{P}$ in eq. (10), without considering the correction terms in squared brackets, by $f_a d^3p/(2\pi)^3 n_a$ and $f_b d^3p/(2\pi)^3 n_b$, the number of particles with momenta between $p_i$ and $p_i + dp_i$ divided by the total number of particles for species $a$ and $b$, respectively, we then can obtain the interaction rate per $a$-particle:

$$\begin{aligned}\Gamma_{int,a} =& \frac{1}{n_a}\int_{\bar{\mathbf{p}}}\frac{1}{2E_a}\int_{\bar{\mathbf{q}}}\frac{1}{2E_b}\int_{\bar{\mathbf{p}}'}\frac{1}{2E_c}\int_{\bar{\mathbf{q}}'}\frac{1}{2E_d}(2\pi)^4\delta^{(4)}(p+q-p'-q') \\ & f_a(\mathbf{p})f_b(\mathbf{q})[1\pm f_c(\mathbf{p}',t)][1\pm f_d(\mathbf{q}',t)]\,|\mathcal{T}_{a+b\to c+d}|^2\end{aligned} \tag{13}$$

Also notice that $\Gamma_{int,a}$ is independent of the number density $n_a$ (a change of factor $n_a^{-1}$ is compensated by a change of $f_a(\mathbf{p})$) and that we have $\Gamma_{int,a} n_a = \Gamma_{int,b} n_b$.

We next define the thermally averaged cross section as:

$$\langle\sigma v\rangle_{ab\to cd} \equiv \frac{\Gamma_{int,a}}{n_b} = \frac{\Gamma_{int,b}}{n_a}, \qquad \langle\sigma v\rangle_{cd\to ab} \equiv \frac{\Gamma_{int,c}}{n_d} = \frac{\Gamma_{int,d}}{n_c} \tag{14}$$

Using the definition of the number density of particles, eq. (2), and the collision term, result (11), we can turn the Bolzmann eq. (8) into:

$$\dot{n}_a + \Gamma^i_{i0} n_a + \nabla_i u^i_a = \sum_{b,c,d}\left[\langle\sigma v\rangle_{cd\to ab}\, n_c n_d - \langle\sigma v\rangle_{ab\to cd}\, n_a n_b\right] \tag{15}$$

where we have used the velocity field $u^i_a$ defined as:

$$u^i(\mathbf{x},t) \equiv \frac{1}{\sqrt{-detg}}\int_{\mathbf{p}} f(\mathbf{x},\mathbf{p},t)\frac{p^i}{E} \tag{16}$$

The relation (15) will be useful to estimate the number of particles during various stages of the hot Big-Bang universe.

# 5. Thermal history of the universe

Our first task to discover the thermal history of the universe is to deduce the particle distributions in the approximation that thermal equilibrium is mantained as the universe expands.

## 5.1. Bose-Einstein and Fermi-Dirac statistics

In strict thermal equilibrium, $f_a = f_a(\mathbf{p})$, and then $C[f(\mathbf{p})] = 0$. This means that interactions appearing in the collision term are efficient enough to keep the bath thermalized. If we set the collision term given by eq. (11) of section 4 to zero, we find that $\frac{f_c(\mathbf{p}',t)f_d(\mathbf{q}',t)}{[1\pm f_c(\mathbf{p}',t)][1\pm f_d(\mathbf{q}',t)]} = \frac{f_a(\mathbf{p},t)f_b(\mathbf{q},t)}{[1\pm f_a(\mathbf{p},t)][1\pm f_b(\mathbf{q},t)]}$ and then $\frac{f_a(\mathbf{p})}{1\pm f_a(\mathbf{p})} = e^{A_a}$ where $A_a$ is a quantity satisfying a conservation equation. Next, given that $f_a$ is homogeneous and isotropic ($\partial_i f_a(\mathbf{p}) = 0$), $A_a$ it must depend on $\mathbf{p}$ through $p^2$. We therefore write $A_a = -\beta(E_a - \mu_a)$ where we have used the thermodynamical $\beta$ of the bath.

Finally, we are able to find:

$$f_a(\mathbf{p}) = \frac{1}{e^{(E_a(\mathbf{p})-\mu_a/kT)} \pm 1} \tag{1}$$

where "+" stands for fermions and "-" for bosons.

## 5.2. Macroscopic observables

The distribution function $f(\mathbf{p},t)$ defined by eq. (1) can be used to derive several macroscopic quantities in a homogeneous universe with interacting particles. Recall we have expressions for $n(t)$, eq. (2) and for $T^{\mu\nu}$, eq. (3) from which we can lower the indices (both equations reffer to the ones in section 4). Given that $f(\mathbf{p},t)$ is invariant under rotations, the energy momentum tensor will have the form of that of a perfect fluid. Next, we can integrate using spherical coordinates in the Fourier space of momentum, so $\int d\Omega_p \hat{p}_i = 0$ and $\int d\Omega_p \hat{p}_i \hat{p}_j = \frac{4\pi}{3}\delta_{ij}$. These results imply that $T_{i0} = T_{0j} = 0$ and that $T_{ij} \propto \delta_{ij}$. Then, changing the integration variable from momentum to energy, we have the number density, energy density and pressure, to be, respectively:

$$n = \frac{1}{2\pi^2}\int_m^\infty dE \frac{E\sqrt{E^2-m^2}}{e^{(E-\mu)/kT} \pm 1}; \rho = \frac{1}{2\pi^2}\int_m^\infty dE \frac{E^2\sqrt{E^2-m^2}}{e^{(E-\mu)/kT} \pm 1}; p = \frac{1}{6\pi^2}\int_m^\infty dE \frac{(E^2-m^2)^{3/2}}{e^{(E-\mu)/kT} \pm 1} \tag{2}$$

For instance, we have:

Table 2: Number and energy density, and pressure in different limits.

| | Number density (n) | Energy density ($\rho$) | Pressure (p) |
|---|---|---|---|
| Fermions ($m << kT$) | $n_f = \frac{3}{4}\frac{\zeta(3)(kT)^3}{\pi^2}$ | $\rho_f = \frac{7}{8}\frac{\pi^2(kT)^4}{30}$ | $p_f = \frac{7}{8}\frac{\pi^2(kT)^4}{90}$ |
| Bosons ($m << kT$) | $n_b = \frac{\zeta(3)(kT)^3}{\pi^2}$ | $\rho_b = \frac{\pi^2(kT)^4}{30}$ | $p_b = \frac{\pi^2(kT)^4}{90}$ |
| $m >> kT$ | $n = e^{-(m-\mu)/kT}\left(\frac{mkT}{2\pi}\right)^{3/2}$ | $\rho = mn$ | $p = nkT$ |

Notice that the two first rows are for the ultra-relativistic limit, so we have to separate between fermions and bosons, whereas for the non-relativistic limit (third row) it does not matter if we are talking about fermions or bosons because the $\pm 1$ contribution is negligible with respect to the exponential.

Table 2 help us to compute that for a fluid of ultra-relativistic particles (for instance, radiation) is given by $w \equiv \frac{p}{\rho} = \frac{1}{3}$ and for a non-relativistic fluid we have $w \equiv \frac{p}{\rho} = \frac{kT}{m} << 1$. Then, if the number of particles per co-moving volume of some species is fixed, then its number density in the bath is not determined by the temperature. This implies that the chemical potential must be fixed goverving by the rule $n \propto a^{-3}$.

On the other hand, to study the evolution of distribution of species we will assume that collisions are very efficient for relativistic species, so that particles are created and destroyed without hindrance. This implies that $\mu << kT$, and so we can neglect the chemical potentials for relativistic species.

## 5.3. Entropy, evolution of temperature and effective number of relativistic species

Let us consider a region of unit co-moving volume, so its proper volume results to be $V = a^3$. An infinitesimal change of the volume and of the energy density define the entropy via the relation $TdS = d(\rho V) + pdV = Vd\rho + (\rho + p)dV$, where $\rho$ and $p$ are local quantities, i.e., they can explicitly depend on the temperature but not on the volume. Extracting $\partial_T S$ and $\partial_V S$ from the previous relation and by imposing the condition $\partial_T \partial_V S = \partial_V \partial_T S$ one can obtain $\frac{dp}{dT} = \frac{\rho+p}{T}$, and then the entropy contained in a volume $V$ is $S = V(\rho + p)/T$. So, the entropy density is $s \equiv \frac{\rho+p}{T}$. Finally, by virtue of the continuity eq. (17) of section 3, we can show that:

$$dS = d\left[\frac{V(\rho + p)}{T}\right] = 0 \tag{3}$$

That is, as long as energy and momentum are both conserved, the entropy contained in a volume $V$ is also conserved.

Let us suppose that our universe contained a hot bath with a single relativistic species, then $E = p/a$ and we are going to consider this value in eq. (2). So, according to the Boltzmann equation, which reads $\dot{f} = 0$, the shape of distribution remains invariant. In particular, this means that the temperature of the bath decreases as the universe expands and the bath dilutes:

$$T = T_* \frac{a_*}{a} \tag{4}$$

Our purpose is to find the temperature in function of time, so we can use eq. (3) to assert with the dependence of $T$ as a function of $a(t)$, given that $\rho$ and $p$ are additive for the complete universe. According to this last reasoning we can infer that during the radiation dominated era the energy density and pressure will necessarily be dominated by photons and possibly other light particles, and then:

$$d\left[\frac{a^3}{T}\left(\frac{4}{3}\rho_R + \rho_M\right)\right] = 0 \tag{5}$$

Given that $\rho_R(T) \propto T^4$, as long as $\frac{4}{3}\rho_R >> \rho_M$ then the temperature will follow the law (4).

As a summary, as the universe expands, it cools down. This means that a species of massive particles can transit from being relativistic to non-relativistic. Given that the temperature of the universe is determined by relativistic species, it is useful to keep track of the effective number of relativistic species defined as:

$$\rho_R \equiv g_* \frac{\pi^2 (kT)^4}{30} \quad \Rightarrow \quad g_* = g_b + \frac{7}{8} g_f \tag{6}$$

This result allows us to derive a relation between time and temperature valid after decoupling. Given that $a(t) \propto t^{1/2}$ during radiation domination (with small corrections of order $t/t_{eq}$), then $H = 1/2t$, and the Friedmann equation reads:

$$\left(\frac{1}{2t}\right)^2 = \frac{8\pi G}{3} g_* \frac{\pi^2 (kT)^4}{30} \quad \Rightarrow \quad t(T) = \left(\frac{45}{16\pi^3 g_* G}\right)^{1/2} \frac{1}{(kT)^2} \tag{7}$$

Then, it is possible that above some temperature the universe is filled with a given species of relativistic particles that below some temperature they decay into other particles of the bath. That's because the effective number of relativistic species is a function of temperature.

## 5.4. Principal phases and interactions of particles with the hot bath

The effective number of relativistic degrees of freedom helps us to determine the temperature after a species become non-relativistic, decay into other particles of the bath or annihilate. Let us now focus on those temperatures:

- **Decay of particles:** Taking for instance the case of a fermion particle $a$ that becomes non-relativistic, we have that the entropy stays constant through the transition. Then, eq. (3) implies $\frac{a_i^3}{T_i}(\rho_i + p_i) = \frac{a_f^3}{T_f}(\rho_f + p_f) \Rightarrow \frac{4}{3}\frac{a_i^3}{T_i}\rho_{R,i} = \frac{4}{3}\frac{a_f^3}{T_f}\rho_{R,f}$, where $i$ and $f$ stands for quantities before and after the transition, respectively, and the last relation is due to the dominant species are relativistic before and after the decay. In terms of the effective relativistic degrees of freedom, we have:

$$T_f = \left(\frac{g_*^i}{g_*^f}\right)^{1/3} \frac{a_i}{a_f} T_i \tag{8}$$

  So, after the transition, the temperature of the bath is slightly larger than it would have been if the species did not become non-relativistic. Given that the overall entropy per co-moving volume is constant, this can be understood as the transfer of entropy from the new non-relativistic species to the rest of the bath.

- **Decoupling from the bath:** It is possible that a given species of particles stop interacting with the rest of the bath, what we call decoupling. Consider a species of particle $a$ interacting with a bath through a process of type $a+b \to a+b$ (where $b$ stands for some other species in the bath), so, the total number of interactions can be computed as $N_{int,a}(t) = \int_t^\infty \Gamma_{int,a} dt$. Suppose that the interaction mentioned before is characterized by a scattering matrix that scales with power of momentum of order $n$. Since $f_a$ and $f_b$ are relativistic, we have:

$$\Gamma_{int,a} = \frac{1}{n_a} \int_{\bar{\mathbf{p}}} \frac{a}{2p} \int_{\bar{\mathbf{q}}} \frac{a}{2q} \int_{\bar{\mathbf{p}}'} \frac{a}{2p'} \int_{\bar{\mathbf{q}}'} \frac{a}{2q'} (2\pi)^4 \delta^4(p+q-p'-q') e^{-\frac{(p+q)}{akT}} \left|\mathcal{T}_{a+b\to a+b}\right|^2 \tag{9}$$

  Then, using the fact that $n_a \propto T^3$ we conclude that $\Gamma_{int,a} \propto T^{n+1}$.

