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Preliminary Insights into Optimal Pricing and Space Allocation at Intermodal Terminals with Elastic Arrivals and Capacity Constraint

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Abstract This paper discusses derivations, and implications of, formulae to compute optimal space allocation and pricing for storage at container terminals. The case discussed in the paper considers elastic arrivals and container dwelling times, which is a more general version of the case considered by the authors in their first publication on the subject. In general terms, the optimal prices have three components that capture the different facets of the process. The first element captures the combined effect of willingness to pay and marginal cost, i.e., the classic solution after Ramsey (1927). The second element represents the contribution of the capacity constraint, i.e., the element introduced in Holguín-Veras and Jara-Díaz (1999). The third element captures the role of elastic arrivals and represents the main contribution of this paper. The role of the cost structure is also analyzed.

Keywords Optimal pricing · Price differentiation · Container transportation

1. Introduction

The intermodal container transportation industry has come a long way since the inception of containers in 1956. In the last decades, a number of trends have spurred the development and implementation of highly sophisticated operations, encompassing multiple transportation modes in global networks. The most

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important trends to the purposes of this paper are those related to the increasing acceptance and the corresponding implementation of service and price differentiation. In this context, intermodal freight operations have evolved from being solely concerned with operational efficiency to a situation in which the industry recognizes and, more importantly, it is willing to implement different types of services tailored to customer needs. In addition to the service differentiation trends, the intermodal container system faces another challenge: increasing ship sizes. The increased proportion of large containerships poses a great challenge to both the marine container terminals and the intermodal container system as a whole. This challenge arises from the operational difficulties of loading and unloading the increasingly large number of containers carried by these vessels. In this context, container terminal operators have to optimize the use of scarce land and expensive handling equipment, while the intermodal system needs to find a way to absorb the pulse in demand that is created by the arrival at port of the mega-containerships. All of this means that the intermodal container transportation system will be expected to keep providing, and expand, service differentiation programs in a context of increasing ship sizes and land constraints. This major challenge provides the background for this paper.

In Holguín-Veras and Jara-Díaz (1999), the authors derived formulae to obtain optimal prices for priority service at intermodal terminals. The optimal prices, representing a general case to the traditional solution obtained by Ramsey (1927), extended price differentiation theory in a number of different ways. First, the resulting models explicitly illustrated the role played by the capacity constraint. Second, the models demonstrated the role of opportunity costs. Third, the formulation presented enabled the simultaneous solution of the space allocation and the optimal storage pricing problem, thus ensuring conceptual and analytical consistency between these two problems. The objective of this paper is to expand the line of work initiated by Holguín-Veras and Jara-Díaz (1999) by considering optimal pricing in the context of an elastic number of arrivals, as opposed to the original assumption of constant arrival of containers. The knowledge gained through the examination of these results will shed light into optimal pricing at container terminals, and will enable terminal managers to examine the conceptual consistency of the pricing schemes they have implemented. The paper is organized as follows. Section 2 provides the general analysis framework. Section 3 presents the analytical derivations and Section 4 the conclusions.

2. Analytical Framework and Assumptions

The analytical framework used in this paper expands the one developed by Holguín-Veras and Jara-Díaz (1999). As before, the intermodal terminal is assumed to have a known and fixed capacity, N, measured in number of spaces. The space allocation is determined by the way in which the available number of spaces are allocated to the different container classes, N_i. The optimal space allocation, N_i^*, is the one associated with the maximization of the pricing rule. The number of arrivals (containers) in each class, I_i, and the corresponding dwelling times, Q_i, are assumed to be a function of the storage charges, P_i, as there is no reason to consider cross elasticities. As shown in Holguín-Veras and Jara-Díaz

(1999), logistic opportunity costs can be easily considered. However, in order to maintain the focus on the elastic arrivals, they will not be considered in this paper. An asterisk as a super index is used to denote optimal values. The complete notation used in the paper follows.

Notation:

W = Total welfare;
CS = Consumer surplus;
Π = Producer surplus;
$\mathbf{I}$ = Vector of arrival rates, I_i;
$I = \sum_i I_i$ = Total arrival rate;
$C(I, Q)$ = Total production cost;
$\eta_{Q_i} = \frac{P_i}{Q_i}\frac{\partial Q_i}{\partial P_i}$ = Elasticity of dwell time;
$\eta_{I_i} = \frac{P_i}{I_i}\frac{\partial I_i}{\partial P_i}$ = Elasticity of arrivals;
$\mathbf{P}$ = Vector of prices P_i, i.e., price per unit storage time;
$\mathbf{Q}$ = Vector of dwelling times Q_i;
$\mathbf{N}$ = Vector of space allocation N_i;
N_i = Spaces allocated to class i;
N = Total number of available spaces;
H_i = Average stack height for class i;
$\omega_{IQ_j} = \frac{\eta_{I_j}}{\eta_{Q_j}}$ = Tradeoff ratio between the elasticities of arrival and dwell times

