Equilibrium in an exchange economy with consumption rights and consequences on redistribution of wealth

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Abstract

In an exchange economy we introduce a parameter called a consumption right, which is freely tradable and so modifies the budget constraint of individuals but does not enter in the consumers preferences. This parameter provides a mechanism to implement lump-sum transfers of wealth among individuals, maintaining the desirable property of avoiding price distortions. An added advantage is that via a market mechanism the role of the government is reduced to defining and assigning the initial amount of rights to each individual, no agent collects and distributes wealth because this is obtained as a byproduct of the Walrasian equilibrium that exists in the economy with consumption rights.

Keywords: competitive equilibrium, consumption rights, redistribution.

JEL Classification: D31, D51, D63, H21.

1 Introduction

Since Arrow-Debreu’s existence of equilibrium theorem (see [1], [8]), a great deal of effort has been made to generalize the hypotheses required to prove this classical result. Given that the equilibrium allocation satisfies a set of desirable properties, including Pareto optimality, belonging to the core of the economy, individual rationality, etc., a fundamental conclusion is that, under very general hypotheses on the economy, the

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2 See Hammond [11] as a general reference on the equilibria properties. For more general economic frameworks, for instance, with increasing returns in production and/or infinite many agents, see Bonniseau and Cornet [6] and references there in.
invisible hand leads to a solution that guarantees all these properties, and is regarded as an efficient outcome without the need of intervention by a policy maker.

However, despite the merits of the competitive allocation, the view that the market is capable of assigning resources appropriately is not generally shared by most economists, mainly because the outcome does not necessarily fulfill a complementary set of consensus on equity criteria. Clearly, the vast literature on social choice supports the need for some complementary requirements.

In fact, economic theory has developed different methods to obtain efficient but also equitable outcomes. Among these methods, a lump-sum transfer of initial endowments has the advantage of avoiding undesired distortions in the assignment, thus maintaining efficiency in the economy that justifies why it is preferred to other tax systems.

However, in practice there are serious and well established difficulties with the implementation of lump-sum taxes. First, tax collection is far from costless, as was shown in the poll tax of the UK (see Myles [18], p. 46), which reduces their efficient performance. Secondly, and more fundamentally, optimal lump-sums depend on all the relevant variables in the economy, many of them only known by individuals and not directly observable by the government, which ultimately relies on reported information.

Mirrlees [14] presents theorems that prove the impossibility of designing non-mani-pulable lump-sum taxes, which makes it ultimately impossible to define optimal lump-sums. Confronting such implementation difficulties we propose an alternative method to re-distribute endowments.

In this paper we consider an exchange economy, with a finite number of consumers and goods, without public goods or externalities. We extend the Arrow-Debreu model (see Arrow and Debreu [1] and Debreu [8]) introducing a parameter that modifies the budgetary set but does not affect consumer preferences. We decided to interpret this parameter as tradable “consumption rights”, although another plausible interpretation is as a parallel currency. In this new economy, called r-economy, the initial endowment of consumers consists not only of commodities (as usual), but of a number of rights centrally distributed to individuals at no cost to them; however, these rights have no role in the utility function. Every transaction of commodities implies two payments: a price in wealth (as usual) and a certain amount of rights. These rights are tradable at a price and are generic, i.e. they are valid to purchase any commodity.

An existence of equilibrium theorem in the r-economy is proved under usual assumptions on the economy. We also prove that, at equilibrium, tradable rights induce a redistribution of the initial endowments among consumers. We analyze this economy assuming positive rights, which has the effect of restricting the points on the contract curve that can be decentralized to a subset. Fortunately, this subset contains those points of interest, obtained from social welfare functions widely used in the literature, but it does not allow swapping initially rich consumers to poor (or vice versa). We also extend the results to a partial r-economy, where only a subset of goods are subject to rights for consumption, thus making the r-economy eventually more politically acceptable.

In our mind, this approach adds a complementary market mechanism that allows an equilibrium allocation intended to comply with some exogenous equity criteria. In this
way, a double objective -efficiency and equity- can be achieved by combining market mechanisms, each one specifically designed for each criteria but interacting without reducing their capability of reaching their underpinning objectives.

With these results, the exchange process with consumption rights has similar merits as the lump-sum method to reassign resources in the economy: both of them avoid price distortions and decentralize desirable Pareto optima. However, a significant difference between these methods is that by exchanging consumption rights wealth is redistributed without the intervention of a policy maker that collects and assigns resources in the economy. This difference eventually could imply significant cost reductions.

