MUSSA: a behavioural land use equilibrium model with location externalities, planning regulations and pricing policies.

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Introduction

MUSSA (Martínez, 1996) was designed to forecast the expected location of agents, residents and firms, in the urban area. The location problem assumes that real estates are allocated to the highest bidder by auctions and market equilibrium is attained by the condition that all agents are located somewhere, therefore, supply satisfies demand. This process produces rents for each real estate and levels of satisfaction (benefits) to located agents. The model is specified by discrete units: land by zones, dwellings by types and households and firms by categories; the number of discrete units is defined by the modeller. Consumers’ agents, households and firms, are rational and their idiosyncratic differences are modelled by a stochastic behaviour.

A significant difference with other land use models is that in MUSSA the interaction between consumer agents – households and firms – is explicitly described in the equilibrium. We call these interactions location externalities, which represent local attributes that depends on the agents’ valuation of neighbor residents and firms’ valuations of agglomeration economies. These interactions makes each agents location choice dependant on all other agents choices, making the calculation of equilibrium a highly complex mathematical problem solved in MUSSA by ad-hoc algorithms. It is worth noting the tremendous dynamic in the land use pattern introduced by location externalities, because each choice affects all other choices and, in theory, the whole location pattern. We shall see however, that in MUSSA these dynamic is as smooth as it is observed in reality, but reflects a real phenomena.

A second key difference is a useful tool for urban planning: the explicit representation of the set of physical constraints (e.g. land available) and planning regulations that supply must comply with. Additionally, the model allows the direct simulation of the effects of pricing incentives (taxes or subsidies) introduced by the modeler. These features provides MUSSA with tools to asses a larger number of issues in urban planning, like the evaluation of regulatory and/or pricing scenarios.

Improvements have been continuously made in the algorithm that calculates equilibrium, exploiting more efficiently the structure of the equations system. This makes current MUSSA not only faster than the previous published versions (Martínez, 1996) but also more precise, while the information about the stability of the solution is more accurate.

The calibration of MUSSA is made by econometric methods, which provides the set of parameters required by functions that describes the behaviour of demand and supply agents. This procedure maximises the likelihood that choices actually made by agents and observed by the modeller, is best reproduced by the set of parameters obtained conditional on the functions specified and the data used. Stated preferences data may also be used. The main advantage of this methodology is that parameters can be defined as mutually consistent, considering correlation dependencies.
A word is worth saying regarding the place that MUSSA has from a theoretical perspective in the context of other land use models. It is an static equilibrium model. Within this class a useful classification from the modeller point of view, is one based on the analytical problem solved by each model. A first generation of land use models was designed under the assumption that agents locates as to minimise the transport cost to other activities located elsewhere, which may be called the maximum access model, where the transport system has a predominant role. Several models of this class where developed following either one or a combination of the Alonso's (1964) bid-rent approach and the Lowry's (1964) gravity -then entropy- approach. A second generation introduced more market elements into the location problem by including rents and good prices, what we call the linear market model; for economists this approach treats pecuniary externalities in the market. Rents have been introduced in several ways: when the assumption is that the market is one with perfect competition, the most preferred approach is the use of an hedonic rent function based on average zone attraction indices; when the assumption is that the location options are quasi-unique so the market is one of imperfect competition, rents are the result of simulating an auction process known as the bid-auction approach. Differential prices on goods have been used successfully for regional studies, hence they have been also introduced in the urban context, however, have a more limited real effect; several models of this class use the input-output approach. A third generation introduces an important amount of complexity into the model by incorporating an explicit representation of the direct interaction between agents decisions, that is the interaction that affect behavior in addition to the price effects. These interactions describes that fact that location options are valued, by all agents and in a significant degree, by their built environment, usually called zone attributes. In the economic literature (see Mas-Colell at al 1995, page 350) this type of interactions are defined as a multilateral public externality, because it involves all agents and public or non-rivalrous goods, which we call the location externality. The significance of this phenomena to the model formulation is that the built environment is generated by the solution of the location problem itself, then zone attributes are endogenous variables of the location problem. This describes a non linear location equilibrium problem with a large number of endogenous variables whose solution requires more sophisticated mathematical techniques than previous generations of models. On the other hand, the advantage of modelling location externalities explicitly is in the significant model enhancement to describe the real inherent dynamic of the location process. This complex location problem is what is specified in MUSSA and presented in the next section.