For example, in the case of weak interactions one finds that the averaged squared scattering matrix is given by $|\overline{\mathcal{T}_{a+b\to a+b}}|^2 \propto G_F^2(p\cdot p')(q\cdot q')$, where $G_F$ corresponds to the Fermi constant; with this $\Gamma_{int,a} \simeq G_F^2T^5$. Another example corresponds to Thomson scattering, in which at low energy photons interact with non-relativistic electrons, for what we have $|\overline{\mathcal{T}_{e\gamma\to e\gamma}}|^2 \simeq 16\pi m_e^2\sigma_T\left[1+\frac{1}{2}P_2(\hat{p}\cdot\hat{p}')\right]$, where $\sigma_T = 0.0017eV^{-2}$ is the total cross section and $P_2(x) = \frac{1}{2}(3x^2-1)$ is the Legendre polynomial of order 2. It follows that the interaction rate per photon is $\Gamma_{int,\gamma} = n_e\sigma_T$, so $\Gamma_{int,\gamma} \propto T^3$.

Finally, we can find:

$$N_{int,a}(t) = \frac{\Gamma_{int,a}(t)}{(n-1)H(t)} \tag{10}$$

Therefore, for $n > 1$, after $\Gamma \sim H$ in average the particle $a$ interacts less than once. Thus we see that a condition to estimate when a species of particle decouples from the thermal bath is provided by $\Gamma \simeq H$.

- **Freezing after decoupling:** If after decoupling the constituents of the species are unable to experience collisions among themselves, their distribution function will necessarily satisfy the collisionless Boltzmann equation, which in a homogeneous universe corresponds to $\partial_t f = 0$, what does it means that the shape of the distribution freezes. Consider, for instance, a massless fermion $c$ that decouples from the bath, then, its distribution will be of the form:

$$f_c(p) = \frac{1}{exp(p/akT)+1} \tag{11}$$

Because of $\partial_t f = 0$ it follows $T_c(t) \propto a^{-1}$, so, after decoupling, the temperature of the thermal bath $T$ may evolve in a way that differs from $T(t) \propto a^{-1}$. This means that the temperature of the decoupled fermion can, in general, differ from the temperature of the thermal bath after the process.

## 5.5. A basic thermal history of our universe

The basic building blocks of our present universe are photons ($\gamma$), neutrinos ($\nu$), electrons ($e^-$), protons ($p$) and neutrons ($n$). However, at temperatures much larger than $m_e = 0.5$MeV (in units where $k = 1$) there was enough energy to maintain pair creation/annihilation $e^+ + e^- \rightleftharpoons \gamma+\gamma$ so positrons ($e^+$) must have been abundant too. As soon as the temperature drops below $m_e = 0.5$MeV pair creation mentioned before becomes kinematically suppressed, and electrons and positrons start to annihilate.

(I) **Neutrino decoupling (kT∼1MeV):** Start considering a temperature slightly below $kT \sim 10^3$MeV; at such temperature $\{\gamma,\nu,e^\pm\}$ are relativistic, but $\{p,n\}$ are not. Thus, the starting effective number of relativistic degrees of freedom is $g_* = g_\gamma + \frac{7}{8}(g_\nu + g_{e^+} + g_{e^-})$, where $g_\gamma = 2$, $g_\nu = 6$ because of the three type of neutrinos, and $g_{e^\pm} = 2$, so we have $g_* = 10.75$. At this time, the main reactions keeping neutrinos coupled to the thermal bath are pair creation/annihilation between the three types of neutrinos/anti-neutrinos with electrons/positrons. As the universe cools down, the interaction rate per neutrino $\Gamma_{int,\nu} \simeq G_F^2T^5$ drops below $H$. Then, by virtue of the Friedmann equation $3H^2 = 8\pi G\rho_R$, it follows that:

$$\frac{\Gamma_{int,\nu}}{H} \simeq \left(\frac{kT}{1MeV}\right)^4 \tag{12}$$

Thus, neutrino decouples at about 1MeV. From that point on, the distribution of neutrinos freezes, and evolves as if it were a thermal bath with its own temperature $T_\nu \propto a^{-1}$. The rest of the particles $\{\gamma, e^\pm, p, n\}$ remain in thermal contact with a temperature $T \propto a^{-1}$.

(II) **Pair annihilation (kT∼0.5MeV):** At this point, the relativistic degrees of freedom continue to be $g_* = 10.75$, however, from here on, the relativistic degrees of freedom consist of the two sectors: $g_{\nu *} = \frac{7}{8} \times 6 = 5.25$ and $g_{e^\pm,\gamma *} = 2 + \frac{7}{8} \times 4 = 5.5$. Both sectors are unable to share entropy. So, the energy density of the universe can be written as $\rho_R = \frac{\pi^2 (kT)^4}{30} \left( g_{e^\pm,\gamma *} + \frac{7}{8} g_{\nu *} \right)$ with $T_\nu = T$. As the universe continues to cool down, electrons and positrons start to annihilate. This happens roughly at about kT∼0.5MeV. This changes the number of degrees of freedom $g_{e^\pm,\gamma *}$ from $g^i_{e^\pm,\gamma *} = 5.5$ to $g^f_{e^\pm,\gamma *} = g_{\gamma *} = 2$. By virtue of eq. (8) for this values, we find:

$$T_f = \left( \frac{11}{4} \right)^{1/3} \frac{a_i}{a_f} T_i \quad \Rightarrow \quad T_\nu = \left( \frac{4}{11} \right)^{1/3} T \tag{13}$$

Recalling $\rho_R$ mentioned in the last paragraph, and combining it with the previous equation we can compute the effective number of relativistic degrees of freedom $g_* = 3.363$.

(III) **Abundance of neutrons (kT≲1MeV):** At temperatures $kT >> 1$MeV the main reactions involving neutrons are: $n + \nu \rightleftharpoons p + e^-$; $n + e^+ \rightleftharpoons p + \bar{\nu}$; $n \rightleftharpoons p + \bar{\nu} + e^-$. At these temperatures both neutrons and protons are non-relativistics, so their number densities are given by the values of Table 2, each one with its indices. If we neglect the chemical potential of relativistic leptons $(e^\pm, \nu)$, then $\mu_n = \mu_p$, and if we define $Q \equiv m_n - m_p = 1.293$MeV, then we have $\frac{n_n}{n_p} = \left( \frac{m_n}{m_p} \right)^{3/2} e^{-Q/kT} \simeq e^{-Q/kT}$. Therefore, the fraction $X_n = n_n/n_N$, where $n_N = n_p + n_n$ is the number density of the nucleons, is given by $X_n = \frac{1}{exp(Q/kT)+1}$. Notice that $X_n$ becomes small for temperatures below $kT \sim 1$MeV. This is because processes in which neutrons are created become kinematically suppressed. It can be shown that $\Gamma_{int,n} \sim H$ at $T$'s about 1MeV, when $X_n \sim 0.3$. From there on, neutrons drop out of equilibrium and start to decay via beta decay $n \rightleftharpoons p + \bar{\nu} + e^-$. Next, using the Boltzmann equation describing the evolution of number densities we find that $a^{-3} \partial_t (a^3 n_n) = \Gamma_{p \to n} n_p - \Gamma_{n \to p} n_n$. If we want to include the beta decay we just redefine the interaction rate as $\Gamma_{n \to p} = \Gamma_n^{n\nu \to pe^-} + \Gamma_n^{ne^+ \to p\bar{\nu}} + \Gamma_n^{n \to p\bar{\nu} \pm e^-}$, where the last term was not been considered before. On the other hand, we can obtain $\dot{X}_n = \Gamma_{p \to n} X_p - \Gamma_{n \to p} X_n$, and it is possible to explicitly find that $\Gamma_{p \to n} = \Gamma_{n \to p} e^{-Q/kT}$, and using that $X_p = 1 - X_n$ we have:

$$\dot{X}_n = \Gamma_{n \to p} \left[ (1 - X_n e^{-Q/kT} - X_n) \right] \tag{14}$$

So, for large $Q/kT$ (about $kT < 0.2$MeV) the interaction rate under study must be given by $\Gamma_{n \to p} = 1/\tau_n$, where $\tau_n = 885.7s$ corresponds to neutron's lifetime in vacuum (between 14 and 15 minutes). This is because positrons annihilated and neutrinos decoupled leaving $n \to p$ to be dominated by pure beta decay. A fairly good result is offered by:

$$X_n(t) \simeq 0.15 e^{-t/885.7s} \tag{15}$$

This gives neutrons enough time to bind with protons to form light atoms.

(IV) **Big-Bang nucleosynthesis (kT∼0.07MeV):** Before neutrons are completely depleted, they start to bind with protons allowing the synthesis of atoms. The first reaction involves the creation of deuterium D (one proton and one neutron) $n + p \to D + \gamma$. Deuterium then gives way to the creation of Tritium $H^3$ (one proton and two neutrons) and Helium-3 $He^3$ (two protons and one neutron) $D + D \to H^3 + p, D + D \to He^3 + n$. Then Helium-4 $He^4$ (two protons and two neutrons) is formed through $D + H^3 \to He^4 + n$ and $D + He^3 \to He^4 + p$. After these reactions occur, most of neutrons will end up in the form of $He^4$ with small fractions in the form of D, $H^3$ and $He^3$. Given that deuterium is necessary to reach the generation of $He^4$, let us limit ourselves to estimate the amount of deuterium produced in the first stage. The fraction of nucleons in the form of $He^4$ is given by $Y_p \equiv 4n_{He^4}/n_N$. To proceed, notice that if the interaction $n + p \rightleftharpoons D + \gamma$ is efficient enough (we have also notice that the chemical potential always must respect the relation $\mu_n + \mu_p = \mu_D$, because we are assuming that $\mu_\gamma = 0$), then we can consider that the generation of deuterium takes place under thermal equilibrium. For the quantities in Table 2 we have to consider spin degrees of freedom for each species given by $g_n = 2, g_p = 2$ and $g_D = 3$. Then it immediately follows that the number of densities are related as $\frac{n_D}{n_n n_p} = e^{(m_p+m_n-m_D)/kT}\left(\frac{2\pi m_D}{m_p m_n kT}\right)^{3/2}$. These can be rewritten in terms of the fractions $X_D \equiv \frac{n_D}{n_N}, X_n \equiv \frac{n_n}{n_N}$ and $X_p \equiv \frac{n_p}{n_N}$ as:

$$X_D \simeq X_n X_p \eta_N \frac{\zeta(3)}{\pi^2} e^{B_D/kT}\left(\frac{4\pi kT}{m_p}\right)^{3/2} \tag{16}$$

where $B_D \equiv m_p + m_n - m_D = 2.22$MeV is the binding energy of deuterium. In the previous expression $\eta_N \equiv \frac{n_N}{n_\gamma}$ is the number of nucleons per photon, which remains constant throughout the expanding universe.

To continue, neutrons become trapped in deuterium roughly when $X_D \sim X_n$. This happens once the temperature acquires a value $T_{nucl}$ such that $\frac{B_D}{kT_{nucl}} + \frac{3}{2}ln\left(\frac{kT_{nucl}}{m_p}\right) + ln(\eta_b) = 0$. It follows that $T_{nucl} = 0.07$MeV, so $X_n(T_{nucl}) = 0.12$. Probably one of the most relevant conclusions, considering all the things mentioned before, is that $Y_p \simeq 0.24$.

(V) **Freezing (kT∼20eV):** As we saw, after $kT \sim 0.07$MeV the universe consists of photons, electrons, protons and $He^4$. The main reaction taking place allowing the bath to stay hot is Thomson scattering $e^- + \gamma \to e^- + \gamma$. Recall that $\Gamma_\gamma = n_e \sigma_T$ for this process. This leads to $\Gamma_\gamma \simeq 1.38 \times 10^{-12}\left(\frac{\Omega_B h^2}{0.022}\right)\frac{(kT)^3}{(eV)^4}$. If we compare this with $\rho_R$ and Friedmann's equation we obtain $\frac{\Gamma_\gamma}{H} \simeq 8.8 \times 10^2\left(\frac{\Omega_B h^2}{0.022}\right)\frac{kT}{eV}$. However, to correctly assess whether photons are in thermal equilibrium with electrons, we must check whether the energy transfer between both species is enough to allow collisions change the energy of an individual photon in a sizable way (keeping the bath thermalized through collisions). The rate of energy transfer compared with H is:

$$\frac{kT}{m_e}\frac{\Gamma_\gamma}{H} = 1.72 \times 10^{-3}\left(\frac{\Omega_B h^2}{0.022}\right)\left(\frac{kT}{eV}\right)^2 \tag{17}$$

We see that about $kT \sim 20$eV photons stop being in thermal contact with electrons. From then on, photons continue colliding with electrons, but the distribution freezes.

(VI) **Matter-Radiation equality (kT∼0.86eV):** At temperatures lower than 0.1MeV the energy density of photons is given by $\rho_\gamma = g_{\gamma *}\frac{\pi^2 (kT)^4}{30}$. The CMB radiation consists of a thermal distribution of photons with the form $f_\gamma = (e^{(p/kT_{CMB})} - 1)^{-1}$, where $T_{CMB} = 2.725$K. This is evidence that the photon bath decoupled from electrons and kept its thermal distribution ever since. Recall that for matter-radiation equality era $z_{eq} \simeq 3676$; this implies that $T_{eq} =$

$1.002 \times 10^4$K, i.e., $kT_{eq} = 0.86$eV. This, together with $\rho_R = \frac{\pi^2 (kT)^4}{30} \left( g_{e^\pm,\gamma *} + \frac{7}{8} g_{\nu *} \right)$, allow us to obtain the energy density of radiation (or matter) during equality.