In order for the intermodal terminal not to overflow, the output rates—determined by the space allocated to each class, N_i, the average stack height for container class i, H_i, and the corresponding dwelling times, Q_i—must be greater than or equal to the corresponding arrival rates, I_i (Holguín-Veras and Jara-Díaz, 1999). For a known and constant total number of spaces, N, the capacity constraint becomes:

$$N \geq \sum_i \frac{I_i Q_i}{H_i}, \forall_i \tag{1}$$

Denoting the pricing rule by the symbol $Z(\mathbf{P})$, the optimization problem consists of obtaining the vector of optimal prices, $\mathbf{P}^*$, that maximizes $Z(\mathbf{P})$ subject to the system constraints. Mathematically:

$$MAX\, Z(\mathbf{P}) \tag{2}$$

subject to:

$$N \geq \sum_i \frac{I_i Q_i}{H_i}, \forall_i \tag{3}$$

$$P_j \geq 0, \forall j \tag{4}$$

As indicated in Holguín-Veras and Jara-Díaz (1999), the optimal space allocation is:

$$N_i^* = \frac{I_i^* Q_i^*}{H_i} \tag{5}$$

The objective functions $Z(\mathbf{P})$ considered in this paper are:

$$\prod = \sum_i I_i P_i Q_i - C \quad \text{(Profit maximization)} \tag{6}$$

$$W = \sum_i \int_{P_i}^{\infty} I_i Q_i dP_i + \sum_i I_i P_i Q_i - C \quad \text{(Welfare maximization)} \tag{7}$$

The next section presents the solutions obtained for the different pricing problems.

3. Optimal Prices

In this section, the optimal prices corresponding the major pricing rules are derived. The general solutions obtained here were found using Karush-Kuhn-Tucker (KKT) conditions (Karush, 1939; Kuhn and Tucker, 1951).

3.1. Optimal Prices Under Profit Maximization

In profit maximization, the main objective is to maximize producer's profits, disregarding the effects on users' welfare. In this case, the problem is to obtain the set of prices, $\mathbf{P}^*$, that maximizes the profits, Π, subject to the capacity constraint. In mathematical terms:

$$MAX\ \Pi = \sum_i I_i P_i Q_i - C \tag{8}$$

subject to:

$$g(P) = N - \sum_i \frac{I_i Qi}{H_i} \geq 0, \forall i \tag{9}$$

Applying the first KKT condition:

$$\frac{\partial \Pi}{\partial P_j} - \lambda \frac{\partial g}{\partial P_j} \leq 0, \forall j \tag{10}$$

Where λ is the multiplier of the capacity constraint.

$$\frac{\partial}{\partial P_j}\left\{\sum_i I_i P_i Q_i - C\right\} - \lambda \frac{\partial}{\partial P_j}\left\{\sum_i \frac{I_i Q_i}{H_i} - N\right\} \leq 0 \tag{11}$$

$$I_j P_j \frac{\partial Q_j}{\partial P_j} + I_j Q_j + P_j Q_j \frac{\partial I_j}{\partial P_j} - \frac{\partial C}{\partial P_j} - \lambda\left\{\frac{I_j}{H_j}\frac{\partial Q_j}{\partial P_j} + \frac{Q_j}{H_j}\frac{\partial I_j}{\partial P_j}\right\} \leq 0 \tag{12}$$

Note that $\frac{\partial C}{\partial P_j} = \frac{\partial C_j}{\partial I_j}\frac{\partial I_j}{\partial P_j} + \frac{\partial C_j}{\partial Q_j}\frac{\partial Q_j}{\partial P_j}$ (because C is a function of I_j and Q_j). Dividing by $I_j \frac{\partial Q_j}{\partial P_j}$ and grouping terms:

$$\left[I_j P_j \frac{\partial Q_j}{\partial P_j} + I_j Q_j + P_j Q_j \frac{\partial I_j}{\partial P_j} - \frac{\partial C_j}{\partial I_j}\frac{\partial I_j}{\partial P_j} - \frac{\partial C_j}{\partial Q_j}\frac{\partial Q_j}{\partial P_j} - \lambda \left\{ \frac{I_j}{H_j}\frac{\partial Q_i}{\partial P_j} + \frac{Q_j}{H_j}\frac{\partial I_j}{\partial P_j} \right\} \right] \Big/ I_j \frac{\partial Q_j}{\partial P_j} \leq 0 \tag{13}$$

Letting

$$\omega_{IQ_j} \equiv \frac{Q_j}{I_j}\frac{\partial I_j}{\partial P_j} \Big/ \frac{\partial Q_j}{\partial P_j} = \left\{ \frac{P_j}{I_j}\frac{\partial I_j}{\partial P_j} \right\} \Big/ \left\{ \frac{P_j}{Q_j}\frac{\partial Q_j}{\partial P_j} \right\} = \frac{\eta^P_{I_j}}{\eta^P_{Q_j}}; \quad m^Q_j = \frac{\partial C}{\partial Q_j};$$

$$\text{and } m^I_j = \frac{\partial C}{I_j} P_j - P_j \frac{1}{|\eta_{Q_j}|} + P_j \omega_{IQ_j} - \frac{m^Q_j}{I_j} - \frac{m^I_j}{Q_j}\omega_{IQ_j} - \frac{\lambda}{H_j}\left(1 + \omega_{IQ_j}\right) \leq 0 \tag{14}$$