Finally, MUSSA operates under Windows with its own user friendly interface, both for inputs and outputs manipulations, including usual geographical information systems facilities.

**The theoretical background**

The basic theoretical element in the model is based on the observation that a location in the urban context is a highly scarce resource because the right to use it (by renting or buying) provides access to enjoy the neighbour amenities generated by the built and natural environment. This makes each location a quasi-unique or differentiable good, which yields a monopoly power to the landowner who obtains maximum benefit by an auction process that extract the maximum willingness-to-pay from consumers, as proposed by Alonso (1964). Consumers play in the auction game by making bids for location options, where bids represent their willingness-to-pay. Since Solow (1973) and Rosen (1974), the willingness-to-pay is a function analytically obtained as the inverse in land rents of the correspondent indirect utility function conditional on the location choice. We denote by $V_{hi}$ the indirect utility function conditional on the location option $vi$. Assuming that each agent “consumes” only one location and has an income $I_h$, this function can be expressed as $V_{hi} = V_h \left( I_h - r_{vi}, p, z_{vi} \right)$, with $p$ the price vector for goods and services, excluding the location price (or rent) $r_{vi}$ and $z_{vi}$ the vector of zone and buildings attributes and accessibility indices.
Then, the willingness-to-pay or bid function, conditional on obtaining a given utility level $U_h$, is:

$$B_{hv} = I_h - V_{hv}^{-1}(p, z_{vi}, U_h) \quad (1)$$

which represents the maximum value the agent is willing-to-pay for a location described by $z_{vi}$, to obtain a utility level $U_h$ given the exogenous $I_h$ and $p$. One can understand this function in the context of choice models by thinking that the agent considers a fixed utility level, which is exogenous and defined by market conditions, and assesses her/his monetary value for each available location option in the city using this bid function, which represents the price that would make the agent indifferent on choosing any alternative location since the utility level is assumed fixed across space. An important observation is that from (1) we define $e_h = B_{hv} + V_{hv}^{-1}$, that represents the expenditure function in all goods plus location cost if the consumer actually pays $B$; this is relevant for evaluation of different land use patterns, as discussed below. Another observation is that similar bid functions can be derived for firms directly from their profit functions.

For our purpose it is important to observe that bids functions theoretically embed location externalities, that is the interaction between activities, by means of vector $z$ in (1). Because neighbourhoods are defined by the allocation of residents and firms, and the location of these activities is the result of auctions, it follows that bids depend on bids. This dependency represents a technological externality between agents, defined directly in their utility function, which operates in the urban system in addition to the pecuniary interaction through land prices (rents). Analytically, this interaction generates a non-linear fixed-point problem that describes in the model the dynamics introduced by the explicit representation of location externalities.

Then, in the short run, where supply is assumed exogenous, equilibrium is twofold. First, at each location equilibrium is defined by the auction mechanism, where the auctioneer selects the maximum bidder $h^*$. This is expressed by

$$h^* = \arg \max_{h \in H} B_{hv} \quad (2)$$

with $H$ the set of bidders including households and firms, but excludes activities forbidden by regulations at location $vi$. This best bidder rule is sufficient to assure simultaneously that suppliers maximise profit and agents to maximise their utility or consumers’ surplus (Martínez, 1992, 2000). By this property the best-bidder’s rule simultaneously defines the optimal allocation of agents and rents at each building-location supplied.

The second equilibrium condition relates to the whole market. Unlike markets of products where consumers decide how much –if any- they buy of each good, in this market we assume that all agents (or least residents) consume only one location, but also not less than one. This means that every agent have to be located somewhere at equilibrium, provided that there is sufficient location options. Thus, agents trade each other location alternatives in auctions up to an equilibrium state that defines the maximum utility level attainable by each consumer in the market, represented by $U_h = U_{hv}^*$ in (1).

The above theory leads to a complex model of the city economics, extremely difficult to use for predictions. The fact that supply is discrete (zone system) and differentiable (location externalities) makes it mathematically untreatable for large cities. Moreover, the introduction of externalities represents a phenomenon that induce inherent instability in the model outcomes. This phenomenon has been widely described in social sciences (see Schelling, 1978) and it is

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1 In the urban land market building properties usually have known common values, for example provided by real estate agents, so we expect that the auctioneer receives several similar bids, nevertheless, inevitably the final value is only defined by the auction. On the issue of auctions with common values see for example the review by McAfee and McMillan (1987).
well recognised that it leads to complex non-linear mathematical formulations, with small changes in initial conditions causes dramatic differences in the location pattern and rents.