Previous models employing a parameter that modifies the budgetary set of consumers without participating in their preferences have been used in microeconomic theory to study other problems. For instance, in monetary economics a slack parameter is defined, usually called fiat money, whose sole role is to facilitate the exchange in the presence of frictions in the economy that make it difficult for agents to execute net-exchanges worth exactly zero. See Kocherlakota [10] and Rocheteau and Wright [20] for more details on this relevant aspect of fiat money in the economy. See also Balasko and Shell [4] as a complementary reference.

Another example is one where the non-satiation assumption does not hold in the economy. If for any given price, some consumers wish to consume a commodity bundle in the interior of their budget set, the Walras equilibrium may fail to exist. In this case one may establish existence of an equilibrium by allowing for the possibility that some agents spend more than the value of their initial endowment. This generalization of the Walras equilibrium is called dividend equilibrium or equilibrium with slack (see Aumann and Drèze [3], Balasko [5], Drèze and Müller [9], Mas-Colell [15] and Makarov [13] among others).

The r-economy also has some similarities with the system of pollution rights to control emissions, where pollution rights are also tradable, however, here too, there are some fundamental differences. In the case of pollution rights, the price to trade rights is exogenously given by technical relationships, it affects only one good (more precisely, one bad) and directly affects the production of goods. Conversely, the r-economy endogenously defines consumption right prices, affecting all goods and is defined as an exchange economy without production being affected. See Montero [16], Montgomery [17] and Pezzey [19] for more details on this type of model.

Our notion of consumption rights should not be confused with the concept used by Hammond [11], defined as a right to “choose net trade vectors”. It is also different to Aumann and Kurz’s [2], which represents the right to choose lump-sum transfers according to the individuals's or group’s political power in the society. On the other hand, these concepts bear in common the fact that they are all defined from equity considerations, by means of political institutions.

The main difference of our approach compared with the above mentioned models is that in our model, in addition to the rights price, there are two different prices for each good in the market, with all of them being endogenously determined at the equilibrium. This introduces a second budget constraint, which combined with the usual wealth constraint, defines the budgetary set for any individual. Moreover, the
possibility of trading rights links the two constraints, in a way that the individual faces a “flexible budgetary set”. As a consequence, the demand for rights in the market is implicitly defined, as is the demand for goods.

The new parameter introduced is called a consumption right, name that we prefer instead of fiat money or parallel currency, despite the fact that consumption rights and wealth are convertible in the r-economy. This name emphasizes the idea that in our model, wealth is freely acquired by agents as a result of their initial advantages on differentiated endowments and without constraints on free trading, as in the classical model.

This paper is organized as follows. In Section 2 we introduce a simple example of the model, which is formalized in Section 3. In Section 4 we study the demand for consumption rights. The existence of equilibrium in the r-economy is proved in Section 5, whereas in Section 6 we study a slightly more general case where only a subset of markets are subject to rights. Finally, in Section 7 we analyze the redistribution aspects of the economy.

2 A motivational example

To illustrate how the r-economy works, consider the simple case of an exchange economy with two individuals and two goods. Assume individuals with identical preferences given by the utility function \( u(x_1, x_2) = x_1^\alpha \cdot x_2^{1-\alpha} \), with \( \alpha \in ]0, 1[ \), and with highly unequitable initial endowments given by \( \omega_1 = (1, 1) \in \mathbb{R}^2 \) for the poor and \( \omega_2 = (40, 40) \in \mathbb{R}^2 \) for the rich. In this case, the competitive equilibrium price is

\[
p^* = (1, \frac{1 - \alpha}{\alpha}) \in \mathbb{R}^2
\]

and the income\(^3\) ratio at the equilibrium is

\[
\frac{I_2}{I_1} = 40.
\]

Suppose now that the social choice is to “improve equity” by halving the income ratio. The classical way to achieve this is to directly transfer lump-sums of wealth between individuals (see Myles [18] as a general reference). Another alternative is to collect one half of all resources in the economy and distribute them back to the individuals according to given percentages; this option is related to our proposal below. Denoting by \( r \in [0, 1] \) the fraction returned to the poor individual, it is easy to check that the income ratio at the new equilibrium will be

\[
IR(r) = \frac{I_2(r)}{I_1(r)} = \frac{81 - 41r}{1 + 41r}.
\]

Because \( IR \) is strictly decreasing with \( r \) and \( IR(0) = 81, \) \( IR(1) < 1 \), it follows that any desirable income ratio between 81 and 1 can be reached by choosing an adequate

\(^3\)That is, the value of their endowments at the equilibrium price.
value of $r \in [0,1]$. For example, in order to halve the initial income ratio, $r$ must be equal to 0.06968, that is 7% of the total collected resources must be returned to the poor individual, which corresponds to an equivalent “tax” to be levied on the rich individual of 4.65% of his initial endowments. The following figure depicts the income ratio ($y$-axis) yield by this method at different values of $r$ ($x$-axis), where one can observe that an even distribution of $r$ reduces the income ratio to 2.8.