The complementary slackness condition becomes:

$$P^*_j \left(P^*_j - P^*_j \frac{1}{|\eta_{Q_j}|} + P^*_j \omega_{IQ_j} - \frac{m^Q_j}{I_j} - \frac{m^I_j}{Q_j}\omega_{IQ_j} - \frac{\lambda}{H_j}\left(1 + \omega_{IQ_j}\right) \right) = 0 \tag{15}$$

The optimal prices P^*_j are derived from the complementary slackness condition, (valid for $P^*_j > 0$). To facilitate interpretation, the results are presented in the traditional format of price differentiation theory. From Eq. (15):

$$P^*_j - \frac{m^Q_j}{I_j} - \frac{m^I_j}{Q_j}\omega_{IQ_j} = P^*_j \frac{1}{|\eta_{Q_j}|} - P^*_j \omega_{IQ_j} + \frac{\lambda}{H_j} + \frac{\lambda}{H_j}\omega_{IQ_j} \tag{16}$$

$$\frac{P^*_j - \frac{m^Q_j}{I_j} - \frac{m^I_j}{Q_j}\omega_{IQ_j}}{P^*_j} = \frac{1}{|\eta_{Q_j}|} + \frac{\lambda}{H_j P^*_j} + \frac{\lambda}{H_j P^*_j}\omega_{IQ_j} - \omega_{IQ_j} \tag{17}$$

$$\frac{P^*_j - \frac{m^Q_j}{I_j} - \frac{m^I_j}{Q_j}\omega_{IQ_j}}{P^*_j} = \frac{1}{|\eta_{Q_j}|} + \frac{\lambda}{H_j P^*_j} - \left(1 - \frac{\lambda}{H_j P^*_j}\right)\omega_{IQ_j} \tag{18}$$

Equation (18) shows a number of interesting features. As it can be seen, the left hand side represents the markup ratio (see further discussion in Section 4), and it happens to be a function of three terms. The first term represents the contribution of the price elasticity of dwell time, resembling the traditional solution obtained by Ramsey (1927). The second term reflects the contribution of the active capacity constraint ($\lambda > 0$), factored down by the contribution to gross revenues, $H_j P^*_j$. The third term captures the effect of elastic arrivals. Since the role of the first two terms was discussed in depth in Holguín-Veras and Jara-Díaz (1999), the analysis focuses on the third term, which is one of the new elements introduced in this paper.

The third term in Eq. (18) has some intriguing features. Most notably, it could either reduce or increase the markup depending upon the sign taken by $\left\{1 - \frac{\lambda}{H_j P_j^*}\right\}$, (as opposed to the contributions of the first two terms that are always positive). This new element can be rewritten as a ratio, where the numerator is the difference between $H_j P_j$ and λ, and the denominator is $H_j P_j$. The numerator represents the amount the port would get in excess of the opportunity cost of the land on a per unit time basis, if a container of class j is stacked to its maximum height. Since this is divided by $H_j P_j$, it is referred to as *marginal profitability*. Positive (negative) values of the marginal profitability indicate gross revenues larger (smaller) than opportunity cost of land. In a capacity constrained situation, container classes with negative marginal profitability will be charged more. Conversely, storage charges for container classes with positive marginal profitability will be reduced accordingly.

Another interesting feature is related to the role played by the tradeoff ratio, ω_{IQ_j}, that as can be seen multiplies the marginal profitability. The tradeoff ratio, defined as the relative change in the rate of arrival with respect to the relative rate of change of dwelling times (or alternatively the ratio of the price elasticities of arrival rates and dwelling times), measures the relative response of arrival rates and dwelling times to price. As can be seen, while the sign of the third term is determined by the marginal profitability, the total contribution is jointly determined with the tradeoff ratio ω_{IQ_j}. This multiplier effect is more significant for user classes with $\omega_{IQ_j} > 1$ (arrivals more elastic than dwelling times). The net effect of the third term is to attract user classes with positive marginal profitability and $\omega_{IQ_j} > 1$, while rejecting user classes with negative profitability and $\omega_{IQ_j} > 1$. This result captures the essence of the optimal storage pricing policies under capacity constraint and elastic arrivals, that entails the use of pricing to attract user (container) classes with positive marginal profitability, while discouraging the others.

3.2. Welfare Maximizing Prices

This case is concerned with obtaining the set of prices, $\mathbf{P}^*$, that maximizes welfare, W, subject to the system constraints. In this context, the resulting prices will take into account the welfare of both producers and consumers. The corresponding optimization problem is:

$$MAX\ W = \sum_i \int_{P_i}^{\infty} I_i Q_i dP_i + \sum_i I_i P_i Q_i - C \tag{19}$$

subject to:

$$g(P) = N - \sum_i \frac{I_i Q_i}{H_i} \geq 0, \forall i \tag{20}$$

Applying the first KKT condition:

$$\frac{\partial W}{\partial P_j} + \lambda \frac{\partial g}{\partial P_j} \leq 0, \forall j \tag{21}$$

$$\frac{\partial}{\partial P_j}\left\{\sum_i \int_{P_i}^{\infty} I_i Q_i dP_i + \sum_i I_i P_i Q_i - C\right\} - \lambda \frac{\partial}{\partial P_j}\left\{\sum_i \frac{I_i Q_j}{H_i} - N\right\} \leq 0 \tag{22}$$