In MUSSA an approach to tackle this difficult problem is to introduce continuity by transforming the deterministic problem into its probabilistic equivalent, which smooths the discontinuity associated to the agents’ choice process. A second complementary tool is to choose the Gumbel probability distribution because it has some properties that helps to solve the location problem. Additionally, the probabilistic approach provides the extra benefit of generating a more realistic model because it allows representing the idiosyncratic variability of bids functions across agents within a cluster. Then we can apply the above model to clusters of homogeneous agents rather than to individuals. The idiosyncratic variability on agents’ behaviour takes into account the usual socio-economic and cultural differences considered in random utility theory, but also the variability in information and speculative behaviour in auction processes.

The residential location equilibrium problem

Although MUSSA models residential and non-residential agents, in this chapter we refer only to residential location. The location pattern is predicted by MUSSA for given point in time in the future and for an aggregate representation of the market components defined by the modeller. That is, households are segmented into clusters based on socio-economic characteristics of households. Supply stock is divided into building types, e.g. by land lot size and dwelling types (detached, semidetached, back-to-back), which can be specified for residential use only, non-residential use only or mixed use. Space is partitioned into homogeneous zones with regards to the land use and transport characteristics.

Population is calculated for the forecasting year as the number of households in each cluster. This is obtained from the growth sub-model that applies an input-output matrix that generates forecast of population and economic activity for the city consistent with the nation’s macro-economy at the forecasting year. Additionally, the transport system is modelled by a transport model that interacts with MUSSA. These two inputs represent an scenario for the location equilibrium sub-model, although MUSSA and the transport model interact to adjust for a land use and transport equilibrium (see Figure 1).

For each scenario, a mathematical location problem predicts a distribution of agents and buildings consistent with a static demand-supply equilibrium.

The MUSSA Location Problem:

\[
\begin{align*}
\min & \sum_{b,f} \left( \sum_{v,i} H_{v,i}(b,S) - \bar{H}_b \right)^2 \\
\text{s.a} & \\
& f_{v,i} = \Psi( r_{v,i}(b,S) ) \quad \forall v,i \\
& \sum_{v,i} \alpha_{v,i} s_{v,i} \leq O_{v,i}^{\text{MAX}} \quad \forall i, \forall m \\
& \sum_{v,i} \sum_{n} \lambda_{v,i,n} H_{v,i}(b,S) \leq L_{v,i}^{\text{MAX}} \quad \forall i, \forall n
\end{align*}
\]

includes the following functions:

\[
H_{v,i}(b,S) = S_{v,i} P_{v,i}(b,S) \quad \forall h, v, i
\]
\[ P_{h/v} = \frac{H_h \exp(\mu B_{hv} \nu)}{\sum_g H_g \exp(\mu B_{gv} \nu)} \quad \forall h, v, i \] (5)

\[ r_{v,i} = \frac{1}{\mu} \ln(\sum_g H_g \exp(\mu B_{gvi} \nu)) + \frac{\gamma}{\mu} \quad \forall v, i \] (6)

\[ B_{hv} = b_1^h + b_2^{hv} (\beta_h, X_{hv}, E_i) \quad \forall h, v, i \] (7)

with following notation:

- \( h, v, i \): indices for consumer cluster, dwelling type and zone, respectively. In application to Santiago city MUSSA is applied to 70 clusters, 12 building types (6 residential and 6 non-residential) and 409 zones.
- \( V, C \): set of building types and zones in the study area.
- \( H_{hv} \): number of agents of cluster \( h \) located in a building type \( v \) in zone \( i \).
- \( \HH_h \): exogenous number of agents in cluster \( h \) to be localized.
- \( S = (S_v)_i \): supply variables, with \( S_v \), the number of buildings type \( v \) available in zone \( i \), obtained from the equilibrium solution.
- \( B = (B_{hv})_{hv} \): Bid functions composed by a cluster specific term \( b^1 \) and a location-cluster specific term \( b^2 \) (equation 7). This function is derived upon the assumption of quasi-linear consumers’ utility functions
- \( b^1 = (b^1_h)_h \): terms of bid functions that represent a cluster specific valuation, invariant across locations because it describe the level of utility attained by the cluster, which is obtained from the equilibrium solution.
- \( b^2 = (b^2_{hv})_{hv} \): terms of bid functions that represents consumers’ valuation of attributes, either endogenous (X) or exogenous (E) to the model.
- \( \Psi \): supply function that describes suppliers' behaviour. For Santiago city application, this function was specified for an aggregate of zones, municipalities, according to constraints imposed by data.
- \( P_{h/v} \): location multinomial logit probability of \( h \) in option \((v, i)\) conditional on supply availability.
- \( r_{v,i} \): expected rent of option \((v, i)\).
- \( m, n \): indices to identify constraints.
- \( W_m, J_n, G_a \): subsets of building types and agents where constraints apply.
- \( \alpha_{int}, \lambda_{int} \): exogenous parameters to define linear constraints.
- \( O_{m;n}^{MAX}, L_{m;n}^{MAX} \): zone constraints defined on supply stock and agents location respectively.
- \( X_{hv}, E_i \): vectors of location attributes, \( X \) contains exogenous attributes and \( E \) endogenous attributes.
- \( \beta_h \): bid functions parameters calibrated for each cluster.
- \( g \): Eulers’ constant (\( \gamma \equiv 0.577 \)).

The objective function of the location problem (3) has a theoretical minimum value zero when all consumers in each cluster are located somewhere. The actual solution \((b^*, S^*)\) obtained by the model minimises the dis-equilibrium or excess of supply, which is called the minimum dis-equilibrium approach (MD). The objective function minimises the square difference for each cluster between the estimated number of agents located, given by equation (4), and the exogenous cluster's population. The solution is obtained by adjusting consumers' behaviour \((b^1)\) and suppliers behaviour \((S)\) as to comply with the MD objective, subject to the large set of constraints.
The auction process

The auction mechanism, where a real estate available in the market is assigned to the best bidder (equation 2), is modelled as its stochastic equivalent equation (5). The basic assumption is this equation is that the urban market behaves as if each consumer $h$ submits bids to the auction of the real estate identified by $vi$, with bids defined as the consumers' willingness-to-pay for the location option.

To introduce idiosyncratic variability within consumers in a cluster, we assume that bids are stochastic variables, denoted as $\hat{B}_{hvi}$, given by $\hat{B}_{hvi} = B_{hvi} + \varepsilon_{hvi}$, with the stochastic components $\varepsilon_{hvi}$ identical, independent and distributed Gumbel (0, $\mu$), and the deterministic component $B_{hvi}$ is given by (7). It follows that the probability that cluster $h$ is the best bidder among consumers in a supply option $(v,i)$, called the conditional location probability $P_{h/vi}$, is the multinomial logit expression (5), first proposed by Ellickson (1981). It also follows that rents are directly obtained by the expected value of the maximum bid (equation 6). These equations depend on the deterministic component of bids functions ($B_{hvi}$) previously derived, justified and interpreted. It is worth mentioning that the probabilistic bid auction process also holds the interesting property that each agent achieves the maximum utility given the rents values resulting from the auction, called the bid-choice equivalence demonstrated in Martínez (1992, 2000).

Additionally, equilibrium should be consistent with suppliers' behaviour, which is represented in the model by the first set of constraints where the number of location options $(v,i), f_{vi}$, should equal the expected supply given by $\Psi$. The second and third set of constraints represent planning regulations defined as specific constraints for each zone. $O^{\text{MAX}}$ values represent regulations affecting supply, as for example zones with a maximum number of high buildings, and $L^{\text{MAX}}$ represents regulations on land use, as for example forbidden activities in a zone.

Location externalities

The location choice behaviour is a process where agents compare a set of attributes, assess their relative value to obtain a willingness-to-pay or utility and decide the best option. These attributes can be transport or access related attributes, where the transport system is relevant (Martinez, 1995), and environmental or neighbourhood amenities. For adjusting accessibility and attractiveness indices, MUSSA interacts with transport models sequentially, which generates economic indices of zone accessibility and attractiveness introduced in the and assumes accessibility attributes as exogenous to the location equilibrium model.