Note, however, that both methods, lump-sums and rights require a central agent that collects and distributes resources among individuals. The main contribution of this work is the definition of a competitive market procedure to make lump-sum transfers happen without the intervention of an agent that collects and distribute resources with the benefit of obtaining optimal transfers. Instead, we only need to assign to each individual a real parameter called consumption rights, represented by $r$ in the above example, and let the market attain equilibrium freely. The desirable re-allocation of goods will be the consequence of the competitive exchange of these rights among individuals. Indeed, according to our approach, in the above example to obtain an income ratio 20 at the equilibrium, it is enough to assign 7% of total consumption rights the poor individual and 93% to the rich individual, and let the “market work”.

Intuitively, the market works because the poor’s 7% of the total consumption rights is larger than his/her initial income, that amount to 2% of the total, while the rich’s 93% of consumption rights is lower than the correspondent 98% of the total income. Thus, the poor is willing to sell his rights surplus and the rich to buy, defining a incentive to trade rights that automatically reduces the income ratio from 40 to 20.

3 The model

Following the standard Arrow-Debreu model (see [1]), we assume that in the economy there are $\ell \in \mathbb{N}$ consumption goods and $m > 2$ consumers, indexed by $i \in I = \{1,2,\cdots,m\}$, whose utility functions and initial endowments are given by $u_i : \mathbb{R}_+^\ell \to \mathbb{R}$.
and $\omega_i \in \mathbb{R}^\ell_+$ respectively\(^4\). Define

$$\omega = \sum_{i \in I} \omega_i \in \mathbb{R}^\ell_{++}.$$  

as the total initial resources of the economy. Thus an \textit{exchange economy} is defined by

$$\mathcal{E} = ((u_i), (\omega_i))_{i \in I}.$$  

Hereafter, we assume that utility functions are of class $C^1$, strictly concave and strictly increasing by components.

Any distribution of goods $\{x_i\}_{i \in I}$ is said to be a feasible allocation if

$$\sum_{i \in I} x_i = \sum_{i \in I} \omega_i.$$  

From \cite{1}, if we assume that $\omega_i \in \mathbb{R}^\ell_{++}$, $i \in I$, then there exists a \textit{competitive equilibrium} for the economy $\mathcal{E}$, that is, there exists a vector price $p^* \in \mathbb{R}^\ell_{++}$ and a feasible allocation $\{x^*_i\}_{i \in I}$ such that for each $i \in I$, $x^*_i$ maximize $u_i$ on the budget set

$$B(p^*, \omega_i) = \{x \in \mathbb{R}^\ell_{++} \mid p^* \cdot x \leq p^* \cdot \omega_i\}.$$  

Let us now consider a new parameter in the economy, named \textit{consumption rights}, which can modify the individual budget set in the way described below. We assume that an initial endowment $r_i \in \mathbb{R}^+$ of consumption rights is assigned to each consumer $i \in I$, then the exchange economy with consumption rights, called $r$-\textit{economy}, is defined by

$$\mathcal{E}_r = ((u_i), (\omega_i), (r_i))_{i \in I}.$$  

Consumption rights can be freely traded on the market, but they do not participate in the utility function; we denote by $q \in \mathbb{R}^+$ the price of consumption rights in the market.

Consumption rights modify the consumer budget constraint in the following way. Assume that goods prices are given by $p \in \mathbb{R}^\ell_+$ and that in the economy there are \textit{transformation rates} from goods to consumption rights, which generically will be represented by a vector\(^5\) $s \in \mathbb{R}^\ell_{++}$. The role of vector $s$ in the economy is to restrain the set of consumption possibilities of each agent, in a way that with the initial consumption rights $r_i \in \mathbb{R}^+$ available for the individual $i \in I$, he can only consume those bundles $x_i \in \mathbb{R}^\ell_+$ that comply with

$$s \cdot x_i \leq r_i.$$  

Regarding the wealth constraint, as usual, individual $i \in I$ may consume consumption bundles $x_i \in \mathbb{R}^\ell_+$ such that

\[^4\]In what follows, given a vector $x \in \mathbb{R}^\ell$, $x^j$ will be its $j \in \{1, 2, \cdots, \ell\}$ component and the inner product between vectors $x$, $y \in \mathbb{R}^\ell$ will be denoted by $x \cdot y$.