which after some manipulation yields:

$$\left[I_j Q_j\right]_{P_j}^{\infty} + I_j Q_j + I_j P_j \frac{\partial Q_j}{\partial P_j} + Q_j P_j \frac{\partial I_j}{\partial P_j} - \frac{\partial C}{\partial P_j} - \lambda\left\{\frac{I_j}{H_j}\frac{\partial Q_j}{\partial P_j} + \frac{Q_j}{H_j}\frac{\partial I_j}{\partial P_j}\right\} \leq 0 \tag{23}$$

$$-I_j Q_j + I_j Q_j + I_j P_j \frac{\partial Q_j}{\partial P_j} + Q_j P_j \frac{\partial I_j}{\partial P_j} - \frac{\partial C}{\partial P_j} - \frac{\lambda}{H_j}\left\{I_j \frac{\partial Q_j}{\partial P_j} + Q_j \frac{\partial I_j}{\partial P_j}\right\} \leq 0 \tag{24}$$

Since $\frac{\partial C}{\partial P_j} = \frac{\partial C_j}{\partial I_j}\frac{\partial I_j}{\partial P_j} + \frac{\partial C_j}{\partial Q_j}\frac{\partial Q_j}{\partial P_j}$ (because C is a function of I_j and Q_j):

$$I_j P_j \frac{\partial Q_j}{\partial P_j} + Q_j P_j \frac{\partial I_j}{\partial P_j} - \frac{\partial C_j}{\partial I_j}\frac{\partial I_j}{\partial P_j} - \frac{\partial C_j}{\partial Q_j}\frac{\partial Q_j}{\partial P_j} - \frac{\lambda}{H_j}\left\{I_j \frac{\partial Q_j}{\partial P_j} + Q_j \frac{\partial I_j}{\partial P_j}\right\} \leq 0 \tag{25}$$

Dividing by $I_j \frac{\partial Q_j}{\partial P_j}$ and grouping terms:

$$P_j + P_j \frac{Q_j}{I_j}\frac{\partial I_j}{\partial P_j}\frac{\partial P_j}{\partial Q_j} - \frac{1}{I_j}\frac{\partial C_j}{\partial I_j}\frac{\partial I_j}{\partial P_j}\frac{\partial P_j}{\partial Q_j} - \frac{1}{I_j}\frac{\partial C_j}{\partial Q_j}\frac{\partial Q_j}{\partial P_j}\frac{\partial P_j}{\partial Q_j} - \frac{\lambda}{H_j} - \frac{\lambda}{H_j}\frac{Q_j}{I_j}\frac{\partial I_j}{\partial P_j}\frac{\partial P_j}{\partial Q_j} \leq 0 \tag{26}$$

Since $\omega_{IQ_j} \equiv \frac{Q_j}{I_j}\frac{\partial I_j}{\partial P_j}\Big/\frac{\partial Q_j}{\partial P_j} = \left\{\frac{P_j}{I_j}\frac{\partial I_j}{\partial P_j}\right\}\Big/\left\{\frac{P_j}{Q_j}\frac{\partial Q_j}{\partial P_j}\right\} = \frac{\eta_{I_j}^P}{\eta_{Q_j}^P}$; $\quad m_j^Q = \frac{\partial C}{\partial Q_j}$; and $m_j^I = \frac{\partial C}{\partial I_j}$

$$P_j + P_j \omega_{IQ_j} - \frac{m_j^Q}{I_j} - \frac{m_j^I}{Q_j}\omega_{IQ_j} - \frac{\lambda}{H_j} - \frac{\lambda}{H_j}\omega_{IQ_j} \leq 0 \tag{27}$$

$$P_j\left(1 + \omega_{IQ_j}\right) - \frac{m_j^Q}{I_j} - \frac{m_j^I}{Q_j}\omega_{IQ_j} - \frac{\lambda}{H_j}\left(1 + \omega_{IQ_j}\right) \leq 0 \tag{28}$$

From the complementary slackness condition:

$$P_j^*\left(P_j^*\left(1 + \omega_{IQ_j}\right) - \frac{m_j^Q}{I_j} - \frac{m_j^I}{Q_j}\omega_{IQ_j} - \frac{\lambda}{H_j}\left(1 + \omega_{IQ_j}\right)\right) = 0 \tag{29}$$

Since $P_j^* > 0$

$$P_j^*\left(1 + \omega_{IQ_j}\right) - \frac{m_j^Q}{I_j} - \frac{m_j^I}{Q_j}\omega_{IQ_j} - \frac{\lambda}{H_j}\left(1 + \omega_{IQ_j}\right) = 0 \tag{30}$$

Finally:

$$P_j^* = \frac{\frac{m_j^Q}{I_j} + \frac{m_j^I}{Q_j}\omega_{IQ_j}}{\left(1 + \omega_{IQ_j}\right)} + \frac{\lambda}{H_j} \tag{31}$$

The optimal prices derived are consistent with the ones found by Holguín-Veras and Jara-Díaz (1999). As before, welfare maximizing prices are a function of the marginal costs for each class and the Lagrange multiplier, λ, which is factored down by the average stack height. As can be seen, the welfare maximizing prices—in the context of a capacity constrained process—have an additional term that ensures satisfaction of the capacity constraint.