A notable feature of the urban location problem is the well known fact that consumers value the environment where they consider their location, which induce the agents' interactions called location externalities. This very simple observation has very complex effects on the location equilibrium problem. To understand this, notice that here "the environment" is partly natural (e.g. rivers) but mostly artificial, defined by buildings and the neighbour land use. Because land use is itself the result of the location problem, that is, the output of location equilibrium across all agents, then the artificial environment is an endogenous location attribute denoted by vector $E$ in equation (7), to differentiate them from other exogenous attributes.

The typical examples of attributes in $E$ for residential location are: average socio-economic level of the neighbourhood, residential density, local services, etc. For non-residential activities the example is agglomeration of complementary activities, such as floor space of banking, commerce, services, etc. These environment attributes are affected by every location decision, thus affecting other consumers' decisions. This generates an intense interdependency between all consumers which is non-rivalrous, because affects the environment attributes which is "consumed" equally by all agents.
Mathematically the consumers' interdependence is represented by

\[ E_i = E_i \left( (S_{wi})_w, (P_{h/wi})_{h,w} \right) \]  

(8)

where the built environment is described attributes which are defined by locations probabilities and built stock. Since \( \beta \) and \( X \) are parameters in the model, we obtain

\[ B_{hvi} = b_h + F_{hvi} \left( (S_{wi})_w, (P_{g/wi})_{g,w} \right) \quad \forall h,v,i \]  

(9)

which states that bids depends on utility levels \( \phi_h \), supply and location probabilities. These probabilities are given by:

\[ P_{h/vi} = \frac{H_h \exp\left( \mu [b_h + F_{hvi} \left( (S_{wi})_w, (P_{g/wi})_{g,w} \right)] \right)}{\sum_g H_g \exp\left( \mu [b_g + F_{gvi} \left( (S_{wi})_w, (P_{g/wi})_{g,w} \right)] \right)} \quad \forall h,v,i \]  

(10)

These equations emphasis the dependency of the agent's location on all other consumer agents' location. Moreover, this shows that auctions are mutually interdependent through location externalities, and so are rents. It is worth noting that, to our knowledge, all land use equilibrium models ignore location externalities, probably because equation (10) has a complex non-linear structure which makes the location problem (3) also non-linear. However, ignoring externalities reduces the model forecasting power to a very limited tendency model, where the land use pattern observed for calibrating the model defines the environmental attributes which perpetuates along the forecasting time period despite all the changes in land use predicted by the model itself. This shortcoming in location models is traduced into fixed zone attraction factors in the utility or bid functions.

Notice that probabilities in equation (10) can not be directly calculated for a given set of values \( (b,f) \) in problem (3), because they are implicit functions of \( P \), so to calculate these probabilities we need to solve the location fixed-point problem. Therefore, the mathematical complexity is not only that the logit expression (10) is non-linear in \( (b,f) \), as it is usually the case with logit expressions, but it is also the case that the dependency can not be solved analytically, which implies that the objective function and many constraints in (3) can not be analytically solved for optimisation variables \( (b,f) \).

However, despite all the complexity introduced by equation (10) we can anticipate that the structure of the logit-fixed-point is exploited in MUSSA to solve the location problem. Our study indicates that very different solutions can be generated by introducing changes \( \beta \)'s parameters in bid functions, which indicates that changes in consumers tastes induce a dynamic of location externalities. This implies that the stability of the solution depends on the stability of the consumers' behaviour, hence on the quality of the parameters \( \beta \) calibrated for the model.

**Pricing policies**

The policy of monetary incentives was successfully applied in the 90's to renovate the Santiago's CBD by attracting a significant amount of middle class residents. The extension of this experience to a wider scheme is currently under government consideration.

Incentives can be positive (subsidy) or negative (tax) and may be defined to affect suppliers' behaviour or consumers' behaviour. Moreover, they can be design for specific building types, geographic area and consumers' clusters. All this alternative designs can be modelled with MUSSA.
For example, in the case of supply incentives the following constraint is modified:

\[ \sum_{v \in V} f_{vi} = \Psi(\tau_{ib}(b, f) + K_{ib}) \quad \forall V, C \]  

(11)

where \( K_{ib} \) is the monetary incentive to produce supply of a specific subset of building types defined by \( V \) in municipality \( C \). This modification has the effect of changing the feasible space of equilibrium solutions. In the case of demand incentives, they can be represented as a change in the bid function as:

\[ B_{hvi} = b_h + F_{hvi}(\beta_h, X_{hvi}, E_i) + S_{hvi} \quad \forall h, v, i \]  

(12)

where \( S_{hvi} \) identify the monetary incentive addressed to specific cluster \( h \) if location is on building type \( v \) and zone \( i \). This incentive may be not restraint to a building type \( S_{hi} \), or zone \( S_{hv} \) or both \( S_h \). For more details on modelling incentives with MUSSA see Martínez and Donoso (2000).