\[^5\]In Section 4 we extend the model to consider that a subset of goods is subject to rights for consumption.
\[ p \cdot x \leq p \cdot \omega_i. \]

Now, if the individual \( i \in I \) decides to trade \( \delta_i \in \mathbb{R} \) consumption rights on the market \(^6\), then he now has \( r_i + \delta_i \) consumption rights available, but his wealth is modified in \(-q\delta_i\); thus his consumption set will be

\[
B(p, s, q, \omega_i, r_i, \delta_i) = \left\{ x \in \mathbb{R}^{\ell} \mid p \cdot x \leq p \cdot \omega_i - q\delta_i, \ s \cdot x \leq r_i + \delta_i \right\}.
\]

Then, the consumer’s problem for individual \( i \in I \) is

\[
P_i : \begin{cases} 
\max u_i(x) \\
\text{s.t.} \ x \in B(p, s, q, \omega_i, r_i, \delta_i).
\end{cases}
\]

The solution of this problem is called the *conditioned demand* for individual \( i \in I \), and is denoted by \( x_i(p, s, q, \omega_i, r_i, \delta_i) \).

From the assumptions on the utility function, if the solution exists, it is unique. Additionally, since the constraints defining the budget set are linear, it is clear that, seen as a correspondence, the consumption set is an upper semi-continuous correspondence in the parameters. Therefore, in order to guarantee existence and continuity for the demand function, it is only necessary to have sufficient conditions that guaranty the budget set be compact.

**Definition 3.1** The parameters \((p, s, q, \omega_i, r_i)\) are said regular for the individual \( i \in I \) if \( B(p, s, q, \omega_i, r_i, 0) \) is compact with non empty interior. A transaction of consumption rights \( \delta_i \) is said to be admissible for the individual \( i \in I \) with the regular parameters \((p, s, q, \omega_i, r_i)\) if \( B(p, s, q, \omega_i, r_i, \delta_i) \) is a compact set with non-empty interior. Denote by \( \mathcal{R}_i \) the set of regular parameters for individual \( i \in I \).

**Proposition 3.1** For each individual \( i \in I \)

\[
\mathbb{R}_{++}^{\ell} \times \mathbb{R}_{++}^{\ell} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \subseteq \mathcal{R}_i.
\]

Additionally, given \((p, s, q, \omega_i, r_i) \in \mathcal{R}_i\) such that \( q > 0 \), if

\[-r_i < \delta_i < \frac{p \cdot \omega_i}{q},\]

then the demand \( x_i(p, s, q, \omega_i, r_i, \delta_i) \) is well defined, unique and continuous in the interior of the respective sets.

**Proof.** Direct. \( \square \)

\(^6\)We denote buying rights by \( \delta_i > 0 \), and selling rights is \( \delta_i < 0 \).
4 Demand for rights

Given prices \( p, q \) and \( s \), there will be incentives to trade rights among consumers whenever their indirect utility increases ex post the transaction. What finally determines whether an individual is a buyer or seller of rights is the “relative size” of wealth and rights given the market prices.

This fact forces us to define a demand for rights, which will be an intermediary concept to finally define the demand for goods. Indeed, to determine the equilibrium in the r-economy, we may consider that the exchange takes place in two stages: in the first one, individuals decide to trade rights and, in the second one, they decide to exchange goods given the optimal quantity of rights now available for them. If both processes are finally “compatible” with the total amount of resources in the economy, then an equilibrium allocation can be attained.

Definition 4.1 Given \((p, s, q, \omega_i, r_i) \in \mathcal{R}_i\), the individual \( i \in I \) demand for rights is defined as the admissible transaction \( \bar{\delta}_i \) for the above parameters that maximizes the individual’s indirect utility in the budget set, for any admissible transaction of rights.

Proposition 4.1 Given \((p, s, q, \omega_i, r_i) \in \mathcal{R}_i \) and \( q > 0 \), the demand for rights exist and it is unique.

Proof. Given \( i \in I \), the demand for consumption rights is well defined is direct from both the continuity of the utility function and the compactness of the budget set.