(VII) **Recombination and last scattering (kT∼0.26eV):** As we saw, below $kT \sim 20$eV photons stop interchanging sizable energy with electrons, but collisions continue to be frequent. However, at some point, electrons and protons will start binding into neutral hydrogen and helium, process known as recombination. Once this happens, photons will have a last chance to scatter with electrons in hydrogen atoms, and after that, they will start to stream freely. Recombination and last scattering do not happen exactly at the same time, but to a first approximation we can only study the sharp appearance of neutral hydrogen via the reaction $p + e^- \rightleftharpoons H + \gamma$. The emerging hydrogen can be created in its many excited states, but the reaction will be dominated by hydrogen in its ground state $1s$. The distributions of the non-relativistic species are shown in Table 2, with chemical potentials satisfying $\mu_p + \mu_e = \mu_H$. The ground state $1s$ of the hydrogen atom has two hyperfine states of spin 0 (with one degree of freedom) and 1 (with three degrees of freedom), therefore $g_H = 4$. All of this implies that $\frac{n_H}{n_p n_e} \simeq e^{B_H/kT} \left( \frac{2\pi}{m_e kT} \right)^{3/2}$, where $B_H = m_p + m_e - m_H = 13.6$eV is the binding energy of the ground state of hydrogen. In addition, let us recall that, given that the universe is neutrally charged, the number of electrons and protons per volume must be the same ($n_e = n_p$).

Let us define the fraction of ionized hydrogen (protons) as $X \equiv \frac{n_p}{n_p + n_H}$. Then it follows that $n_H/(n_p + n_H) = 1 - X$. Since the fraction of nucleons in the form of $He^4$ is 0.24, we have $n_p + n_H = 0.76 n_N$. All of this considerations leads to Saha's equation $X(1 + SX) = 1$, where $S \equiv 0.76 n_N e^{B_H/kT} \left( \frac{2\pi}{m_e kT} \right)^{3/2}$. The solution to this equation is $(\sqrt{1 + 4S} - 1)/2S$. Thus, recombination happens at about $kT \sim 0.26$eV, which corresponds to a redshift $z_{rec} = 1100$. This implies that the time of last scattering is $t_L = 474031$ years. To finish, let us mention that a more accurate account of recombination would require us to deal with $X$ away from the equilibrium by deriving a differential equation for $X$ out from the Boltzmann equation. Analogous to the derivation of eq. (14), the resulting equation is found to be:

$$\dot{X} = \langle \sigma v \rangle_{p+e \to H} \left[ (1 - X) \left( \frac{m_e kT}{2\pi} \right)^{3/2} e^{-B_H/kT} - 0.76 n_N X^2 \right] \tag{18}$$

where $\langle \sigma v \rangle_{p+e \to H}$ is the recombination rate:

$$\langle \sigma v \rangle_{p+e \to H} = 9.78 \frac{\alpha_e^2}{m_e^2} \left( \frac{B_H}{kT} \right)^{1/2} ln \left( \frac{B_H}{kT} \right) \tag{19}$$

where $\alpha_e = 1/137.036$ stands for the fine structure constant. Solving eq. (18) provides an accurate profile of $X$ as a function of time or temperature.

# 6. Cosmological perturbations and dynamics of the scalar sector of linear perturbations

We now define the cosmological perturbations describing departures from a homogeneous and isotropic background and derive their equations of motion. Observations allow us to focus the present discussion to the case $K = 0$, i.e., spatially flat geometries.

## 6.1. Cosmological perturbations

- **Metric perturbations:** Let us start by defining the metric perturbation as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and and taking $\bar{g}_{\mu\nu}$ to be composed by $\bar{g}_{00} = -1$, $\bar{g}_{i0} = 0$ and $\bar{g}_{ij} = a^2(t)\delta_{ij}$, then, to linear order we have $h^{\mu\nu} = -\bar{g}^{\mu\rho}\bar{g}_{\nu\sigma}h_{\rho\sigma}$. This leads us to $h^{00} = -h_{00}$, $h^{i0} = \frac{h_{i0}}{a^2(t)}$ and $h^{ij} = -\frac{h_{ij}}{a^4(t)}$. The Christoffel symbols then split between background and perturbed contributions as $\Gamma^{\alpha}_{\mu\nu} = \bar{\Gamma^{\alpha}_{\mu\nu}} + \delta\Gamma^{\alpha}_{\mu\nu}$ where the zeroth order are given by the non vashining terms mentioned in section 4, with the exception that we are taking $\bar{\Gamma^{i}_{jk}} = 0$. So, the non-vanishing perturbations of the Christoffel connections are $\delta\Gamma^0_{00}$, $\delta\Gamma^i_{00}$, $\delta\Gamma^0_{i0}$, $\delta\Gamma^0_{ij}$, $\delta\Gamma^i_{j0}$ and $\delta\Gamma^k_{ij}$. To deal with $h_{\mu\nu}$ we introduce four scalar perturbations $A, B, C$ and $E$, two vector perturbations $F_i$ and $G_i$ and one tensor perturbation $\gamma_{ij}$, such that:

$$h_{00} = -2A, \quad h_{i0} = a(t)(\partial_i C + F_i), \quad h_{ij} = a^2(t)[2B\delta_{ij} + D_{ij}E + \partial_i G_j + \partial_j G_i + \gamma_{ij}] \tag{1}$$

where we have used the differential operator $D_{ij} = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)$ and the vector and tensor perturbations are imposed to satisfy the following conditions:

$$\delta^{ij}\partial_i F_j = 0, \quad \delta^{ij}\partial_j G_i = 0, \quad \delta^{ij}\partial_i\gamma_{ij} = 0, \quad \delta^{ij}\gamma_{ij} = 0 \tag{2}$$

This set of relations ensure that $G_i$, $F_i$ do not contain scalars and that $\gamma_{ij}$ do not contain neither vectors nor scalars. Besides, thanks to those conditions te perturbations $A, B, etc.$ can be inverted and expressed in terms of $h_{i0}$ and $h_{ij}$. For instance, thanks to the first relation, we can write $\nabla^2 C = a^{-1}\delta^{ij} - \partial_i h_{j0}$ which, provided BC's, has a unique solution. Then, $F_i = a^{-1}h_{i0} - \partial_i C$.

- **Energy-momentum tensor perturbations:** The energy-momentum tensor can be split as $T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$, where $\bar{T}_{\mu\nu} = \bar{p}\bar{g}_{\mu\nu} + (\bar{p} + \bar{\rho})\bar{u}_\mu\bar{u}_\nu$ and $\bar{u}_\mu$ is the usual 4-velocity for a flat space metric. To write $\delta T_{\mu\nu}$, we have to consider the full energy-momentum tensor of a perfect fluid. With this in mind, we can split the 4-velocity as $u^\mu = \bar{u}^\mu + \delta u^\mu$, and the the condition $u_\mu u^\mu = -1$ implies the restriction with spatial part, respectively:

$$\delta u^0 = \frac{1}{2}h_{00}, \quad \delta u_i \equiv \partial_i\delta u + \delta u_i^V, \quad \partial_i\delta u_i^V = 0 \tag{3}$$

Given that $\delta u^i = a^{-2}(t)(\delta u_i - h_{i0})$, the previous decomposition implies:

$$\delta u^i_S = a^{-2}(t)\partial_i(\delta u - F), \qquad \delta u^i_V = a^{-2}(t)(\delta u_i^V - G_i) \tag{4}$$

But, given eq. (2) one has $\delta^{ij}\partial_i\delta u_j^V = 0$. Notice that by their definition $\rho$ and $p$ are scalars, so we can directly write $\rho = \bar{\rho} + \delta\rho$ and $p = \bar{p} + \delta p$. Then:

$$\delta T_{\mu\nu} = \delta p \bar{g}_{\mu\nu} + \bar{p} h_{\mu\nu} + (\delta p + \delta\rho)\bar{u}_\mu \bar{u}_\nu + (\bar{p} + \bar{\rho})(\delta u_\mu \bar{u}_\nu + \bar{u}_\mu \delta u_\mu) \tag{5}$$

which in terms of the previous decomposition, imply:

$$\delta T_{00} = \delta\rho - \bar{\rho} h_{00}, \quad \delta T_{0i} = \bar{p} h_{0i} - (\bar{p} + \bar{\rho})(\partial_i \delta u + \delta u_i^V), \quad \delta T_{ij} = a^2(t)\delta_{ij}\delta p + \bar{p} h_{ij} \tag{6}$$

Notice that $\delta T_{ij}$ contains only one scalar degree of freedom $\delta p$, corresponding to its trace. A more general version requires five more degrees of freedom, including an additional scalar and intrinsic vector and tensor components. These can be introduced through the anisotropic stress tensor $\pi_{ij}$ and therefore write $\delta T_{ij} = a^2(t)\delta_{ij}\delta p + \bar{p} h_{ij} + a^2(t)\pi_{ij}$. So, we can decompose $\pi_{ij}$ as follows:

$$\pi_{ij} = D_{ij}\pi^S + \partial_i \pi_j^V + \partial_j \pi_i^V + \pi_{ij}^T, \quad \delta^{ij}\partial_i \pi_j^V = 0, \quad \delta^{ij}\partial_i \pi_{jk}^T = 0, \quad \delta^{ij}\pi_{ij}^T = 0 \tag{7}$$

where the lasts relations are the conditions that the anisotropic tensor must respect.

## 6.2. Gauge transformations

A coordinate transformation changes the forms of the metric and then, of other tensors. These changes can be understood as a transformation of the perturbed terms:

$$\begin{aligned} h'_{\mu\nu} &= h_{\mu\nu} - \bar{\nabla}_\mu \xi_\nu - \bar{\nabla}_\nu \xi_\mu \\ \delta T'_{\mu\nu} &= \delta T_{\mu\nu} - \xi_\lambda \bar{\nabla}^\lambda \bar{T}_{\mu\nu} - \bar{T}_\mu^{\ \lambda} \bar{\nabla}_\nu \xi_\lambda - \bar{T}_\nu^{\ \lambda} \bar{\nabla}_\mu \xi_\lambda \end{aligned} \tag{8}$$

where $\bar{\nabla}$ corresponds to a covariant derivative using background metric Christoffel symbols. On the other hand, we may split the gauge transformation parameter $\xi_i$ into scalar and pure vector parts as $\xi_i \equiv \partial_i \xi^S + \xi_i^V$ with condition $\delta^{ij}\partial_i \xi_j^V = 0$. Then, each component of the perturbed metric transforms as:

$$\begin{aligned} A' = A + \dot{\xi}_0, \quad C' = C - \frac{1}{a}\left(\dot{\xi}^S + \xi_0 - 2H\xi^S\right), \quad F'_i = F_i - \frac{1}{a}\left(\dot{\xi}_i^V - 2H\xi_i^V\right) \\ E' = E - \frac{2}{a^2}\xi^S, \quad B' = B + H\xi_0 - \frac{1}{3a^2}\nabla^2\xi^S, \quad G'_j = G_j - \frac{1}{a^2}\xi_j^V \end{aligned} \tag{9}$$

On the other hand, each component of the perturbed energy-momentum tensor is:

$$\delta\rho' = \delta\rho + \xi_0 \dot{\bar{\rho}}; \quad \delta u' = \delta u - \xi_0; \quad \delta p' = \delta p + \xi_0 \dot{\bar{p}}; \quad \delta u_i'^V = \delta u_i^V; \quad \pi'_{ij} = \pi_{ij} \tag{10}$$

Notice that $\delta u_i^V$ and $\pi_{ij}$ are gauge invariant.

## 6.3. Gauge invariant perturbations

Just from inspecting how eqs. (9) and (10) transform, one finds particular combination of perturbations that remain invariant. For example, by combining only metric perturbations one finds the following three gauge invariant perturbations:

$$\Phi_{GI} = A + \frac{d}{dt}\left(aC - \frac{1}{2}a^2\dot{E}\right), \quad \Psi_{GI} = -B + \frac{1}{6}\nabla^2 E - aHC + \frac{a^2}{2}H\dot{E}, \quad \mathcal{F}_i = F_i - a\dot{G}_i \tag{11}$$

On the other hand, there are other useful combinations that emerge from combining metric and energy-momentum tensor perturbations. For instance, the gauge invariant co-moving curvature perturbation is defined as $\mathcal{R} = B - \frac{1}{6}\nabla^2 E - H\delta u$, whereas the gauge invariant curvature perturbations at uniform densities is defined as $\zeta = B - \frac{1}{6}\nabla^2 E - \frac{H}{\dot{\rho}}\delta\rho$. The curvature perturbation is particularly useful to study the evolution of perturbations during inflation. Also, there are gauge invariant quantities related to the energy-momentum tensor perturbations, which may be defined for an arbitrary species, such as:

$$\delta\rho_{GI} \equiv \delta\rho + \dot{\bar{\rho}}\left(aC - \frac{a^2}{2}\dot{E}\right), \quad \delta u_{GI} = \delta u - aC + \frac{a^2}{2}\dot{E}, \quad \delta p_{GI} \equiv \delta p + \dot{\bar{p}}\left(aC - \frac{a^2}{2}\dot{E}\right) \tag{12}$$

The principal gauges we are considering are:

## 6.4. Main gauges in cosmology

Now we are going to proceed fixing $\xi_0$, $\xi^S$ and $\xi_i^V$ in such a way that some of the terms in equation (9) vanish. Different combinations give us particular gauges.