Incentives will modify the location equilibrium solution, affecting both location \( H_{hvi}(b, f, S, K) \) and rents \( r_{ib}(b, f, S, K) \). The solution algorithm requires a method to assure the existence of a feasible space. Martínez and Manterola (2000) studied the case of state expropriation of urban land for transport investment, defining and calculating shadow prices for expropriating urban land that differs from the market price, where the difference reflects the effects of government projects -typically roads- on the final equilibrium of locating agents. The evidence for Santiago using MUSSA shows that the difference is larger the more developed is the area and they recommend this shadow price for cost-benefit assessment of land expropriation. A second implication analysed is the existence of optimum location patterns not necessarily achieved by the free competitive market.

Planning regulations over building supply and land use.

Planning regulations is the most preferred tool in urban land use management. They control the type of buildings and their characteristics defining the range of building options legally feasible to develop in each zone. They also define a range of activities that can be legally located in each zone, that is they restrain land use, as well control land use and building densities.

MUSSA has been designed to represent explicitly such regulations. For example, in problem (3) parameters \( \alpha_{vim} \) allows the modeller to define supply regulations expressed as linear function of buildings supplied and limits can be defined by \( O_{im}^{\text{MAX}} \) and applied to specific sets defined by \( W_m \), which allows the modeller to specify a variety of regulations expressed as linear constraints. The obvious example is to restrain the occupied land to that available, which is obtained by defining for some \( m \) that \( W_m \) contains all building types, \( \alpha_{vim} = q_{vi} \) the land parcel associated to \( v \) in zone \( i \), and \( O_{im}^{\text{MAX}} = Q_{i}^{\text{MAX}} \) the total land lot available for location of activities in zone \( i \).

Similarly, by defining parameters \( \lambda_{hvin} \) limits \( L_{in}^{\text{MAX}} \) and sets \( J_i \) and \( G_{in} \) in problem (3) it is possible to specify a large set of land use regulations. For example,

- Maximum land use by cluster -households or firms- by zone, expressed by:

\[ \sum q_{vi} f_{vi} P_{hvi} \leq O_{hvi}^{\text{MAX}} \quad \forall h, i \]  

(13)

- Maximum residential density by zone
\[ N_i^{MIN} \leq \sum_{v \in RB} \sum_{h \in RC} k_h f_{vi} P_{h/vi} \leq N_i^{MAX} \quad \forall i \] (14)

with \( Q_h^{MAX} \) the maximum land quantity available for location of cluster \( h \) in zone \( i \). \( RB \) is the set of buildings for residential use and \( RC \) is the set residents clusters; \( k_h \) is the average number of household members in cluster \( h \); \( N_i^{MIN} \) y \( N_i^{MAX} \) are the minimum and maximum number of persons that can be located in zone \( i \).

Although problem (3) is flexible to accommodate linear constraints, most of them are not linear because of the presence of non-linear location probabilities. Additionally, the number of variables and constraints required to represent a usual set of planning regulation is large. For example, in the application to Santiago city MUSSA’s location problem deals with approximately 5,000 variables and 30,000 constraints. Thus, to solve problem (3) MUSSA requires specialised algorithms.

**Solution Algorithms**

The current version of MUSSA solves problem (3) by applying two algorithms sequentially. The first one is an approximated algorithm (AA), whose objective is to obtain an approximation to the solution. Once it converges, the principal algorithm (PA) starts from the approximate solution. This combined method exploits the high efficiency of algorithm AA compared to PA, with the precision in the solution generated with PA, which guarantees optimality but is time demanding.

*The Approximated Algorithm (AA)*

This algorithm exploits two interesting properties of the structure of problem (3).