Then, let \( \bar{\delta}_i \) be the demand for rights at the indicated parameters and be \( \bar{x}_i \) the corresponding conditioned demand for goods. Since the utility function is strictly increasing by component, it follows that at least one constraint of the consumer problem is binding at the optimum. Without loss of generality, let us assume that

\[
p \cdot \bar{x}_i = p \cdot \omega_i - q \bar{\delta}_i, \quad s \cdot \bar{x}_i < r_i + \bar{\delta}_i.
\]

Given \( \epsilon > 0 \) and \( e_1 = (1, 0, 0, \ldots, 0) \in \mathbb{R}^l \), define \( \tilde{x}_i = \bar{x}_i + \epsilon e_1 \) and

\[
d = r_i + \bar{\delta}_i - s \cdot \tilde{x}_i > 0, \quad \epsilon = \frac{d}{s^1 + p^1 q} > 0, \quad \tilde{\delta}_i = \bar{\delta}_i - \frac{p^1}{q} \epsilon.
\]

Given this, it is direct that \( \tilde{x}_i \) complies with

\[
p \cdot \tilde{x}_i = p \cdot \omega_i - q \tilde{\delta}_i, \quad s \cdot \tilde{x}_i = r_i + \tilde{\delta}_i
\]

and hence \( u_i(\tilde{x}_i) > u_i(\bar{x}_i) \). Thus, \( \bar{\delta}_i \) would not be the consumer’s demand for rights and therefore at the optimum both constraints are binding.

From the fact that

\[
p \cdot \bar{x}_i = p \cdot \omega_i - q \bar{\delta}_i, \quad s \cdot \bar{x}_i = r_i + \bar{\delta}_i \Rightarrow (p + qs) \cdot \bar{x}_i = p \cdot \omega_i + qr_i,
\]

we deduce that the consumer’s problem may be equivalently rewritten as
\[ p_i \iff \left\{ \begin{array}{l}
\max u_i(x) \\
\text{s.t. } (p + qs) \cdot x = p \cdot \omega_i + qr_i
\end{array} \right. \]

and therefore, because the utility function is strictly concave, the solution to the above problem is unique, and so is the demand for rights. \( \square \)

5 The equilibrium concept and existence

To define equilibrium in the economy \( \mathcal{E}_r \), we shall consider the demand feasibility in two ways. On the one hand, the condition that requires that total demand is feasible by initial endowments and, on the other hand, the condition that the net trade of rights must be equal to zero. This latter condition is equivalent to saying that total rights in the economy comply with

\[ \mathcal{R} = \sum_i r_i. \]

In what follows we generically refer by prices in the economy to the tupla \((p, s, q) \in \mathbb{R}^2_{+} \times \mathbb{R}_{+}\).

**Definition 5.1** A rights exchange distribution \( \delta_i \in \mathbb{R}, \; i \in I \), is said to be feasible if

\[ \sum_{i \in I} \delta_i = 0. \]

**Definition 5.2** Prices \((p^*, s^*, q^*) \in \mathbb{R}^2_{+} \times \mathbb{R}_{+}\) are said to be an equilibrium price for the economy \( \mathcal{E}_r \) if there exists a feasible distribution of goods \( \{x^*_i\}_{i \in I} \) and a feasible distribution of right exchanges \( \{\delta^*_i\}_{i \in I} \), such that

(a) for each \( i \in I \), \( \delta^*_i \) is the rights demand at prices \((p^*, s^*, q^*)\),

(b) for each \( i \in I \), \( x^*_i \) is the conditioned demand for rights \( \delta^*_i \).

The tupla \((p^*, s^*, q^*,(x^*_i)_{i \in I},(\delta^*_i)_{i \in I})\) constitutes what we call a competitive equilibrium for the economy \( \mathcal{E}_r \).

Now on, previous to continue with the existence of equilibrium analysis, we would like to remark the following things

(i) if \((p^*, s^*, q^*,(x^*_i)_{i \in I},(\delta^*_i)_{i \in I})\) is an equilibrium of \( \mathcal{E}_r \), then

\[ s^* \cdot x^*_i = r_i + \delta^*_i \Rightarrow s^* \cdot \sum_{i \in I} x^*_i = \sum_{i \in I} r_i + \sum_{i \in I} \delta^*_i = \mathcal{R} \]

and thus the condition \( s^* \cdot \omega = \mathcal{R} \) is always verified;

(ii) at the equilibrium one reasonably may assume that the marginal rate of substitution in terms of goods and consumption rights should be the same, that is, prices \( p^* \) and \( s^* \) must be linearly dependent. Otherwise could exist incentives to trade goods and consumption rights inefficiently. In consequence, we can assume the existence of \( \lambda \in \mathbb{R} \) such that \( p^* = \lambda s^* \);
(iii) since at the equilibrium the total wealth of individual \( i \in I \) is \( p^* \cdot \omega_i - q^* \delta^*_i \), whereas the value of his consumption rights is \( q^* [r^*_i + \delta^*_i] \), we can assume that

\[
p^* \cdot \omega_i = q^* r^*_i + 2q^* \delta^*_i \quad \Rightarrow \quad p^* \cdot \omega = q^* R.
\]

Thus, from \( (i) \) and \( (ii) \) follows that \( p^* \cdot \omega = \lambda R \), and then, using \( (iii) \), we conclude that \( \lambda = q^* \), that is, \( p^* = q^* s^* \). This last fact motivates the following definition, which ex post will be a crucial assumption to ensure the existence of equilibrium in our model.