- **Synchronous gauge (A'=C'=F'$_i$ =0):** This choice leaves the perturbations $B'$, $E'$, $G_i'$ and $\gamma_{ij}'$ as non-vanishing quantities to be determined through equations of motion together with initial conditions. We may label them simply as $B$, $E$, $G_i$ and $\gamma_{ij}$. It is also customary to define:

  $$\psi \equiv 3\dot{B} \tag{13}$$

  And notice that there is a residual gauge of the form:

  $$\xi_0 = -\tau(\mathbf{x}), \quad \xi^S = \tau(\mathbf{x})a^2(t)\int^t \frac{dt}{a^2(t)} \tag{14}$$

  Notice that this transformation does not affect the condition $A = C = 0$, however it does affect $B'$ and $E'$ (we can fix this residual gauge by setting $\delta u = 0$ for dark matter).

  The synchronous gauge is particularly useful to study the evolution of perturbations of the photon-baryon plasma during the early universe.

- **Newtonian gauge (E'=C'=F'$_i$ =0):** The remaining scalar perturbations $A'$ and $B'$ are usually labeled as:

  $$A' = \Phi, \qquad B' = -\Psi \tag{15}$$

  The Newtonian gauge is particularly useful to study the evolution of perturbations after decoupling, when the large scale structure starts to form.

- **Co-moving gauge ($\delta u$'=E'=G'$_i$=0):** Non-vanishing perturbations are usually labeled as:

$$B = \mathcal{R}, \qquad A = \delta\mathcal{N}, \qquad C = \frac{1}{a}\left(\Xi - \frac{1}{H}\mathcal{R}\right) \tag{16}$$

  The co-moving gauge is particularly useful to study the evolution of perturbations during inflation.

## 6.5. Newtonian/Synchronous gauges relation

Suppose that we are working with synchronous gauge, but we wish to move on to Newtonian gauge. Then, our initial perturbations satisfy $A = C = F_i = 0$, whereas the remaining perturbations $E$, $B$ and $G_i$ are non-vanishing. The new perturbations in Newtonian gauge must satisfy $E' = C' = F'_i = 0$. Then, suing eq. (9), it follows that the transformations must be such that $\xi^S = \frac{a^2}{2}E$, $\xi_0 = 2H\xi^S - \dot{\xi}^S$ and $\xi^V_i = 0$. They subsequently imply that the new perturbations in Newtonian gauge must be written in terms of the old synchronous gauge perturbations as:

$$\Phi = \dot{\xi}_0, \quad \Psi = -B - H\xi_0 + \frac{1}{6}\nabla^2 E, \quad G_j^{Newt} = G_j^{Sync} \tag{17}$$

Suppose now the opposite case: we are working within the Newtonian gauge, but we wish to move on to synchronous gauge. Then, our initial perturbations satisfy $E = C = F_i = 0$ whereas the non-vanishing perturbations are $A = \Phi$ and $B = -\Psi$. We want to find the gauge transformations given by (9) that lead to $A' = C' = F'_i = 0$. Again, using eq. (9) we see that the transformations must be such that $\dot{\xi}_0 = -\Phi$, $\partial_t(a^{-2}\xi^S) = -a^{-2}\xi_0$ and $\xi^V_i = 0$. Notice that $\Phi$ alone determines $\xi_0$ and $\xi^S$. These transformations further imply that the perturbations $E$, $B$ and $G_i$ in the new synchronous gauge must be written in terms of the old perturbations as:

$$E^{Sync} = -\frac{2}{a^2}\xi^S, \quad B^{Sync} = -\Psi + H\xi_0 - \frac{1}{3a^2}\nabla^2\xi^S, \quad G_i^{Sync} = G_i^{Newt} \tag{18}$$

Also, the perturbation defined in equation (13) is found to be given as:

$$\psi = -3\dot{\Psi} + 3\frac{\partial}{\partial t}(H\xi_0) + \frac{1}{a^2}\nabla^2\xi_0 \tag{19}$$

## 6.6. Dynamics of linear perturbations

Now we are going to derive the equations of motion governing the dynamics of the scalar components of the metric and energy momentum perturbations. We will adopt the synchronous gauge.

Recall that for a given species $a$ the continuity equation is then found to have the general form $\nabla_\mu T_a^\mu{}_\nu = C_{a\nu} = \int_{\bar{p}} C_a p_\mu$ and that $\sum_a C_{a\nu} = 0$, Then, notice that we may decompose the spatial part of $C_{a\nu}$ as $C_{ai} = \partial_i C_a^S + C_{ai}^V$ with $\delta^{ij}\partial_i C_{aj}^V = 0$. So, in terms of perturbations, the continuity equation becomes:

$$\begin{aligned} &\delta\dot{\rho}_a + 3H(\delta\rho_a + \delta p_a) - \nabla^2\left[\frac{\bar{\rho}_a + \bar{p}_a}{a(t)}C - \frac{\bar{\rho}_a + \bar{p}_a}{a^2(t)}\delta u_a\right] + 3(\bar{\rho}_a + \bar{p}_a)\dot{B} = -\delta C_{a0} \\ &\partial_i\left(\delta p_a + \frac{2}{3}\nabla^2\pi_a^S + \partial_0[(\bar{\rho}_a + \bar{p}_a)\delta u_a] + 3H(\bar{\rho}_a + \bar{p}_a)\delta u_a + (\bar{\rho}_a + \bar{p}_a)A\right) = \delta C_a^S \end{aligned} \tag{20}$$

Here $\delta C_0^a$ and $\delta C_S^a$ are the perturbed collision terms (background values vanish in thermal equilibrium). Notice that, because both the pressure perturbation and the stress energy momentum tensor perturbation do not have time derivatives acting on them, we need further equations to complete the system. One of these equations is the equation of state $\delta p_a = w_a \delta\rho_a$. In the case of the stress energy momentum tensor we need to derive additional equations of motion where $\dot{\pi}_{ij}$ plays a role.

On the other hand, recall that Einstein's equations involve the total energy-momentum tensor, so the various perturbations of this tensor are additive, in particular, $\delta u \equiv \frac{1}{\bar{\rho}+\bar{p}} \sum_a (\bar{\rho}_a + \bar{p}_a)\delta u_a$. Then, using the perturbations introduced in equation (1), the scalar components of Einstein's equations are found to be:

$$\begin{gathered}
-H\dot{A} - 2\left(3H^2 + \dot{H}\right)A - \frac{1}{2a^2}\nabla^2\left(2B - \frac{1}{3}\nabla^2 E\right) + \frac{1}{2}\left(2\ddot{B} - \frac{1}{3}\nabla^2\ddot{E}\right) \\
+ 6H\dot{B} - \frac{1}{2}H\nabla^2\dot{E} - \frac{H}{a}\nabla^2 C = 4\pi G\left[\delta\rho - \delta p - \frac{2}{3}\nabla^2\pi^S\right]; \\
\ddot{E} + 3H\dot{E} - \frac{2}{a}\dot{C} - \frac{4H}{a}C - \frac{2}{a^2}A - \frac{1}{a^2}\left(2B - \frac{1}{3}\nabla^2 E\right) = 16\pi G\pi^S; \\
2\dot{B} - \frac{1}{3}\nabla^2\dot{E} - 2HA = 8\pi G(\bar{\rho} + \bar{p})\delta u; \\
3H\dot{A} - 3\ddot{B} - 6H\dot{B} + 6(\dot{H} + H^2)A + \frac{1}{a^2}\nabla^2(A + a\dot{C} + aHC) = 4\pi G(\delta\rho + 3\delta p).
\end{gathered} \tag{21}$$

These equations must be supplemented with a gauge condition reducing the number of fields. A simple consequence from the second equation is that, in the absence of anisotropic stress tensor, it follows that $\Phi_{GI} = \Psi_{GI}$.

## 6.7. Equations of motion in synchronous gauge

Because of the choice we have made in the synchronous gauge the only scalar perturbation to the metric is through the spatial part $h_{ij}$ which is of the form $h_{ij} = a^2(t)(2B\delta_{ij} + D_{ij}E)$. Then, it is convenient to define $\psi \equiv \frac{1}{2}\frac{\partial}{\partial t}\left(\delta^{ij}\frac{h_{ij}}{a^2}\right) = 3\dot{B}$. So, we find the following equations of motion that determine the evolution of the disturbances:

(I) The gravitational potential $\psi$ is determined in terms of the energy density and pressure perturbations as:

$$\frac{1}{a^2}\partial_t(a^2\psi) = -4\pi G(\delta\rho + 3\delta p) \tag{22}$$

(II) We have the continuity equation determining the evolution of $\delta\rho_a$ in terms of $\psi$, $\delta p_a$ and $\delta u_a$:

$$\delta\dot{\rho}_a + 3H(\delta\rho_a + \delta p_a) + \nabla^2\left[\frac{\bar{\rho}_a + \bar{p}_a}{a^2(t)}\delta u_a\right] + (\bar{\rho}_a + \bar{p}_a)\psi = -C_{a0} \tag{23}$$

(III) We have an equation of motion determining the evolution of $\delta u_a$ in terms of $\psi$, $\delta\rho_a$ and $\delta p_a$:

$$\partial_t[(\bar{\rho}_a + \bar{p}_a)\delta u_a] + 3H(\bar{\rho}_a + \bar{p}_a)\delta u_a + \delta p_a + \frac{2}{3}\nabla^2\pi_a^S = C_S^a \tag{24}$$

These equations must be supplemented with an equation of state and with an equation for the anisotropic stress perturbation. Notice that $B$ and $E$ do not appear explicitly. These perturbations can be obtained in terms of $\psi$ and $\rho$ by solving the equations:

$$\dot{B} = \frac{1}{3}\psi\,, \qquad \frac{1}{a^2}\nabla^2\left(2B - \frac{1}{3}\nabla^2 E\right) = 2H\psi - 8\pi G\delta\rho \tag{25}$$

Another useful equation that will become handy is:

$$\frac{1}{3}\nabla^2\dot{E} = \frac{2}{3}\psi - 8\pi G(\bar{\rho}+\bar{p})\delta u \tag{26}$$

It will be also useful to count with a concrete expression for the curvature perturbation $\mathcal{R}$ defined previously. Notice that, using (25) back in the definition of the curvature perturbation, it follows that:

$$\frac{1}{a^2}\nabla^2\mathcal{R} = 2H\psi - 8\pi G\delta\rho - \frac{H}{a^2}\nabla^2\delta u \tag{27}$$

This expression will be useful to determine how the perturbations resulting from inflation fix the initial values of the hot Big-Bang perturbations in synchronous gauge.

## 6.8. Perturbations in Fourier space

The equations of motion that we must deal with are complicated, but, recall equation (3) of section 1, then, for a given equation of motion involving $\phi(\mathbf{x},t)$, we can obtain the equation of motion for $\tilde{\phi}(\mathbf{k},t)$ via: $\phi(\mathbf{x},t) \to \tilde{\phi}(\mathbf{k},t)$, $\partial_i\phi(\mathbf{x},t) \to ik_i\tilde{\phi}(\mathbf{k},t)$ and $\nabla^2\phi(\mathbf{x},t) \to -k^2\tilde{\phi}(\mathbf{k},t)$. So, the evolution of Fourier modes are simple, and can be understood by looking into the following prototype equation, describing the evolution of a scalar field (decoupled from metric fluctuations) in an expanding universe:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{c_s^2}{a^2}\nabla^2\phi = 0 \quad \Rightarrow \quad \ddot{\tilde{\phi}} + 3H\dot{\tilde{\phi}} + c_s^2\frac{k^2}{a^2}\tilde{\phi} = 0 \tag{28}$$

where $c_s$ is the so called sound speed.

There are two relevant scales at play here: The Hubble expansion rate $H$ and the the physical wavenumber $k/a = 2\pi/\lambda_k$. The time evolution of $\phi$ will depend on which of these two scales dominate.

- **Subhorizon modes:** If $c_s k/a >> H$ the physical wavelength $\lambda_k(t) = 2\pi a/k$ of the mode $k$ is smaller than the sound horizon $R_S = c_s R_H$, where $R_H = H^{-1}$ is the Hubble radius. We can neglect the Hubble friction term $H$, and write:

$$\ddot{\tilde{\phi}} + c_s^2\frac{k^2}{a^2}\tilde{\phi} = 0 \tag{29}$$

If we disregard the time evolution of $a$ (which is possible since we are considering that $c_s k/a >> \dot{a}/a$), then we see that a Fourier mode $\tilde{\phi}(\mathbf{k},t)$ propagate at a speed $c_s$.

- **Superhorizon modes:** If $c_s k/a << H$ the physical wavelength $\lambda_k(t) = 2\pi a/k$ of the mode $k$ is larger than the Hubble radius $R_S = c_s R_H$. Here we can neglect the wavenumber and write:

$$\ddot{\tilde{\phi}} + 3H\dot{\tilde{\phi}} = 0 \tag{30}$$

In general, different fluids may have different sound speeds. However, in our treatment of cosmological perturbations a particular wave-vector **k** simultaneously characterizes the wavelength of perturbations related to different classes of fluids.

In both, radiation and matter dominated universes $R_H(t)$ grows faster than $\lambda_k(t)$. If a mode $k$ starts with a wavelength $\lambda_k$ larger than $R_H$ (superhorizon modes), eventually, it will become smaller than $R_H$ (subhorizon modes). Therefore, every mode that we observe today within a volume of radius $R_H$ where once superhorizon (see figure 2). Thus, as the universe evolves, perturbations are subject to a transition from respecting the equation of motion of the type (30) to respect an equation of motion of the type (29).