**Theorem 1:** There exists a vector \( b^* \) (not necessarily a feasible point of the problem 1) that verifies the equilibrium condition:

\[ \sum_{v \in H} f_{vi} P_{h/vi} (b^*_h, f) = \Pi_h \] (15)

Proof: Replacing equation (10) in (15) yields:

\[ \sum_{v \in H} \sum_{g} f_{vi} \exp(\mu(b_h + F_{h/v}((f_{wi})_w,(P_{h/vi})_w))) = \Pi_h \quad \forall h \] (16)

which can be solved for \( b_h \) obtaining

\[ b_h = -\frac{1}{\mu} \ln \left( \sum_{v \in H} \sum_{g} \exp(\mu F_{h/v}((f_{wi})_w,(P_{h/vi})_w)) \right) \] (17)

\[ \Leftrightarrow b = \text{Logsum}(b, f, P) \]

which represents a fixed-point equation system in vector \( b \). This fixed-point formula is well known formula known as the logsum discrete choice modelling and as balancing factors in double constraints entropy models (see, for example, Wilson and Bennett, 1985). The solution \( b^* \) exists conditional on \( f \) and \( P \).
Theorem 2: The problem defined by the location probability (equation 10) has a fixed-point in \( P^* \).

Proof: Is a direct consequence of Brouwer's fixed point theorem for a continuous bounded function in \([0,1]\). The solution is conditional on \( f \) and \( b \).

The approximated algorithm AA is an iteration that generates a succession of approximated variables \( b = (b_h)_h \), \( f = (f_v)_v \) and \( P = (P_{h/v})_{h/v} \). For any iteration \( n \), we denote these variable as \( b^n, f^n \) and \( P^n \), and iteration \( n+1 \) is given by:

- **The first stage**: Given \( f^n \), calculate \( (b^{n+1}, P^{n+1}) \) as the solution of the following combined fixed point problem:

\[
\begin{bmatrix}
b^{n+1} \\
P^{n+1}
\end{bmatrix} = \begin{bmatrix}
\logsum(b^{n+1}, f^n, P^{n+1}) \\
MN\logit(b^{n+1}, f^n, P^{n+1})
\end{bmatrix}
\tag{18}
\]

which is solved by generating a fixed point succession.

- **Second stage**: Given \( (b^{n+1}, P^{n+1}) \), calculate \( f^{n+1} \) by solving the following optimisation problem:

\[
\begin{align*}
\min_{f^{n+1}} & \sum_h \left( \sum_v f_v^{n+1} \frac{P_v^{n+1}}{h/v} - H_h \right)^2 \\
\text{s.t.} & \\
& \sum_v f_v^{n+1} = \Psi(r_C(b^{n+1}, f^{n+1})) \quad \forall V, C \\
& \sum_i \alpha_{vim} f_v^{n+1} \leq O_{im}^{MAX} \quad \forall i \quad \forall m \\
& \sum_v \sum_{h/v} \lambda_{h/v} f_v^{n+1} \frac{P_v^{n+1}}{h/v} \leq L_{in}^{MAX} \quad \forall i \quad \forall n
\end{align*}
\tag{19}
\]

This algorithm decomposes the problem (3) in two sub-problems: a complex non-linear fixed-point problem and a quadratic optimisation problem. The first stage solves the main complexity of the location problem by means of exploiting the high efficiency of the fixed-point succession in obtaining the solution \((b^*, P^*)\). In fact it generates a minimum of the objective function of problem (3), although it may not be a feasible solution.

Then, the second stages adjust supply variables to the feasible space. Here the difficulty is associated with the large number of constraints that define such space. The advantage of problem (19) is that, in contrast to problem (3), where we can only have numerical gradients, it has analytical expression for \( f \) variables and the gradient, the objective function is quadratic and all constraints are linear, except from the supply function (11). This differences enhances significantly the efficiency, in terms of computing time, in solving problem (19), although it is conditional on variables \((b, P)\).

To assure a feasible solution algorithm AA always stops after the second stage, where we know that the solution may not verify the fixed-point problem of stage one and, therefore, exists some level of dis-equilibrium, then AA provides a good approximation of the solution of problem (3).
**The Principal Algorithm (PA)**

This algorithm directly solves problem (3) in all variables simultaneously. It applies a known optimisation algorithm, like Newton-Rapson, which generates a succession of approximations to the solution using the gradient of the objective function.