The above result indicates that if the capacity constraint is not binding ($\lambda = 0$) and the arrivals are inelastic to price ($\omega_{IQ_j} = 0$), the optimal prices reduce to the ones estimated by traditional welfare economics, i.e., optimal price equals marginal cost. If the capacity constraint is binding ($\lambda > 0$) and the arrivals are inelastic to price ($\omega_{IQ_j} = 0$) the optimal prices correspond to the solution obtained by Holguín-Veras and Jara-Díaz (1999). As seen, when $\omega_{IQ_j} \geq 0$, the net effect of elastic arrivals is to reduce the contribution of marginal costs to optimal prices. For that reason, in equality of conditions, container classes with elastic arrival rates and inelastic dwelling times, would pay much less than container classes with inelastic arrival rates and elastic dwelling times. If this discount is not given, user classes with elastic arrivals may opt simply not to call at the port. In essence, this result represents another dimension of the Inverse Elasticity Rule (IER).

3.3. Second Best Pricing

Second best pricing, also known as *constrained welfare maximization*, is concerned with the calculation of prices that maximize welfare, subject to the constraint that gross revenues equal total production cost. This formulation considers social welfare, while ensuring that production cost are met by the gross revenues, thus avoiding the need for unwanted subsidies.

Thus, the optimization problem is:

$$MAX\ W = \sum_i \int_{P_i}^{\infty} I_i Q_i dP_i + \sum_i I_i P_i Q_i - C \tag{32}$$

subject to the revenue and capacity constraints:

$$g_1(P) = \sum_i I_i P_i Q_i \ - \ C \ \geq \ 0, \forall i \tag{33}$$

$$g_2(P) = N - \sum_i \frac{I_i Q_i}{H_i} \geq 0, \forall i \tag{34}$$

Applying the first KKT condition:

$$\frac{\partial W}{\partial P_j} \ + \ \lambda_1 \frac{\partial g_1}{\partial P_j} \ + \ \lambda_2 \frac{\partial g_2}{\partial P_j} \ \leq \ 0, \forall j \tag{35}$$

where λ_1 and λ_2 are the multipliers of the corresponding constraints.

$$\frac{\partial}{\partial P_j}\left\{\sum_i \int_{P_i}^{\infty} I_i Q_i dP_i + \sum_i I_i P_i Q_i - C\right\} - \lambda_1 \frac{\partial}{\partial P_j}\left\{C - \sum_i I_i P_i Q_i\right\} - \lambda_2 \frac{\partial}{\partial P_j}\left\{\sum_i \frac{I_i Q_i}{H_i} - N\right\} \le 0 \tag{36}$$

which after some manipulation yields:

$$\left[I_j Q_j\right]_{P_j}^{\infty} + I_j Q_j + I_j P_j \frac{\partial Q_j}{\partial P_j} + Q_j P_j \frac{\partial I_j}{\partial P_j} - \frac{\partial C}{\partial P_j} - \lambda_1\left\{\frac{\partial C}{\partial P_j} - I_j Q_j - I_j P_j \frac{\partial Q_j}{\partial P_j} - P_j Q_j \frac{\partial I_j}{\partial P_j}\right\} - \lambda_2\left\{\frac{I_j}{H_j}\frac{\partial Q_j}{\partial P_j} + \frac{Q_j}{H_j}\frac{\partial I_j}{\partial P_j}\right\} \le 0 \tag{37}$$

Grouping terms:

$$I_j P_j \frac{\partial Q_j}{\partial P_j}(1+\lambda_1) + P_j Q_j \frac{\partial I_j}{\partial P_j}(1+\lambda_1) - \frac{\partial C_j}{\partial P_j}(1+\lambda_1) + \lambda_1 I_j Q_j - \lambda_2 \frac{I_j}{H_j}\frac{\partial Q_j}{\partial P_j} - \lambda_2 \frac{Q_j}{H_j}\frac{\partial I_j}{\partial P_j} \le 0 \tag{38}$$

Dividing by $I_j \frac{\partial Q_j}{\partial P_j}$ and grouping terms:

$$P_j(1+\lambda_1) + P_j \frac{Q_j}{I_j}\frac{\partial I_j}{\partial P_j}\frac{\partial P_j}{\partial Q_j}(1+\lambda_1) - \frac{1}{I_j}\frac{\partial C}{\partial P_j}\frac{\partial P_j}{\partial Q_j}(1+\lambda_1) + \lambda_1 Q_1 \frac{\partial P_j}{\partial Q_j} - \frac{\lambda_2}{H_j} - \frac{\lambda_2}{H_j}\frac{Q_j}{I_j}\frac{\partial I_j}{\partial P_j}\frac{\partial P_j}{\partial Q_j} \le 0 \tag{39}$$