**Definition 5.3** We say that prices \( p^* \), \( q^* \) and \( s^* \) are in arbitrage if

\[
p^* = q^* s^*.
\]

The above, in addition to the characterization of the consumer’s problem proved in the previous Section, allow us to characterize the equilibrium in the economy \( E_r \) without using rights transactions. This formalizes the previously mentioned comment regarding the auxiliary role of the rights demand in our model. The following proposition is direct from previous considerations.

**Proposition 5.1** \( (p^*, s^*, q^*, (x^*_i)_{i \in I}, (\delta^*_i)_{i \in I}) \) is an equilibrium of \( E_r \) if, and only if,

\[
(a) \quad s^* \cdot \omega = R,
\]

\[
(b) \quad \text{for each } i \in I, \ x^*_i \text{ maximize } u_i \text{ on the budget set}
\]

\[
\Gamma(p^*, s^*, q^*, \omega_i, r_i) = \{ x \in \mathbb{R}^\ell_+ \mid (p^* + q^* s^*) \cdot x \leq p^* \cdot \omega_i + q^* r_i \}.
\]

**Theorem 5.1** If the utility functions are strictly concave, class \( C^1 \), and strictly increasing by components, if the total amount of initial endowments in the economy is strictly positive and if the initial amount of rights assigned to each individual are also strictly positive, then a competitive equilibrium exists in the economy \( E_r \), with the prices satisfying the arbitrage condition.

**Proof.** Given the prices \( p, q \) and \( s \), suppose that \( p = q s \) (arbitrage condition). In such case, we can readily deduce that the consumer’s problem for individual \( i \in I \) can be rewritten as

\[
\begin{aligned}
\max & \quad u_i(x) \\
\text{s.t.} & \quad p \cdot x = p \cdot (\omega_i/2) + q(r_i/2).
\end{aligned}
\]

Now, be \( z_i \in \mathbb{R}^\ell_+ \), \( i \in I \) (to be defined) such that for each \( i \in I \),

\[
p \cdot z_i = (qr_i)/2 \quad \text{[*].}
\]

Since in this case we have that\(^7\)

\[
\tilde{\delta}_i = \frac{p \cdot \omega_i}{2q} - \frac{qr_i}{2q} = \frac{p \cdot \omega_i}{2q} - \frac{p \cdot z_i}{q}
\]

\(^7\)At the optimum, \( p \cdot \bar{x}_i = p \cdot \omega_i - q \bar{\delta}_i \) and \( s \bar{x}_i = r_i + \bar{\delta}_i \iff qs \bar{x}_i = qr_i + q \bar{\delta}_i \). Therefore, \( qr_i + q \bar{\delta}_i = p \cdot \omega_i - q \bar{\delta}_i \).
imposing the condition that at the optimum rights transactions add up to zero, we conclude that

\[
p \cdot \omega - \frac{p \cdot \sum_{i \in I} z_i}{q} = 0
\]

which, as a particular case, is fulfilled if

\[
\sum_{i \in I} z_i = \frac{\omega}{2} \quad [**].
\]

On other hand, from [*] again it follows that for all \( i, j \in I \)

\[
p \cdot \left( \frac{2z_i}{r_i} - \frac{2z_j}{r_j} \right) = 0 \iff p \cdot \left( \frac{z_i}{r_i} - \frac{z_j}{r_j} \right) = 0,
\]

which, as a special case, implies that for all \( i, j \in I \)

\[
z_j = \left( \frac{r_j}{r_i} \right) z_i \in \mathbb{R}^d.
\]

Imposing the condition [**] yields

\[
\sum_{j \in I} z_j = \left[ \sum_{j \in I \setminus \{i\}} z_j \right] + z_i = \left[ \sum_{j \in I \setminus \{i\}} \frac{r_j}{r_i} z_i \right] + z_i = \frac{R}{r_i} z_i,
\]

and therefore, for all \( i \in I \)

\[
z_i = \left( \frac{r_i}{2R} \right) \omega \in \mathbb{R}^d_+.
\]