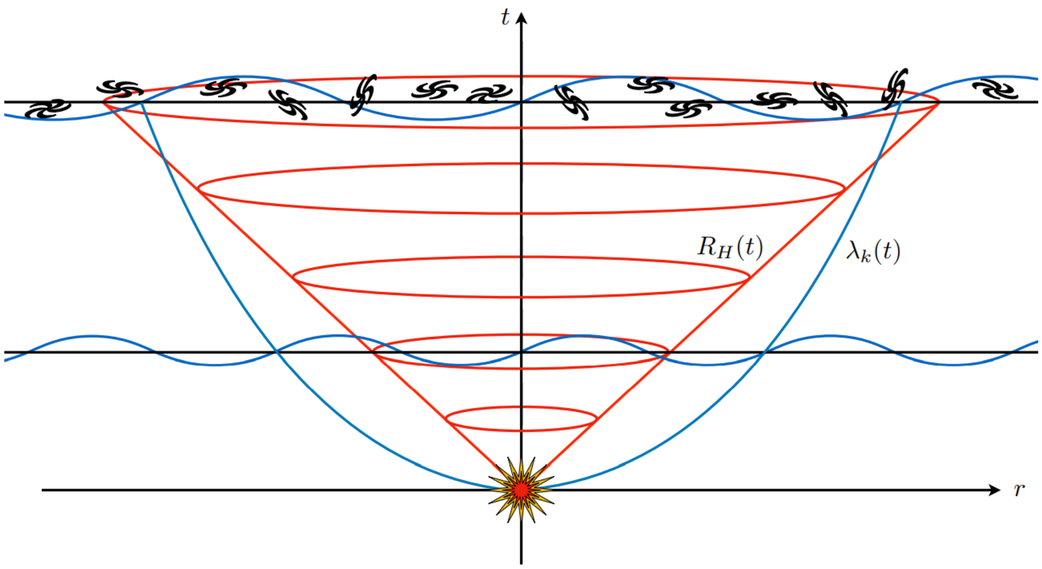


Figure 2: Schematic evolution of the horizon (red line) against the wavelength of a mode describing the distribution of galaxies.

# 7. Kinetic theory for perturbations

Let us have a look into the definition of perturbations from the kinetic theory point of view, preserving our assumptions for the metric and the synchronous gauge choice.

## 7.1. Perturbed distribution function and temperature fluctuations

In a perturbed universe the distribution function describing a species $a$ will be perturbed about its background value, then omitting the label, $\bar{f} \equiv (e^{\bar{E}/k\bar{T}} \pm 1)^{-1}$ with $\bar{E} = \sqrt{m^2 + p^2/a^2}$. To define a perturbed distribution we can write $f = (e^{E/k\bar{T}} \pm 1)^{-1} + \Delta f$ (notice that the first term does not correspond to the background distribution $\bar{f}$, because it depends on $E \simeq \bar{E} + \frac{1}{2\bar{E}} h^{ij} p_i p_j$). Then, expanding the first term about $E = \bar{E}$ we obtain:

$$f = \bar{f} + \delta f, \quad \delta f = \frac{\partial \bar{f}}{\partial \bar{E}} \frac{h^{ij} p_i p_j}{2\bar{E}} + \Delta f \quad \Rightarrow \quad \frac{\partial}{\partial t} \Delta f + \frac{p^i}{E} \partial_i \Delta f = \delta C_a[f] - \frac{\partial}{\partial t} \left( \frac{\partial \bar{f}}{\partial \bar{E}} \frac{h^{ij} p_i p_j}{2\bar{E}} \right) \tag{1}$$

where we have used that $\delta f$ must respect the perturbed Boltzmann equation and then we already write it in terms of $\Delta f$. Using the definition for the perturbed energy-momentum tensor and the perturbed metric it is possible to derive $\delta T_{00} = \frac{1}{a^3} \int_{\mathbf{p}} \Delta f \bar{E}$, $\delta T_{0i} = -\frac{1}{a^3} \int_{\mathbf{p}} \Delta f p_i$ and $\delta T_{ij} = \bar{p} h_{ij} + \frac{1}{a^3} \int_{\mathbf{p}} \frac{\Delta f}{\bar{E}} p_i p_j$. In terms of the hydrodynamical quantities these perturbations can be re-expressed as:

$$\begin{gathered} \delta\rho = \frac{1}{a^3} \int_{\mathbf{p}} \Delta f \bar{E} \,, \quad \delta u = \frac{1}{a(\bar{p} + \bar{\rho})} \int_{\mathbf{p}} p^i \frac{1}{\nabla^2} \partial_i \Delta f \,, \quad \delta p = \frac{1}{3a^3} \int_{\mathbf{p}} \Delta f \frac{1}{\bar{E}} \frac{p^2}{a^2} \,, \\ \delta u_i^V = \frac{1}{a(\bar{p} + \bar{\rho})} \int_{\mathbf{p}} p^j \left( \delta_{ij} - \frac{1}{\nabla^2} \partial_i \partial_j \right) \Delta f \,, \quad \pi_{ij} = \frac{1}{a^3} \int_{\mathbf{p}} \Delta f \frac{1}{a^2 \bar{E}} \left( p_i p_j - p^2 \frac{1}{3} \delta_{ij} \right) \end{gathered} \tag{2}$$

Perturbed Boltzmann equation together with initial conditions should be enough to follow the evolution of $\Delta f$ and therefore any fluctuation deduced from it. To have a more physical understanding of what it is this quantity and decide the appropriate set of initial conditions, it is useful to identify $\Delta f$ as the result of inhomogeneities of the temperature.

Thus, we can write the full distribution $f_a$ as $f_a(e^{E_a/kT_a} \pm 1)^{-1}$ with $T_a \equiv \bar{T} + \Delta T_a$ been interpreted as the temperature parametrizing the thermal distribution of the species $a$, with $\bar{T}$ the temperature of the bath common to all the interacting particles. This is because, in general, $\Delta T_a = \Delta T_a(t, \mathbf{x}, \mathbf{p})$. So, expanding this expression with respect to $\Delta T_a$ and equating with the left hand side of the implicancy in eq. (1) we have:

$$\Delta f_a = -\bar{E}_a \frac{\partial f_a}{\partial \bar{E}_a} \Theta_a \,, \qquad \Theta_a \equiv \frac{\Delta T_a}{\bar{T}} \tag{3}$$

Notice that as long as $\frac{\partial f_a}{\partial \bar{E}_a}$ is well defined this identification is not restrictive. With this, the Boltzmann equation takes the form:

$$\dot{\Theta}_a + \frac{p}{a^2 \bar{E}} \hat{p}^i \partial_i \Theta_a + \frac{p^2}{2a^4 \bar{E}^2} \frac{\partial}{\partial t} \left( \frac{h_{ij}}{a^2} \right) \hat{p}^i \hat{p}^j = - \left( \bar{E} \frac{\partial \bar{f}_a}{\partial \bar{E}} \right)^{-1} \delta C_a[f] \tag{4}$$

The power of this identification relies in that it provides a systematic expansion of $\Delta f_a$ in terms of momenta. We expect that this dependence on momenta to be close to the background value $\bar{f}$, i.e., that the distribution of the species $a$ is almost locally thermal.

We can now assert with an expansion for $\Theta_a(\mathbf{x},\mathbf{p}),t)$, then, omitting labels:

$$\Theta(\mathbf{x},\mathbf{p},t) = A(\mathbf{x},t,p) + aA^i(\mathbf{x},t,p)\hat{p}_i + \frac{a^2}{2!}A^{ij}(\mathbf{x},t,p)\hat{p}_i\hat{p}_j + \frac{a^3}{3!}A^{ijk}(\mathbf{x},t,p)\hat{p}_i\hat{p}_j\hat{p}_k + ... \tag{5}$$

The tensors $A^{i...j}$ must satisfy the trace-free conditions $\delta_{ij}A^{ij}=0$, $\delta_{ij}A^{ijk}$, $\delta_{ik}A^{ijk}$, $\delta_{jk}A^{ijk}$, etc. As usual, we can decompose these tensors into a basis of sclaras, pure vectors, pure second rank tensors, etc. For concreteness, let us focus only on the scalar contributions to $A_a^{i...j}$. Then, we can write them in terms of scalar functions $\{A^{(1)}, A^{(2)}, A^{(3)}, \cdots$ such that:

$$\begin{gathered} A^i = \frac{\delta^{ij}}{a^2}\partial_j A^{(1)}\,, \quad A^{ij} = \frac{\delta^{ik}}{a^2}\frac{\delta^{jl}}{a^2}\left(\partial_k\partial_l - \frac{1}{3}\nabla^2\delta_{kl}\right)A^{(2)} \\ A^{(ijk)} = \frac{\delta^{il}}{a^2}\frac{\delta^{jm}}{a^2}\frac{\delta^{kn}}{a^2}\left(\partial_l\partial_m\partial_n - \frac{1}{5}\delta_{lm}\nabla^2\partial_n - \frac{1}{5}\delta_{mn}\nabla^2\partial_l - \frac{1}{5}\delta_{nl}\nabla^2\partial_m\right)A^{(3)} \end{gathered} \tag{6}$$

and similar expressions for tensors of higher rank. To appreciate how these tensors are involved in physical quantities, notice that $\frac{1}{4\pi}\int\Omega_p\Theta(\mathbf{x},\mathbf{p},t) = A(\mathbf{x},t,p)$, $\frac{1}{4\pi}\int\Omega_p\Theta(\mathbf{x},\mathbf{p},t)\hat{p}^i = \frac{a}{3}A^i(\mathbf{x},t,p)$, $\frac{1}{4\pi}\int\Omega_p\Theta(\mathbf{x},\mathbf{p},t)\left(\hat{p}^i\hat{p}^j - \frac{1}{3}\delta^{ij}\right) = \frac{a^2}{15}A^{ij}(\mathbf{x},t,p)$. Then, the scalar sector of the hydrodynamical quantities in eq. (2) are written in terms of this expansion as:

$$\begin{gathered} \delta\rho = -\frac{1}{2\pi^2a^3}\int dpp^2\frac{\partial\bar{f}}{\partial\bar{E}}\bar{E}^2A(\mathbf{x},t,p)\,, \quad \delta u = -\frac{a}{6\pi^2(\bar{p}+\bar{\rho})a^3}\int dpp^3\frac{\partial\bar{f}}{\partial\bar{E}}\bar{E}\frac{1}{\nabla^2}\partial_jA^j(\mathbf{x},t,p)\,, \\ \delta p = -\frac{1}{6\pi^2a^5}\int dpp^4\frac{\partial\bar{f}}{\partial\bar{E}}A(\mathbf{x},t,p)\,, \quad \pi_{ij} = -\frac{1}{30\pi^2a^3}\delta_{im}\delta_{jn}\int dpp^4\frac{\partial\bar{f}}{\partial\bar{E}}A^{mn}(\mathbf{x},t,p) \end{gathered} \tag{7}$$

If the system is well thermalized, the full distribution will locally preserve the momentum dependence of the thermal background $\bar{f}$ and $A$ will be independent of $p$. It then follows that:

$$\delta\rho = T\frac{\partial\rho}{\partial T}A\,, \qquad \delta p = T\frac{\partial p}{\partial T}A \quad \Rightarrow \quad \delta p = w\delta\rho \tag{8}$$

In particular, relativistic matter will obey $\delta p = 0$. Similarly, notice that if $A^{mn}$ is independent of the momenta, then:

$$\pi_{ij} = \frac{1}{5}T\frac{\partial p}{\partial T}\delta_{im}\delta_{jn}A^{mn} \tag{9}$$

This shows that if $A^{mn}$ does not have a strong dependence of momenta, then the anisotropic stress tensor $\pi_{ij}$ for non-relativistic species is kinematically suppressed (in the same way as $\delta p$).

## 7.2. Multipole expansion of temperature fluctuations

The expansion (5) is equivalent to a multipole expansion with respect to Legendre polynomials. So, in Fourier space:

$$\tilde{\Theta}(\mathbf{k},\mathbf{p},t) = \tilde{A}(\mathbf{k},t,p) + a\tilde{A}^i(\mathbf{k},t,p)\hat{p}_i + \frac{a^2}{2!}\tilde{A}^{ij}(\mathbf{k},t,p)\hat{p}_i\hat{p}_j + \frac{a^3}{3!}\tilde{A}^{ijk}(\mathbf{k},t,p)\hat{p}_i\hat{p}_j\hat{p}_k + ... \tag{10}$$

Then, if we express the tensors $A^{i...j}$ in Fourier space and using this last relation we can find:

$$\tilde{\Theta}(\mathbf{k},\mathbf{p},t) = \sum_l i^l \frac{2^l l!}{(2l)!} P_l(\mu)\frac{k^l}{a^l}\tilde{A}^{(l)}(\mathbf{k},t,p) \tag{11}$$

where $\mu \equiv \hat{k}\cdot\hat{p}$ parametrizes the angle between the particle momenta and the Fourier wavenumber. In more explicit terms, notice that a general function $\tilde{\Theta}(\mathbf{k},\mathbf{p},t)$ of $\mathbf{k}$ and $\mathbf{p}$ can be expressed as a function $\tilde{\Theta}(\mathbf{k},\mu,p,t)$ of $\mathbf{k}$, $p$ and $\mu$. Then, by defining the Legendre moment as $\tilde{\Theta}_l(\mathbf{k},p,t) = \frac{i^l}{2}\int_{-1}^{1} d\mu\tilde{\Theta}(\mathbf{k},\mu,p,t)P_l(\mu)$, we can expand:

$$\tilde{\Theta}(\mathbf{k},\mu,p,t) = \sum_l \frac{(2l+1)}{i^l}P_l(\mu)\tilde{\Theta}_l(\mathbf{k},p,t) \Rightarrow \tilde{A}^{(l)}(\mathbf{k},p,t) = (-1)^l(2l+1)\frac{(2l)!}{2^l l!}\frac{a^l}{k^l}\tilde{\Theta}_l(\mathbf{k},p,t) \tag{12}$$

In the case where the species under study is relativistic and the temperature fluctuation $\tilde{\Theta}(\mathbf{k},\mu,p,t)$ is independent of momenta, one finds:

$$\begin{aligned} \delta\tilde{\rho} &= \left(\bar{T}\frac{\partial\bar{\rho}}{\partial\bar{T}}\right)\tilde{\Theta}_0(\mathbf{k},t)\,, \quad \delta\tilde{p} = \left(\bar{T}\frac{\partial\bar{p}}{\partial\bar{T}}\right)\tilde{\Theta}_0(\mathbf{k},t)\,, \\ \delta\tilde{u} &= -3\frac{a}{k}\tilde{\Theta}_1(\mathbf{k},t)\,, \quad \tilde{\pi}^S = 3\left(\bar{T}\frac{\partial\bar{p}}{\partial\bar{T}}\right)\frac{a^2}{k^2}\tilde{\Theta}_2(\mathbf{k},t) \end{aligned} \tag{13}$$

These expressions will be relevant to study the behavior of photons interacting with electrons via Thomson scattering (which makes $\tilde{\Theta}_l(\mathbf{k},p,t)$ to be independent of momenta).