Since $\frac{\partial C}{\partial P_j} = \frac{\partial C_j}{\partial I_j}\frac{\partial I_j}{\partial P_j} + \frac{\partial C_j}{\partial Q_j}\frac{\partial Q_j}{\partial P_j}$:

$$P_j(1+\lambda_1) + P_j \frac{Q_j}{I_j}\frac{\partial I_j}{\partial P_j}\frac{\partial P_j}{\partial Q_j}(1+\lambda_1) - \frac{1}{I_j}\frac{\partial C_j}{\partial I_j}\frac{\partial I_j}{\partial P_j}\frac{\partial P_j}{\partial Q_j}(1+\lambda_1) - \frac{1}{I_j}\frac{\partial C_j}{\partial Q_j}\frac{\partial Q_j}{\partial P_j}\frac{\partial P_j}{\partial Q_j}(1+\lambda_1) + \lambda_1 Q_j \frac{\partial P_j}{\partial Q_j} - \frac{\lambda_2}{H_j} - \frac{\lambda_2}{H_j}\frac{Q_j}{I_j}\frac{\partial I_j}{\partial P_j}\frac{\partial P_j}{\partial Q_j} \le 0 \tag{40}$$

Since $\omega_{IQ_j} = \frac{Q_j}{I_j}\frac{\partial I_j}{\partial P_j}\frac{\partial P_j}{\partial Q_j} = \left\{\frac{P_j}{I_j}\frac{\partial I_j}{\partial P_j}\right\}\left\{\frac{Q_j}{P_j}\frac{\partial P_j}{\partial Q_j}\right\} = \frac{\eta_{I_j}^P}{\eta_{Q_j}^P}$; $\quad m_j^Q = \frac{\partial C}{\partial Q_j}$; and $\quad m_j^I = \frac{\partial C}{\partial I_j}$

$$P_j(1+\lambda_1) + P_j \omega_{IQ_j}(1+\lambda_1) - \frac{m_j^I}{Q_j}\omega_{IQ_j}(1+\lambda_1) - \frac{m_j^Q}{I_j}(1+\lambda_1) - \lambda_1 P_j \frac{1}{\left|\eta_{Q_j}\right|} - \frac{\lambda_2}{H_j}\left(1+\omega_{IQ_j}\right) \le 0 \tag{41}$$

Finally, from the complementary slackness condition:

$$P_j^*\left\{P_j^*(1+\lambda_1)+P_j^*\omega_{IQ_j}(1+\lambda_1)-\left(\frac{m_j^I}{Q_j}\omega_{IQ_j}+\frac{m_j^Q}{I_j}\right)(1+\lambda_1)\right.$$
$$\left.-\lambda_1 P_j^*\frac{1}{\left|\eta_{Q_j}\right|}-\frac{\lambda_2}{H_j}\left(1+\omega_{IQ_j}\right)\right\}=0 \tag{42}$$

Dividing by $P_j^*(1+\lambda_1)$:

$$\frac{P_j^*-\frac{m_j^I}{Q_j}\omega_{IQ_j}-\frac{m_j^Q}{I_j}}{P_j^*}=\frac{\lambda_1/(1+\lambda_1)}{\left|\eta_{Q_j}\right|}+\frac{\lambda_2/(1+\lambda_1)}{H_jP_j^*}\left(1+\omega_{IQ_j}\right)-\omega_{IQ_j} \tag{43}$$

where $k=\lambda_1/(1+\lambda_1)$ is the Ramsey number.

$$\frac{P_j^*-\frac{m_j^I}{Q_j}\omega_{IQ_j}-\frac{m_j^Q}{I_j}}{P_j^*}=\frac{k}{\left|\eta_{Q_j}\right|}+\frac{\lambda_2/(1+\lambda_1)}{H_jP_j^*}\left(1+\omega_{IQ_j}\right)-\omega_{IQ_j} \tag{44}$$

or, alternatively,

$$\frac{P_j^*-\frac{m_j^I}{Q_j}\omega_{IQ_j}-\frac{m_j^Q}{I_j}}{P_j^*}=\frac{k}{\left|\eta_{Q_j}\right|}+\frac{\lambda_2/(1+\lambda_1)}{H_jP_j^*}-\left(1-\frac{\lambda_2/(1+\lambda_1)}{H_jP_j^*}\right)\omega_{IQ_j} \tag{45}$$

The right hand side of Eq. (45) has three terms. The first term resembles the classical result obtained by Ramsey (1927), that expresses the contribution of the budget constraint and price elasticities to the markups. The second term captures the contribution of the capacity constraint, first obtained by Holguín-Veras and Jara-Díaz (1999). The third term takes into account the effect of elastic arrivals on optimal prices.

There are a number of different cases, depending upon the values taken by the Lagrange multipliers, λ_1, λ_2, and the tradeoff ratio, ω_{IQ_j}. If the capacity constraint is not binding, $\lambda_2 = 0$, and arrivals are inelastic, $\omega_{IQ_j}=0$, the optimal prices shown in Eq. (45) reduce to the traditional solution for second best pricing. If $\omega_{IQ_j}=0$, corresponding to inelastic arrivals, and the capacity constraint is binding, $\lambda_2 > 0$, the optimal prices reduce to the ones obtained by Holguín-Veras and Jara-Díaz (1999).