From all the above, the consumer’s problem can now be equivalently re-written as

\[
P_i : \begin{cases} 
\max & u_i(x) \\
\text{s.t} & p \cdot x = p \cdot \left( \frac{\omega_i}{2} + \frac{r_i}{2R} \omega \right).
\end{cases}
\]

Define now the economy

\[
\tilde{E} = ((u_i), (\tilde{\omega}_i))_{i \in I}
\]

where

\[
\tilde{\omega}_i = \frac{\omega_i}{2} + \frac{r_i}{2R} \omega \in \mathbb{R}^d_{++}.
\]

From Arrow and Debreu [1], the economy \( \tilde{E} \) has a competitive equilibrium, say \((\tilde{p}, (\tilde{x}_i)_{i \in I})\). Finally, for all \( i \in I \), let us define

\[
x_i^* = \tilde{x}_i, \quad p^* = \tilde{p}, \quad q^* = \frac{\tilde{p} \cdot \omega}{R}, \quad s^* = \frac{1}{q^*} p^*, \quad \delta_i^* = \frac{p^* \cdot \omega_i - q^* r_i}{2q^*}.
\]

Given that, it is easy to see that such prices and allocations constitute a competitive equilibrium for \( E_r \). □
Remark 5.1

(a) The presence of consumption rights in the economy can finally be interpreted as a
lump-sum transfer among individuals. This transfer consists of taking half of the
total resources in the economy and redistributing them back to the individuals
according to the percentage of rights assigned to them.

(b) Since we have assumed that the initial rights assigned are strictly positive, all
consumers have a strictly positive initial endowment in the economy, which fi-
nally ensures that all consumers can participate in the exchange, even if his/her
initial endowments are equal to zero. Additionally, this condition on initial rights
limits the range of feasible redistribution of the initial endowments to a subset of
distributions, compared to those that, in principle, can be obtained by a general
tax system. In fact the maximum equivalent tax is one half of the individual’s
initial endowment. See Section 7 for more details on this.

(c) In our equilibrium notion, we impose feasibility according to the total resources
of the economy, and so the resulting equilibrium allocation in the r-economy will
be a Pareto optimum allocation for the economy without rights $E$ (and of course
for the $E_r$ economy).

(d) The arbitrage condition employed here can be imposed as an exogenous condition
by a policy maker in order to implement a desired policy using consumption
rights. Indeed, this condition could be interpreted as an obligation in order to
achieve efficient transaction in rights and goods, in the sense that the marginal
rate of substitution in rights and wealth must be equal and the richness of any
individual in terms of wealth from endowments and value of consumptions rights
must be the same.

(e) It is worth commenting that the equivalent problem with prices in arbitrage
yields a unique constraint with an intriguing factor, $1/2$ of initial -wealth and
rights- endowments. This constraint accepts the interpretation that a central
agent collects half of the endowments, as in the example in Section 2 above.
Another interpretation is that the mechanism is equivalent to buying goods with
two types of currencies (wealth and rights), which under the arbitrage condition
doubles the value in terms of wealth.

6 The market partially restrained to rights

A more general case than that presented above is obtained by assuming that only
$k \in \{1, 2, \cdots, \ell\}$ goods are restricted by consumption rights, that is, when the vector
$s \in \mathbb{R}^\ell$ has the form

$$s = (s_1, s_2, \cdots, s_k, 0, 0, \cdots, 0) \in \mathbb{R}^\ell.$$  

For any vector $z = (z^j) \in \mathbb{R}^\ell$, define $z(k) = (z^1, z^2, \cdots, z^k, 0, 0, \cdots, 0) \in \mathbb{R}^\ell$ and
$z(-k) = (0, 0, \cdots, 0, z^{k+1}, z^{k+2}, \cdots, z^{\ell}) \in \mathbb{R}^\ell$. Abusing the notation consensus, we set
$z = (z(k), z(-k))$.  