The problem, however, is the fact that the Jacobian matrix $\nabla P(b, f)$ has no analytical expression due to the existence of a probability fixed point problem (equation 8), hence MUSSA uses an approximation of this matrix.

**Implementation in MUSSA**

The implementation of these algorithms imposes the additional challenge of obtaining a solution for a case of large dimensions. In the application for Santiago city with 409 zones, 12 building types, 65 household clusters by income, family size and car ownership and 5 firm types, generates probability matrices with 343,560 elements which are updated in the fixed-point problem, 70 $b_h$ and 4,908 $f_{vi}$ variables obtained in the optimisation problem. Additionally, for Santiago the model handles 30,000 regulation constraints.

**The software capabilities**

The model forecasts, for any future year conditional on availability of exogenous data, the urban equilibrium. The basic output includes:

- Property monthly rents, by building type and zone.
- Location distribution of agents, that is the land use pattern, by cluster: income, car ownership and household type for residents clusters and by firm type for non-residence clusters.
- Buildings distribution by zone, including houses of different sizes and types and building blocks by height levels, which defines density and average heights.
- Benefits by agent clusters, which define distribution of welfare across agents.
- Active planning regulations and regulation slackness for non-active; this provides a tool to analyse proposals for regulation changes.

The model requires the following inputs for each forecasting year:

- Accessibility indexes by zone and preferably by cluster.
- Total city households population by cluster and total activity for non-residential agents, also by cluster.
- An initial observation of the land use variables to initiate attributes of the built environment (by zone): average income of residents, commercial, education and services floor space.
- The set of planning regulations and pricing (tax/subsidy) policies.
- The variables set that describe dwelling types: lot size, floor space and building type (house or building).

With MUSSA outputs it is theoretically justified to make rigorous economic assessments of land use scenarios. Benefits may be calculated using income – compensated and equivalent-variations as the variation of bid values associated with the scenario changes (Martínez, 2003). Thus, MUSSA is a useful tool to obtain economic assessment of planning options, such as: land use regulatory schemes, location pricing policies, transport-accessibility projects, etc. A particularly interesting application is to assess the social benefit (or cost) of each planning regulation, therefore dressing the planning process with an economic viewpoint. It can also be used by the private sector to assess expected profits from real estate investments, land acquisition, location of retail and services, etc.
Some remarks. First, it may seem that all agents, residents and non residents are allocated at each forecasting period without dependency to the previous period allocation. This interpretation is wrong, since it is possible to write the probability distribution (10) as an incremental multinomial logit (similarly for rents), which makes evident the inter-temporal dependency. Second, land use variables are endogenously updated within the equilibrium algorithm, which makes that zone attraction attributes are endogenous to the model and modified in each forecasting period. Third, residents location behaviour consider multiple attributes, including access to most relevant activities (work, educations, shopping, etc.), which are defined and calculated as transport users benefits. The trade off between attributes is defined through the bid functions by their calibrated parameters.

Performance of MUSSA

The model was validated by testing forecasts with observed data from Santiago City. In a first test, which is the usual practice for model validation, consisted in reproducing with the model a sample of cross-section data set used for calibrating the model. We regard this test as very weak since it reproduces what the model was calibrated for.

A more significant test consisted in comparing the forecast for 1997 with observed data, while the model was calibrated using a data set from 1991. The conclusion of this forecast was enlightening, because we learned that the model performed satisfactorily if the supply for 1997 was provided exogenously. In that case significant changes in land rents and location and building patterns where satisfactorily reproduced. However, the calibration of a new supply model required a significant enhancement. Particularly rewarding was the model replication of significant changes in non residential land use, because for this clusters we had highly detailed 1977 data to compare with, obtaining good results.

Running the model for 70 households clusters and 10 building types, plus a full land use regulation plan, using a PC Pentium III with 128 RAM takes 10 minutes in the case of fairly aggregated spatial representation (34 zones) and 60 minutes if the user considers a fully dis-aggregated spatial application (318 zones).

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References


FIGURE 1: The model structure

- **Macroeconomy**
  - **Growth**
    - Population & firms by cluster
      - Number of households & firms
  - **Land Use**
    - Equilibrium with location externalities
      - Accessibility & attractiveness
      - Location of households & firms
  - **Transport Model**