## 7.3. Boltzmann equation for photons and neutrinos

- **Photons:** In this particular case we have $\dot{\Theta} + \frac{1}{a}\hat{p}^i\partial_i\Theta + \frac{1}{2a^2}\partial_t\left(\frac{h_{ij}}{a^2}\right)\hat{p}^i\hat{p}^j = -\left(\bar{E}\frac{\partial\bar{f}_\gamma}{\partial\bar{E}}\right)^{-1}\delta C_\gamma[f]$, where we have used $\bar{E}\partial_{\bar{E}}\partial\bar{f}_\gamma$ remains constant for relativistic particles. Recall that $h_{ij} = a^2(2B\delta_{ij} + D_{ij}E)$ and $\psi = 3B$ we are lead to:

$$\dot{\tilde{\Theta}} + i\frac{k}{a}P_1(\mu)\tilde{\Theta} + \frac{1}{3}\left(\dot{\tilde{\psi}} - k^2P_2(\mu)\dot{\tilde{E}}\right) = -\left(\bar{E}\frac{\partial\bar{f}_\gamma}{\partial\bar{E}}\right)^{-1}\delta\tilde{C}_\gamma[f] \tag{14}$$

  Let us now specialize this Boltzmann equation to the relevant case in which photons interact with non-relativistic electrons via Thomson scattering $e^- + \gamma \rightleftharpoons e^- + \gamma$. So, recall that $\delta C_\gamma[f(\mathbf{p})] = \frac{a^2}{16p}\int_{\vec{q}}\int_{\vec{p}'}\int_{\vec{q}'}(2\pi)^4\delta^{(4)}(p+q-p'-q')\frac{|\overline{\mathcal{T}_{e\gamma\rightleftharpoons e\gamma}}|^2}{p'E_e(\mathbf{q})E_e(\mathbf{q}')}\left(f_\gamma(\mathbf{p}',t)f_e(\mathbf{q}',t) - f_\gamma(\mathbf{p},t)f_e(\mathbf{q},t)\right)$. If we manipulate the $\delta$-term and also recalling the previous results for Thomson scattering, we then obtain:

$$
\begin{aligned}
\delta C_\gamma[f(\mathbf{p})] =& \frac{2\pi^2\sigma_T n_e}{\bar{p}}\int_{\bar{\mathbf{p}}'}\frac{1}{\bar{p}'}(\delta(\bar{p}-\bar{p}')+\mathbf{u}_e\cdot(\bar{\mathbf{p}}-\bar{\mathbf{p}}')\partial_{\bar{p}'}\delta(\bar{p}-\bar{p}')) \\
&\times\left[1+\frac{1}{2}P_2(\hat{p}\cdot\hat{p}')\right](f_\gamma(\bar{\mathbf{p}}',t)-f_\gamma(\bar{\mathbf{p}},t))
\end{aligned}
\tag{15}
$$

where we have used $n_e = \bar{\rho}_e/m_e$ and $\delta u_e^i = \bar{\rho}_e^{-1}\int_{\bar{q}}\bar{q}^i f_e$ to identify the velocity field of electrons. Next, recall that to work with the collision term we must treat it as a local function without including gravitational corrections. Then, $f_\gamma = \bar{f}_\gamma + \Delta f = \bar{f}_\gamma - \bar{E}\frac{\partial f_\gamma}{\partial \bar{E}_\gamma}\Theta$. So, we can finally have:

$$
\delta\tilde{C}_\gamma[f(\mathbf{p})] = -\bar{E}\frac{\partial f_\gamma}{\partial\bar{E}_\gamma}\sigma_T n_e\left(\tilde{\Theta}_0(\mathbf{k},t)-\frac{1}{2}P_2(\mu)\tilde{\Theta}_2(\mathbf{k},t)-\tilde{\Theta}(\mathbf{k},\mathbf{p},t)+\delta\tilde{\mathbf{u}}_e\cdot\hat{p}\right)
\tag{16}
$$

Then, we obtain the full Boltzmann equation for photons interacting with electrons via Thomson scattering to be:

$$
\begin{aligned}
&\dot{\tilde{\Theta}}+i\frac{k}{a}P_1(\mu)\tilde{\Theta}+\frac{1}{3}\left(\tilde{\psi}-k^2P_2(\mu)\dot{\tilde{E}}\right)= \\
&-\dot{\tau}\left[\tilde{\Theta}_0(\mathbf{k},t)-\frac{1}{2}P_2\tilde{\Theta}_2(\mathbf{k},t)-\tilde{\Theta}(\mathbf{k},\mathbf{p},t)+\delta\tilde{\mathbf{u}}_e\cdot\hat{p}\right]
\end{aligned}
\tag{17}
$$

where we have introduced $\dot{\tau}\equiv-\sigma_T n_e$; $\tau(t)\equiv\int_t^\infty\sigma_T n_e dt$ stands for the so called optical depth, which is nothing but the number of interactions $N_{int,\gamma}$ that photons have with electrons from a given time $t$ up until now.

On the other hand, if we recall the expansion of $\tilde{\Theta}(\mathbf{k},\mathbf{p},t)$ and using the useful property of Legendre polynomials $(2l+1)_l(\mu) = (l+1)P_{l+1}(\mu)+lP_{l-1}(\mu)$, one finally find:

$$
\begin{aligned}
&i\frac{k}{a}\left(\sum_{l\geq3}\frac{l}{i^{l-1}}P_l(\mu)\tilde{\Theta}_{l-1}(\mathbf{k},t)+\sum_{l\geq1}\frac{(l+1)}{i^{l+1}}P_l(\mu)\tilde{\Theta}_{l+1}(\mathbf{k},t)\right)+\sum_{l\geq2}\frac{(2l+1)}{i^l}P_l(\mu)\dot{\tilde{\Theta}}_l(\mathbf{k},t) \\
&+\left(\dot{\tilde{\Theta}}_0(\mathbf{k},t)+i\frac{k}{a}\mu\tilde{\Theta}_0(\mathbf{k},t)\right)-3i\mu\dot{\tilde{\Theta}}_1(\mathbf{k},t)-\frac{k^2}{3a^2}[2P_2(\mu)+1]\delta\tilde{u}_\gamma+\frac{1}{3}(\psi-P_2(\mu)k^2\dot{\tilde{E}}) \\
&=-\dot{\tau}\left[i\mu\left(3\tilde{\Theta}_1(\mathbf{k},t)+\frac{k}{a}\delta\tilde{u}_B\right)+\frac{9}{2}P_2(\mu)\tilde{\Theta}_2(\mathbf{k},t)-\sum_{l\geq3}\frac{(2l+1)}{i^l}P_l(\mu)\tilde{\Theta}_l(\mathbf{k},t)\right]
\end{aligned}
\tag{18}
$$

Thanks to the linear independence of Legendre polynomials, this equation implies the following equations for $l=0, l=1, l=2$ and $l\geq3$, respectively:

$$
\begin{aligned}
&\dot{\tilde{\Theta}}_0(\mathbf{k},t)+\frac{k}{a}\tilde{\Theta}_1(\mathbf{k},t)+\frac{1}{3}\tilde{\psi}=0 \\
&-3\frac{a}{k}\dot{\tilde{\Theta}}_1(\mathbf{k},t)+\tilde{\Theta}_0(\mathbf{k},t)-2\tilde{\Theta}_2(\mathbf{k},t)=-\dot{\tau}\left(3\frac{a}{k}\tilde{\Theta}_1(\mathbf{k},t)+\delta\tilde{u}_e\right) \\
&5\dot{\tilde{\Theta}}_2(\mathbf{k},t)-2\frac{k}{a}\tilde{\Theta}_1(\mathbf{k},t)+3\frac{k}{a}\tilde{\Theta}_3(\mathbf{k},t)+\frac{k^2}{3}\dot{\tilde{E}}=\frac{9}{2}\dot{\tau}\tilde{\Theta}_2(\mathbf{k},t) \\
&(2l+1)\dot{\tilde{\Theta}}_l(\mathbf{k},t)-\frac{k}{a}l\tilde{\Theta}_{l-1}(\mathbf{k},t)+\frac{k}{a}(l+1)\tilde{\Theta}_{l+1}(\mathbf{k},t)=\dot{\tau}(2l+1)\tilde{\Theta}_l(\mathbf{k},t)
\end{aligned}
\tag{19}
$$

Notice that for wave numbers $k/a$ much smaller than $\dot{\tau}$, we are allowed to truncate this tower of equations. Besides, we can see that in the second of this equations $\tilde{\Theta}_2$ must be suppressed with respect to $\delta\tilde{u}_\gamma k/a$. But this in turns implies that $\tilde{\Theta}_2$ must be suppressed with respect to $\tilde{\Theta}_0$ too, and so we can neglect the presence of $\tilde{\Theta}_2$ in the equation mentioned. Given that $\tilde{\Theta}_2 = k^2\pi^S_\gamma/4$, this is equivalent to neglect the anisotropic stress tensor for photons.

- **Neutrinos:** Neutrinos can be treated in a similar way as with photons, but there are crucial differences. First, the absence of collision terms, and, second, the presence of non-vanishing masses. Let us designate $\mathcal{N}$ as the temperature fluctuation $\Theta_\nu$ for neutrinos, so $\Delta f_\nu = -\bar{E}\frac{\partial \bar{f}}{\partial \bar{E}}\mathcal{N}$, then, the Boltzmann equation takes the form:

$$\dot{\mathcal{N}} + \frac{p}{a^2\bar{E}}\hat{p}^i\partial_i\mathcal{N} + \frac{p^2}{a^4\bar{E}^2}\left(\frac{1}{3}\delta_{ij}\psi + \frac{1}{2}D_{ij}\dot{E}\right)\hat{p}^i\hat{p}^j = 0 \tag{20}$$

Neutrino masses are known to be smaller than about 0.1eV, so neutrinos are still relativistic at the time of recombination , so we may therefore assume that they are massless and write $\bar{E} = p/a$. Then repeating the same steps followed with photons, we may write the Legendre moments as $\tilde{\mathcal{N}}(\mathbf{k},\mathbf{p},t) = \sum_l \frac{(2l+1)}{i^l}P_l(\mu)\tilde{\mathcal{N}}_l(\mathbf{k},t)$. Then, the equations of motion governing the evolution of the $\mathcal{N}_l$'s, for $l=0, l=1, l=2$ and $l \geq 3$, respectively are:

$$\begin{aligned}
\dot{\tilde{\mathcal{N}}}_0(\mathbf{k},t) + \frac{k}{a}\tilde{\mathcal{N}}_1(\mathbf{k},t) + \frac{1}{3}\tilde{\psi} = 0 \\
-3\frac{a}{k}\dot{\tilde{\mathcal{N}}}_1(\mathbf{k},t) + \tilde{\mathcal{N}}_0(\mathbf{k},t) - 2\tilde{\mathcal{N}}_2(\mathbf{k},t) = 0 \\
5\dot{\tilde{\mathcal{N}}}_2(\mathbf{k},t) - 2\frac{k}{a}\tilde{\mathcal{N}}_1(\mathbf{k},t) + 3\frac{k}{a}\tilde{\mathcal{N}}_3(\mathbf{k},t) + \frac{k^2}{3}\dot{\tilde{E}} = 0 \\
(2l+1)\dot{\tilde{\mathcal{N}}}_l(\mathbf{k},t) - \frac{k}{a}l\tilde{\mathcal{N}}_{l-1}(\mathbf{k},t) + \frac{k}{a}(l+1)\tilde{\mathcal{N}}_{l+1}(\mathbf{k},t) = 0
\end{aligned} \tag{21}$$

The lack of collision terms means that there is no obvious way of truncating the tower of equations. At long wavelengths ($k/a << H$),it is clear that the higher multipole terms are suppressed by $k/a$. However, at short wavelengths higher multipole terms become relevant. In particular, the anisotropic stress tensor $\pi_{ij}$ for neutrinos cannot be disregarded on short wavelengths. This reflect that neutrinos, being massless and collisionless, have no incentive to behave as a perfect fluid.