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Following the same reasoning used in the demonstration of Proposition 4.1, it is easy to check that in this new context, both income and rights restrictions must be binding at the optimum, and therefore the consumer’s problem can be put in the same form as we already know it, that is,

\[
P_i : \max_{x} u_i(x) \quad \text{s.t.} \quad (p + qs) \cdot x = p \cdot \omega_i + qr_i.
\]

Additionally, assuming the arbitrage condition \( p(k) = qs \), the consumer’s problem \( P_i \) can be expressed equivalently as

\[
P_i \iff \max_{x} u_i(x) \quad \text{s.t.} \quad (2p(k), p(-k)) \cdot x = (2p(k), p(-k)) \cdot \left( \frac{\omega(k)}{2}, \omega_i(-k) \right) + qr_i.
\]

From the demonstration of Theorem 5.1, if we define \( z_i \in \mathbb{R}^\ell, i \in I \), as

\[
z_i = \frac{r_i}{2R} \sum_{j \in I} \omega_j(k) \equiv \frac{r_i}{2R} \omega(k) \in \mathbb{R}^\ell,
\]

it is easy to check that

\[
2p(k) \cdot z_i = qr_i, \quad \sum_{i \in I} z_i = \frac{\omega(k)}{2}.
\]

Consequently, defining

\[
\Omega_{ik} = \left( \frac{\omega_i(k)}{2} + \frac{r_i}{2R} \omega(k), \omega_i(-k) \right) \in \mathbb{R}^\ell, \quad P_k = (2p(k), p(-k)) \in \mathbb{R}^\ell,
\]

the consumer’s problem of individual \( i \in I \) can be re-formulated as

\[
P_i \iff \max_{x} u_i(x) \quad \text{s.t.} \quad P_k \cdot x = P_k \cdot \Omega_{ik}.
\]

Finally, assuming that \( \omega \in \mathbb{R}^\ell_{++}, r_i > 0 \) and \( \omega_i(-k) \in 0_{\mathbb{R}^{\ell-k}} \times \mathbb{R}^{m-k}_{++}, i \in I \), it follows immediately that there exists a competitive equilibrium for this exchange, namely a price equilibrium \( P_k^* \) and optimal allocations \( X_{ik}^*, i \in \{1, 2, \cdots, m\} \). Thus, if we employ this equilibrium point in a similar manner as in the demonstration of Theorem 5.1, we can finally conclude that there exists an equilibrium set of prices \( p^*, s^*, q^* \) and an equilibrium allocation for the economy with \( k \) restricted markets. These points are given by

\[
x_i^* = X_{ik}^*, \quad p^* = P_k^*, \quad q^* = \frac{P_k^* \cdot \sum_{i=1}^{m} \Omega_{ki}}{R}, \quad s^* = \frac{1}{q^*} p^*(k).
\]

With all the above, we have proved the following proposition.

**Proposition 6.1** Under the same hypotheses of Theorem 5.1, if we additionally assume that there are \( 1 \leq k \leq \ell \) markets restricted to rights, such that the initial endowments of goods subject to rights are strictly positive for any individual, then there exists a competitive equilibrium in the economy, with prices satisfying the arbitrage condition in those components that correspond with the restricted markets.
7 Decentralization

A central question dealing with public choice arises when we try to characterize the Pareto allocations that can be decentralized through the competitive process. We discuss this very relevant issue for the r-economy whenever we are interested in implementing a social policy defined ex ante. Unfortunately, the mechanism described above can not reach any point on the contract curve, because our scheme corresponds to an specific lump-sum transfer: one that collects only half of the total resources in the economy and assigns them to individuals according to the percentage of rights they own. Thus, in the extreme case an individual can be “taxed” with only 50% of his resources, which is not enough to reach any Pareto allocation.

The following figure of an Edgeworth box, with individual’s initial endowments given by \( \omega_1 \) and \( \omega_2 \), illustrates to us that there are some points on the contract curve (passing trough \( DAB \)) that can not be decentralized with any assignment of consumption rights. Indeed, any Pareto optimum allocation such that its supporting line intersects the shadowed region can be decentralized through an adequate assignment of rights. This is the case, for instance, with allocation \( A \), because the segment from \( \omega_1/2 \) to \( E \) intersects the supporting line through \( A \) at the point \( C \). Conversely, the Pareto allocation \( B \) can not be decentralized because the intersection between its supporting line defined by price \( \bar{p} \) does not intersect the shadowed region. A point like \( D \), closer to the shadowed region, is a “natural” candidate to be decentralized.

Thus, the question that arises is which are the Pareto allocations that can be decentralized using consumption rights to implement the lump-sum transfers? From the previous figure, and considering the nature of the transfers here defined, this procedure can not decentralize Pareto optimum allocations that correspond to a “radical redistribution” of wealth, that is, all that make a “poor individual” to become “rich” and...
a rich to become a poor. This is the case, for instance, with allocation $B$ in the above figure: via our mechanism we are unable to reach this type of allocation. Our procedure transfers wealth in such a way that it reduces the income inequality but prohibits extreme progressive outcomes. This result is in the line with Roemer’s [21] notions of equalitarian societies, since they correspond to allocations located in shadowed region of the Edgeworth box depicted in the above figure.

References