The most relevant case to the purposes of this paper is when the Lagrange multipliers, λ_1, λ_2, and the tradeoff ratio, ω_{IQ_j}, are greater than zero. In such a case, the elastic arrivals term could either reduce or increase the markup depending upon the significance of the system constraints vis a vis the gross revenues (as in profit maximization). The marginal profitability, in this case defined as $\left\{1-\frac{\lambda_2}{(1+\lambda_1)H_jP_j^*}\right\}$, determines whether the markup will increase or decrease. The elastic arrivals term will increase the markup when $\lambda_2/(1+\lambda_1) > H_jP_j^*$, i.e., negative marginal profitability, and decrease it otherwise. This contribution is then multiplied by the tradeoff ratio ω_{IQ_j}.

The first case, $\lambda_2/(1+\lambda_1) > H_jP_j^*$, could arise at severely congested intermodal terminals where the cost of land is so high that overwhelms the gross revenues.

Equation (45) also indicates that container classes with $\omega_{IQ_j} > 1$, i.e., arrivals more sensitive than dwelling times, will be penalized even more, as to discourage them from using the terminal. The second case, $\lambda_2/(1+\lambda_1) < H_j P_j^*$, represents the situation in which the opportunity cost of the land is less than the corresponding gross revenues. In such a case, the net contribution of the third term will be to reduce the corresponding markup.

As it is customary in price differentiation theory, it would be interesting to see whether or not the solutions to the welfare maximization and the profit maximization cases could be derived from the second best prices represented by Eq. (46), in which the Ramsey number k was replaced by its definition $\lambda_1/(1 + \lambda_1)$. The optimal prices for welfare maximization correspond to the particular case when $\lambda_1 = 0$, while the optimal prices for profit maximization correspond to $\lambda_1 \rightarrow \infty$.

$$\frac{P_j^* - \frac{m_j^I}{Q_j}\omega_{IQ_j} - \frac{m_j^Q}{I_j}}{P_j^*} = \frac{\lambda_1/(1+\lambda_1)}{\left|\eta_{Q_j}\right|} + \frac{\lambda_2/(1+\lambda_1)}{H_j P_j^*} - \left(1 - \frac{\lambda_2/(1+\lambda_1)}{H_j P_j^*}\right)\omega_{IQ_j} \tag{46}$$

From Eq. (46), when $\lambda_1 = 0$, the following result is obtained after some manipulations:

$$P_j^* = \frac{\frac{m_j^I}{Q_j}\omega_{IQ_j} + \frac{m_j^Q}{I_j}}{\left(1+\omega_{IQ_j}\right)} + \frac{\lambda_2}{H_j} \tag{47}$$

As shown, Eq. (47) is the same expression found in the paper [see Eq. (31)]. However, when attempting to derive the profit maximizing prices, letting $\lambda_1 \rightarrow \infty$, the following result is obtained:

$$\frac{P_j^* - \frac{m_j^I}{Q_j}\omega_{IQ_j} - \frac{m_j^Q}{I_j}}{P_j^*} = \frac{1}{\left|\eta_{Q_j}\right|} - \omega_{IQ_j} \tag{48}$$

As can be seen, the above result is not consistent with the optimal prices derived from the profit maximization rule, shown in Eq. (18). This indicates that the mathematical linkage among the optimal prices for the three pricing roles does not represent a general result, holding only for the simplest formulations of price differentiation theory (the linkage among optimal prices does not hold either for the optimal prices found in Holguín-Veras and Jara-Díaz (1999).

4. An Extension

An interesting case arises if one assumes that the cost function is of the form outlined in Eq. (49). In this case, one could consider the product IQ to be the demand for space expressed in units of container-time at the port.

$$C(\mathbf{IQ}) = C(I_1Q_1, I_2Q_2 \ldots I_KQ_K) \tag{49}$$

where K is the number of classes.

Letting $D_i = I_i Q_i$ Eq. (49) becomes

$$C(\mathbf{D}) = C(D_1, D_2 \ldots D_K) \tag{50}$$

In this context, the following equalities hold:

$$m_j^Q = \frac{\partial C}{\partial Q_j} = \frac{\partial C}{\partial D_j} I_j \tag{51}$$

$$m_j^I = \frac{\partial C}{\partial I_j} = \frac{\partial C}{\partial D_j} Q_j \tag{52}$$

Thus:

$$m_j = \frac{\partial C}{\partial D_j} = \frac{m_j^Q}{I_j} = \frac{m_j^I}{Q_j} \tag{53}$$

Equation (53) represents the total marginal cost with respect to the total demand in container-unit of time, referred to as m_j. It is interesting to reinterpret the previous findings under the light of Eq. (53). Equations (54) through (56) show the optimal prices for the different pricing rules after substituting Eq. (53).

$$\frac{P_j^* - m_j(1 + \omega_{IQ_j})}{P_j^*} = \frac{1}{|\eta_{Q_j}|} + \frac{\lambda}{H_j P_j^*} - \left(1 - \frac{\lambda}{H_j P_j^*}\right)\omega_{IQ_j}\,(\text{Profit maximization}) \tag{54}$$

$$P_j^* = m_j + \frac{\lambda}{H_j} \quad (\text{Welfare maximization}) \tag{55}$$

$$\frac{P_j^* - m_j(1 + \omega_{IQ_j})}{P_j^*} = \frac{k}{|\eta_{Q_j}|} + \frac{\lambda_2/(1+\lambda_1)}{H_j P_j^*} - \left(1 - \frac{\lambda_2/(1+\lambda_1)}{H_j P_j^*}\right)\omega_{IQ_j}\,(2^{\text{nd}}\ \text{Best pricing}) \tag{56}$$

As shown, under the assumptions of this cost function, if the arrivals are inelastic to price ($\omega_{IQ_j} = 0$) Eqs. (54) and (56) revert back to the ones found in Holguín-Veras and Jara-Díaz (1999). Interestingly enough, the solution for welfare maximization is the same found in Holguín-Veras and Jara-Díaz (1999), regardless of the elasticity of arrivals. In this context, a reasonable assumption on the cost structure, i.e., that the demand for space during a period determines port expenses, permits a clear interpretation and a link to previous results.