The initial conditions for neutrinos will be such that $\mathcal{N}_l = 0$ for $l \geq 2$. Thus, initially, the amplitude of neutrino fluctuations will be exclusively in the form $\mathcal{N}_0$. As time goes on, and wavelengths transit from being long to short, the higher multipole fluctuations $\mathcal{N}_i$, for $i \geq 1$, will start to quickly borrow amplitude from $\mathcal{N}_0$, producing a damping of the full set of $\mathcal{N}_l$.

# 8. Hydrodynamical approximation of ΛCDM perturbations

We are finally in a position to write down the complete set of equations of motions governing the dynamics of perturbations during the early universe. So, lets examine each relevant species and rewrite the continuity equation for each of them.

## 8.1. Continuity equation for species compounding our universe

- **Dark matter:** Modeling DM as a non-relativistic species following a thermal distribution requires us to adopt $\delta p_{DM} = 0$. Besides, as DM is only coupled to gravity, this implies that the collision terms $C_0^{DM}$ and $C_S^{DM}$ can be neglected. The anisotropic stress tensor $\pi_{ij}^{DM}$ can be also be disregarded. All of this implies that the dark matter fluid must respect the following equations of motion:

$$\begin{gathered}\delta\dot{\tilde{\rho}}_{DM} + 3H\delta\tilde{\rho}_{DM} - \frac{\bar{\rho}_{DM}}{a^2}k^2\delta\tilde{u}_{DM} = -\bar{\rho}_{DM}\tilde{\psi}\,, \\ \partial_t(\bar{\rho}_{DM}\delta\tilde{u}_{DM}) + 3H\bar{\rho}_{DM}\delta\tilde{u}_{DM} = 0\end{gathered} \tag{1}$$

However, given that $\dot{\bar{\rho}}_{DM} = -3H\rho_{DM}$ it follows that $\delta\dot{u}_{DM} = 0$. Therefore, the velocity field of dark matter is independent of time. This implies that we can use the residual gauge transformation eq. (15) of section 6 to conveniently fix $\delta\tilde{u}_{DM} = 0$. This leads to the following equation of motion for the dark matter density perturbation:

$$\delta\dot{\tilde{\rho}}_{DM} + 3H\delta\tilde{\rho}_{DM} = -\bar{\rho}_{DM}\tilde{\psi} \tag{2}$$

- **Neutrinos:** Similar to DM, neutrinos remain coupled to the rest of the bath only through gravity, so its collision terms can also be disregarded, but its velocity field can not. The, we must consider the following equations of motion to study their evolution:

$$\begin{gathered}\delta\dot{\tilde{\rho}}_\nu + 4H\delta\tilde{\rho}_\nu - \frac{4\bar{\rho}_\nu}{3a^2(t)}k^2\delta\tilde{u}_\nu = -\frac{4}{3}\bar{\rho}_\nu\tilde{\psi} \\ \delta\tilde{\rho}_\nu + 2\nabla^2\tilde{\pi}_a^S + \partial_t[\bar{\rho}_\nu\delta\tilde{u}_\nu] + 12H\bar{\rho}_\nu\delta\tilde{u}_\nu = 0\end{gathered} \tag{3}$$

Unlike photons (which remain tightly coupled to non-relativistic baryons), the anisotropic stress tensor is important for neutrinos. $\nabla^2\tilde{\pi}_a^S$ can be neglected except at small scales, where fast oscillations have the effect of damping the evolution of neutrino perturbations. For this reason, we will adopt the approach whereby $\nabla^2\tilde{\pi}_a^S = 0$, and simply set $\delta\tilde{\rho}_\nu \to 0$ and $\delta\tilde{u}_\nu \to 0$ in those situations where the anisotropic stress tensor becomes relevant.

- **Photons:** Collision terms are given by $C_{\gamma 0} = 0$ and $C_\gamma^S = \frac{4}{3}\dot{\tau}\bar{\rho}_\gamma[\delta\tilde{u}_\gamma - \delta\tilde{u}_e$. So, under Thomson scattering there is little interchange of energy between photons and electrons, but a relevant interchange of momenta. As a consequence, the equations of motion governing the evolution of photon perturbations are:

$$\begin{aligned} \delta\dot{\tilde{\rho}}_\gamma + 4H\delta\tilde{\rho}_\gamma - \frac{4\bar{\rho}_\gamma}{3a^2(t)}k^2\delta\tilde{u}_\gamma &= -\frac{4}{3}\bar{\rho}_\gamma\tilde{\psi} \\ \partial_t[\bar{\rho}_\gamma\delta\tilde{u}_\gamma] + 3H\bar{\rho}_\gamma\delta\tilde{u}_\gamma - \frac{1}{2}k^2\tilde{\pi}^S_\gamma + \frac{1}{4}\delta\tilde{\rho}_\gamma &= \dot{\tau}\bar{\rho}_\gamma[\delta\tilde{u}_\gamma - \delta\tilde{u}_e] \end{aligned} \tag{4}$$

The anisotropic stress $\pi^S_\gamma$ may be disregarded thanks to the tight coupling with electrons offered by Thomson scattering. If we wanted to take the effects of the anisotropic stress into account, we would need to notice that $\frac{5k^2}{4}\dot{\tilde{\pi}}^S_\gamma + \frac{2k^2}{3a^2}\delta\tilde{u}_\gamma - 2\dot{H}\delta\tilde{u}_{tot} - \frac{2}{3}\tilde{\psi} = \frac{9k^2}{8}\dot{\tau}\tilde{\pi}^S_\gamma$ where we have disregarded $\Theta_3$, whihc, thanks to Thomson scattering, is even more suppressed.

- **Baryons:** Prior to recombination, electrons and protons remain tightly coupled through the interactions $e^- + p \rightleftharpoons e^- + p$. Thus, they can be treated as a single fluid, the baryon fluid. Baryons interact with photons via Thomson scattering, and so the collision terms for baryons describing Thomson scattering can be worked out following the same steps followed in the last section. The result, unsurprisingly, is found to be $C_{B0} = 0$ and $C^S_B = -C^S_\gamma = \frac{4}{3}\dot{\tau}\bar{\rho}_\gamma[\delta\tilde{u}_e - \delta\tilde{u}_\gamma$. The velocity field of electrons is just the velocity field of baryons, so we may just write $\delta\tilde{u}_e = \delta\tilde{u}_B$. We therefore find that the continuity equations for baryons are:

$$\begin{aligned} \delta\dot{\tilde{\rho}}_B + 3H\delta\tilde{\rho}_B - \frac{\bar{\rho}_B k^2}{a^2(t)}k^2\delta\tilde{u}_B &= -\bar{\rho}_B\tilde{\psi} \\ \partial_t[\bar{\rho}_B\delta\tilde{u}_B] + 3H\bar{\rho}_B\delta\tilde{u}_B &= \frac{4}{3}\dot{\tau}\bar{\rho}_B[\delta\tilde{u}_B - \delta\tilde{u}_\gamma] \end{aligned} \tag{5}$$

  where we are neglecting the anisotropic stress scalar $\tilde{\pi}^S_B$.

- **Photon-Baryon plasma:** We can study photons and baryons together by treating them as a single fluid. To do so, we may simply join the equations of motion written in the previous sections to find the following set of equations

$$\begin{aligned} \delta\dot{\tilde{\rho}}_\gamma + 4H\delta\tilde{\rho}_\gamma - \frac{4\bar{\rho}_\gamma}{3a^2(t)}k^2\delta\tilde{u}_{\gamma B} + \frac{4}{3}\bar{\rho}_\gamma\tilde{\psi} &= 0 \\ \delta\dot{\tilde{\rho}}_B + 3H\delta\tilde{\rho}_B - \frac{\bar{\rho}_B k^2}{a^2(t)}k^2\delta\tilde{u}_{\gamma B} + \bar{\rho}_B\tilde{\psi} &= 0 \\ \partial_t\left[\left(\frac{4}{3}\bar{\rho}_\gamma + \bar{\rho}_B\right)\delta\tilde{u}_{\gamma B}\right] + 3H\left(\frac{4}{3}\bar{\rho}_\gamma + \bar{\rho}_B\right)\delta\tilde{u}_{\gamma B} + \frac{1}{3}\delta\tilde{\rho}_\gamma - \frac{2}{3}k^2\tilde{\pi}^S_\gamma &= 0 \\ \frac{5k^2}{4}\dot{\tilde{\pi}}^S_\gamma + \frac{2k^2}{3a^2}\delta\tilde{u}_{\gamma B} - 2\dot{H}\delta\tilde{u}_{tot} - \frac{2}{3}\tilde{\psi} &= \frac{9k^2}{8}\dot{\tau}\tilde{\pi}^S_\gamma \end{aligned} \tag{6}$$

  where we defined the common velocity field perturbation $\delta\tilde{u}_{\gamma B} \equiv \left(\frac{4}{3}\bar{\rho}_\gamma + \bar{\rho}_B\right)^{-1}\left(\frac{4}{3}\bar{\rho}_\gamma\delta\tilde{u}_\gamma + \bar{\rho}_B\delta\tilde{u}_B\right)$. Notice that because $\dot{\tau}$ is large, the difference $\delta\tilde{u}_B - \delta\tilde{u}_\gamma$ must be small, an do $\delta\tilde{u}_B \simeq \delta\tilde{u}_\gamma \simeq \delta\tilde{u}_{\gamma B}$.

## 8.2. Metric dimensionless perturbations

The equation of motion determining the evolution of the metric perturbation $\psi$ is simply given by $\frac{1}{a^2}\frac{\partial}{\partial t}(a^2\tilde{\psi}) = -4\pi G(2\delta\tilde{\rho}_\gamma + 2\delta\tilde{\rho}_\nu + \delta\tilde{\rho}_B + \delta\tilde{\rho}_{DM})$. To deal with these equations, it is useful to define a dimensionless perturbation that simplifies the treatment $\delta_a \equiv \frac{\delta\rho_a}{\bar{\rho}_a}$. Then, the equations of motion for the perturbations $\delta_{DM}$, $\delta_\gamma$, $\delta_B$, $\delta_\nu$, $\delta u_\gamma$ and $\delta u_\nu$ are found to be:

$$\begin{aligned} \dot{\tilde{\delta}}_{DM} = -\tilde{\psi}\,, \quad \dot{\tilde{\delta}}_\gamma - \frac{4k^2}{3a^2(t)}\delta\tilde{u}_{\gamma B} = -\frac{4}{3}\tilde{\psi}\,, \quad \dot{\tilde{\delta}}_\nu - \frac{k^2}{3a^2(t)}\delta\tilde{u}_\nu = -\frac{4}{3}\tilde{\psi} \\ \dot{\tilde{\delta}}_B - \frac{k^2}{a^2(t)}\delta\tilde{u}_{\gamma B} = -\tilde{\psi}\,, \quad \frac{1}{4}\delta\tilde{u}_\gamma + a\partial_t\left[(1+R)\frac{\delta\tilde{u}_{\gamma B}}{a}\right] = 0\,, \quad \frac{1}{4}\tilde{\delta}_\nu + a\partial_t\left[\frac{\delta\tilde{u}_\nu}{a}\right] = 0 \end{aligned} \tag{7}$$

where we have introduced $R = \frac{3\bar{\rho}_B}{4\bar{\rho}_\gamma}$. Notice that we are disregarding the anisotropic stress scalar $\tilde{\pi}^S_\gamma$ for convenience. On the other hand, we finally have:

$$\frac{1}{a^2}\frac{\partial}{\partial t}\left(a^2\tilde{\psi}\right) = -4\pi G(2\bar{\rho}_\gamma\tilde{\delta}_\gamma + 2\bar{\rho}_\nu\tilde{\delta}_\nu + \bar{\rho}_B\tilde{\delta}_B + \bar{\rho}_{DM}\tilde{\delta}_{DM}) \tag{8}$$

Our challenge now is to solve these equations, but before doing so, we need initial conditions.

## 8.3. Adiabatic initial conditions

All of the modes relevant for cosmology today must have been superhorizon sometime in the past. We can start the evolution of the perturbed universe at a time $t_{ini}$ when all the modes had their wavelengths outside the horizon ($K/a >> H$). So, let us consider a particular species $a$ with an energy density $\rho_a$, and let us take into account only the long wavelength modes, then, the perturbed energy density reads $\rho_a(\mathbf{x},t) = \bar{\rho}_a(t) + \delta\rho_a(\mathbf{x},t) = \bar{\rho}_a(t + \delta t_a(\mathbf{x},t))$. The last equality is just because $\delta\rho_a(\mathbf{x},t)$ consist on a superposition of only superhorizon modes, then the energy density must satisfy the same background equations of motion respected by $\bar{\rho}_a(t)$, i.e., at wavelengths smaller than $R_H$ there is no way of telling that the energy density is a perturbed quantity, so we write it as if it were a background solution. On the other hand, check that $\delta t_a(\mathbf{x},t)$ is a shift in time that informs us that at different Hubble patches the background will look slightly different, but still will be a solution of the same background equations of motion (though with a slightly different initial condition); we then obtain:

$$\frac{\delta\rho_a(\mathbf{x},t)}{\dot{\bar{\rho}}_a(t)} = \delta t_a(\mathbf{x},t) \tag{9}$$

Now, requiring that at length-scales smaller than $R_H$ the perturbed solutions respect the same background equations of motion as the background solutions (where abundances are dictated by local thermodynamical equilibrium), requires us to demand that $\delta t_a(\mathbf{x},t)$ be common to all species. In effect, if we had two species $a$ and $b$ with $\delta t_a(\mathbf{x},t) \neq \delta t_b(\mathbf{x},t)$, then these two species would have distributions characterized by different temperatures. Therefore, we impose:

$$\frac{\delta\rho_a}{\dot{\bar{\rho}}_a} = \frac{\delta\rho_b}{\dot{\bar{\rho}}_b} \quad \Rightarrow \quad \frac{\delta\rho_a}{\bar{\rho}_a + \bar{p}_a} = \frac{\delta\rho_b}{\bar{\rho}_b + \bar{p}_b}\,, \quad a \neq b \tag{10}$$

where we have used the background equation $\dot{\bar{\rho}} + 3H(\bar{\rho} + \bar{p}) = 0$.

Perturbations respecting this relation are called adiabatic. If the first of these relations is not respected, then the abundances of different species within a Hubble patch are not those of a background universe in local equilibrium, and the perturbations are non-adiabatic. However, through micro-physics, non-adiabatic perturbations should decay with time, in such a way that local equilibrium is reached within a patch. Of course, we can repeat the argument for other perturbed quantities such as the pressure, temperature, etc. That is:

$$\frac{\delta\rho_a(\mathbf{x},t)}{\dot{\bar{\rho}}_a(t)} = \frac{\delta p_a(\mathbf{x},t)}{\dot{\bar{p}}_a(t)} = \frac{\delta T(\mathbf{x},t)}{\dot{\bar{T}}} = \delta t(\mathbf{x},t)\,, \quad \forall a \tag{11}$$

Thus, at very early times, and particularly at a time $t_{ini}$, all perturbations must be determined by a single perturbation $\delta t(\mathbf{x},t)$.

## 8.4. Primordial curvature perturbation and some properties

Let us now attempt to identify the perturbation $\delta t(\mathbf{x},t)$ with a specific gauge invariant perturbation. Notice that we can write eq. (10) in terms od the dimensionless perturbations as $\frac{3}{4}\delta_\gamma = \frac{3}{4}\delta_\nu = \delta_{DM} = \delta_B = 3H\delta t$. At very early times, we have $R = \frac{3\bar{\rho}_B}{4\bar{\rho}_\gamma} << 1$, $a(t) = \sqrt{t}$ and $H = 1/2t$. Therefore, in Fourier space, taking $k/a \to 0$, eq. (7) imply:

$$\frac{3}{4}\dot{\tilde{\delta}}_\gamma = \frac{3}{4}\dot{\tilde{\delta}}_\nu = \dot{\tilde{\delta}}_{DM} = \dot{\tilde{\delta}}_B = -\tilde{\psi} \quad \Rightarrow \quad \frac{\partial}{\partial t}\left(t\dot{\tilde{\delta}}_\gamma\right) - \frac{1}{t}\tilde{\delta}_\gamma = 0 \tag{12}$$

where we have used the Friedmann's equation together wit (8) to find the last relation. The solutions to this equation is $\delta_\gamma \propto t^{\pm 1}$. Let's keep the relevant mode (the growing one, which also implies that $\psi$ is independent of time) and denote it as $\psi_0(\mathbf{x})$. Now consider its Fourier transform, it then follows that $\frac{3}{4}\tilde{\delta}_\gamma = \frac{3}{4}\tilde{\delta}_\nu = \tilde{\delta}_{DM} = \tilde{\delta}_B = -t\tilde{\psi}_0(\mathbf{k})$. Equation (7) again implies:

$$\partial_t\left[\frac{1}{t^{1/2}}\delta\tilde{u}_{\gamma B}\right] = \frac{1}{3}\tilde{\psi}_0(\mathbf{k})t^{1/2} = \partial_t\left[\frac{1}{t^{1/2}}\delta\tilde{u}_\nu\right] \quad \Rightarrow \quad \delta\tilde{u}_\gamma = \delta\tilde{u}_B = \delta\tilde{u}_\nu = \frac{2}{9}\tilde{\psi}_0(\mathbf{k})t^2 \tag{13}$$

Recall our definition for the gauge invariant co-moving curvature perturbation $\mathcal{R}$, whose Fourier transform satisfies $\frac{k^2}{a^2}\tilde{\mathcal{R}} = -H\tilde{\psi}_0 + 4\pi G\delta\tilde{\rho} + \frac{k^2}{a^2}H\delta\tilde{u}$. Then, using that at the initial time $t_{ini}$ the universe is radiation dominated, we have $\delta\tilde{\rho} = \bar{\rho}_\gamma\delta_\gamma + \bar{\rho}_\nu\delta_\nu + \bar{\rho}_{DM}\delta_{DM} + \bar{\rho}_B\delta_B \simeq \bar{\rho}_R\delta_\gamma \simeq \bar{\rho}_{tot}\delta_\gamma$, and, as a consequence, we find:

$$\tilde{\psi}_0(\mathbf{k}) = -t\frac{k^2}{a^2}\tilde{\mathcal{R}}(\mathbf{k},t_{ini}) \tag{14}$$

From here, we finally see that initially, every perturbation can be expressed in terms of $\mathcal{R}$ as:

$$\begin{gathered}\frac{3}{4}\tilde{\delta}_\gamma = \frac{3}{4}\tilde{\delta}_\nu = \tilde{\delta}_{DM} = \tilde{\delta}_B = \frac{1}{4H^2}\frac{k^2}{a^2}\tilde{\mathcal{R}}(\mathbf{k},t_{ini})\,,\\ \delta\tilde{u}_{\gamma B} = \delta\tilde{u}_\nu = -\frac{1}{36H^3}\frac{k^2}{a^2}\tilde{\mathcal{R}}(\mathbf{k},t_{ini})\,, \quad \tilde{\psi}_0(\mathbf{k}) = -\frac{1}{2H}\frac{k^2}{a^2}\tilde{\mathcal{R}}(\mathbf{k},t_{ini})\end{gathered} \tag{15}$$

Every higher multipole is taken to vanish initially. From here on, we will omit the initial time and simply write $\tilde{\mathcal{R}}(\mathbf{k})$. In this way, the primordial curvature fluctuation $\mathcal{R}(x)$ evaluated at the initial time is given by the inverse Fourier transform of $\tilde{\mathcal{R}}(\mathbf{k})$. Given the primordial curvature is real, it must be fulfilled that $\tilde{\mathcal{R}}^*(-\mathbf{k}) = \tilde{\mathcal{R}}(\mathbf{k})$.

As a summary, we are now able to express every perturbation in terms of a single perturbation $\mathcal{R}(\mathbf{x})$. Let us talk now about cosmic inflation, a process that predated the hot Big-Bang universe and let to a thermalized universe with adiabatic perturbations satisfying equations (15). This process is just a proposal to understand the origin of $\mathcal{R}$ and other physical mechanisms may have led to the existence of adiabatic perturbations.

Notice that it is not possible to predict a particular realization of our inhomogeneous universe (a particular $\mathcal{R}(\mathbf{x})$), so the curvature must be treated as a stochastic field. Schematically, we can think on the curvature as a collection containing each conceivable realization of the primordial curvature fluctuation (similar to an ensemble). Thus, each time we reproduce the birth of our universe, a particular realization $\mathcal{R}_i(\mathbf{x})$ will come up, and if we were to repeat the experiment a number of times $N >> N_{tot}$, some particular realizations would be found to be more frequent than others. For instance, we have the average value $\langle\mathcal{R}(\mathbf{x})\rangle = \frac{1}{N}\sum_{i=1}^{N_{tot}} n_i\mathcal{R}_i(\mathbf{x})$, where $n_i$ is the number of time that a particular realization $i$ appeared, and so $n_i/N$ is the probability assigned to that particular realization. On the other hand, as the metric must coincide with the background one, then require us to impose $\langle\mathcal{R}(\mathbf{x})\rangle = 0$. However, higher correlation functions are not forced to satisfy this conditions. To deal with non-numerable number of realizations we have to sum using functional integrals.

Let us denote the density functional as $\rho[\mathcal{R}(\mathbf{x})]$, so, an arbitrary $n$-point correlation function is then computed as:

$$\langle\mathcal{R}(\mathbf{x}_1)...\mathcal{R}(\mathbf{x}_n)\rangle = \int D\mathcal{R}(\mathbf{x})\rho[\mathcal{R}(\mathbf{x})]\mathcal{R}(\mathbf{x}_1)\cdots\mathcal{R}(\mathbf{x}_n) \tag{16}$$

Typically, it is convenient to introduce a generator functional $Z[J(x)]$ defined as:

$$Z[J(x)] = \int D\mathcal{R}(\mathbf{x})\rho[\mathcal{R}(\mathbf{x})]e^{\int_{\mathbf{x}} J(\mathbf{x})\mathcal{R}(\mathbf{x})} \Rightarrow \langle\mathcal{R}(\mathbf{x}_1)...\mathcal{R}(\mathbf{x}_n)\rangle = \left(\frac{\delta}{\delta J(\mathbf{x}_1)}...\frac{\delta}{\delta J(\mathbf{x}_n)}Z[J(x)]\right)_{J=0} \tag{17}$$

While each realization of $\mathcal{R}(\mathbf{x})$ depends on $\mathbf{x}$, we request that averages of the form $\langle\mathcal{R}(\mathbf{x}_1)...\mathcal{R}(\mathbf{x}_n)\rangle$ must be equal to one with the arguments evaluated for some arbitrary displacement $\mathbf{b}$. In the particular case of two-point correlation functions this condition implies that $\langle\mathcal{R}(\mathbf{x})\mathcal{R}(\mathbf{y})\rangle$ must be a function of the difference **x**-**y**. Also, it can not change under rotations (average are isotropic). Thus, we can write:

$$\langle\mathcal{R}(\mathbf{x})\mathcal{R}(\mathbf{y})\rangle = \langle\mathcal{R}\mathcal{R}\rangle(|\mathbf{x}-\mathbf{y}|) \Rightarrow \left\langle\tilde{\mathcal{R}}(\mathbf{k})\tilde{\mathcal{R}}(\mathbf{q})\right\rangle \equiv (2\pi)^3\delta^3(\mathbf{k}+\mathbf{q})P_{\mathcal{R}}(k) \tag{18}$$

where $P_{\mathcal{R}}(k)$ is called the power spectrum. It follows that:

$$\langle\mathcal{R}\mathcal{R}\rangle(|\mathbf{x}-\mathbf{y}|) = \int_{\mathbf{k}} P_{\mathcal{R}}(k)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \tag{19}$$

### 8.4.1. Gaussianity

A simple choice for the functional density described before is the form of Gaussian statistics. So:

$$\rho[\mathcal{R}(\mathbf{x})] = \mathcal{N}exp\left[-\frac{1}{2}\int_{\mathbf{x}}\int_{\mathbf{y}}\mathcal{R}(\mathbf{x})\Sigma^{-1}(\mathbf{x},\mathbf{y})\mathcal{R}(\mathbf{y})\right] \tag{20}$$

where we have defined the covariance matrix to be $\Sigma^{-1}(\mathbf{x},\mathbf{y}) \equiv \int_{\mathbf{k}}\frac{exp(i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}))}{P_{\mathcal{R}}(k)}$ and we have choose a normalization constant $\mathcal{N}$ such as $\int D\mathcal{R}(\mathbf{x})\rho[\mathcal{R}(\mathbf{x})] = 1$. Then, the generator $Z[J(x)]$ becomes:

$$Z[J(\mathbf{x})] = e^{-\frac{1}{2}\int_{\mathbf{x}}\int_{\mathbf{y}} J(\mathbf{x})J(\mathbf{y})\langle\mathcal{R}\mathcal{R}\rangle(|\mathbf{x}-\mathbf{y}|)} \tag{21}$$

Then, it directly follow that:

$$\langle\mathcal{R}_1\mathcal{R}_2...\mathcal{R}_n\rangle = \begin{cases}\langle\mathcal{R}_1\mathcal{R}_2\rangle\cdots\langle\mathcal{R}_{n-1}\mathcal{R}_n\rangle + perms. & \text{if } n \text{ is even}\\ 0 & \text{if } n \text{ is odd}\end{cases} \tag{22}$$

That is, every correlation function consist of the product of every possible permutation of two point correlation functions.

### 8.4.2. Scale invariance

As we have already seen, the distribution of $\mathcal{R}(\mathbf{x})$ is parametrized by the power spectrum $P_{\mathcal{R}}(k)$, a single function of the scale $k$. Given that $P_{\mathcal{R}}(k)$ has dimension of volume, we can rewrite it in terms of the so called dimensionless power spectrum $\Delta_{\mathbf{R}}(k)$ as:

$$P_{\mathcal{R}}(k) \equiv \frac{2\pi^2}{k^3}\Delta_{\mathbf{R}}(k) \tag{23}$$

The simplest choice of the shape of the dimensionless power spectrum consists of a scale invariant function, this severely restricts $\Delta_{\mathbf{R}}(k)$ to be a constant. Another less restrictive choice consists of parametrizing small deviations of scale invariance as:

$$\Delta_{\mathbf{R}}(k) = A_{\mathcal{R}}\left(\frac{k}{k_*}\right)^{n_s-1} \tag{24}$$

where $A_{\mathcal{R}}$ and $n_s$ are the amplitude and spectral index of the primordial power spectrum, respectively. The scale $k_*$ is an arbitrary pivot scale that must be chosen in order to constrain $\Delta_{\mathbf{R}}(k)$ observationally.