In both profit maximization and second best pricing, the tradeoff ratio reduces the markup because ω_{IQ_j} is always positive. As shown in Eqs. (54) and (56), ω_{IQ_j} reduces the values of the optimal prices with respect to the inelastic case, as expected. This is, in essence, an elaborate form of the Inverse Elasticity Rule. In the case of welfare maximization in Eq. (55), the elasticity of arrivals plays no role whatsoever, which is consistent with the classic solutions.

5. Conclusions

This paper analyzed optimal storage pricing for intermodal terminals. The models developed in this research extend price differentiation theory along the line of work initiated by Holguín-Veras and Jara-Díaz (1999), to the case in which the numbers of containers arriving at the terminal are elastic to price.

The optimal prices found represent general solutions to the classic result after Ramsey (1927). The models presented here, although developed in the context of intermodal transportation, are of general applicability to situations in which: (a) there are multiple user classes; (b) there is a capacity constraint; (c) the number of arrivals in each class is elastic to price; and, (d) the amount of capacity utilized by each user class is a function of the variable being priced, e.g., a parking lot.

In general terms, the optimal prices have three components that capture the different facets of the process. The first element captures the combined effect of willingness to pay and marginal cost, i.e., the classic solution after Ramsey (1927). The second element represents the contribution of the capacity constraint, i.e., the element introduced in Holguín-Veras and Jara-Díaz (1999). The third element captures the role of elastic arrivals and represents the main contribution of this paper. Since the first two elements were sufficiently discussed in Holguín-Veras and Jara-Díaz (1999), the conclusions focus on the role of elastic arrivals.

The optimal prices for the different pricing rules are conditioned by parameters depending upon the elasticity of arrivals. It was found that the tradeoff ratio, a parameter that measures the relative rate of change of arrivals and dwelling times to price changes, has a direct impact upon the prices. The possible values taken by the tradeoff ratio gives rise to three cases: $\omega_{IQ_j} = 0$ (inelastic arrivals to price); $0 < \omega_{IQ_j} \leq 1$, i.e., the rate of change of arrivals is smaller than the rate of change of dwelling times with respect to price; and, $\omega_{IQ_j} > 1$, i.e., the rate of change of arrivals is larger than the rate of change of dwelling times with respect to price. The limit case corresponds to perfectly elastic dwelling times, for which $\omega_{IQ_j} \rightarrow \infty$.

The results indicated that, in welfare maximization, the consideration of elastic arrivals translate into optimal prices that are lower that they would be otherwise, because the tradeoff ratio reduces the contribution of the marginal costs. This result represents a new dimension of the Inverse Elasticity Rule (IER), that previously was considered not to play any role in the determination of welfare maximizing prices. However, if the cost structure is a function of the total demand for space (I_jQ_j) for the various user classes, the tradeoff ration plays no role whatsoever and the analytical solutions revert back to the ones found in Holguín-Veras and Jara-Díaz (1999).

In the most general case of both profit maximization and second best pricing, the consideration of elastic arrivals translates into complex interaction terms that could either reduce or increase the mark up, depending upon the sign taken by the parameter defined as *marginal profitability*, i.e., an indicator of the relative values of the opportunity cost of the land vs. the gross revenues. The marginal profitability for the profit maximization case, defined as $\left\{1 - \frac{\lambda}{H_j P_j^*}\right\}$, is positive when the gross benefits associated to that particular user class are larger than the corresponding opportunity cost of land, and negative otherwise. The marginal profitability in second best pricing, defined as $\left\{1 - \frac{\lambda_2}{(1+\lambda_1)\, H_j P_j^*}\right\}$, has similar interpretation. As in profit maximization, positive marginal profitability will reduce the markup, and increase it otherwise. In both pricing rules, the sign of the contribution is determined by the marginal profitability while its absolute value is jointly determined with the tradeoff ratio. However, in the case of a cost structure that is a function of the total time demanded by user class, the markup ratio is clearly lower than the one estimated for the inelastic case.

This paper also opened a set of questions that should be addressed by future research. The nature and characteristics of the marginal cost functions, which are a fundamental input to optimal pricing, also deserves further examination. As shown in the paper, the cost function has an important role in the results. At this moment, very little is known about how operational costs change with increased terminal congestion.

Only after the different components of this analytical puzzle has been put together, the research community will be able to undertake the empirical estimation of the important parameters in optimal pricing, such as the tradeoff ratio, and would be able to closely examine the consistency of the pricing rules implemented at the major container ports.

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