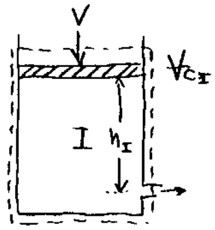
Pauta Ejercicio #3 - CI31A

a)



En el volumen de control $\forall_{c_I}$:

$$\frac{d\forall_I}{dt} = Q_{e_I} - Q_{s_I}$$

$$\forall_I = \frac{\pi D_1^2}{4} \cdot h_I + \forall_{I0}$$ ↖ volumen bajo la cañería

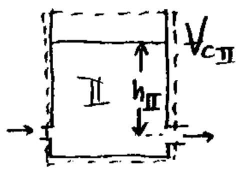
$$\frac{d\forall_I}{dt} = \frac{\pi D_1^2}{4}\frac{dh_I}{dt} = -\frac{\pi D_1^2}{4} \cdot V$$

$$\left(\frac{dh_I}{dt} = -V\right)$$

Además, $Q_{e_I} = 0$

$$\Rightarrow \quad \frac{d\forall_I}{dt} = -\frac{\pi D_1^2}{4} V = -Q_{s_I}$$

$$Q_{s_I} = \frac{\pi D_1^2}{4} \cdot V$$



En el volumen de control $\forall_{c_{II}}$:

$$\frac{d\forall_{II}}{dt} = Q_{e_{II}} - Q_{s_{II}}$$

Lo que entra en II es lo que sale en I:

$$Q_{e_{II}} = Q_{s_I}$$

Además $Q_{s_{II}} = \frac{\pi d^2}{4} \cdot \sqrt{2 g h_{II}}$

$$\forall_{II} = \frac{\pi D_2^2}{4} \cdot h_{II} + \forall_{II_0}$$ ↖ volumen bajo el orificio

$$\frac{d\forall_{II}}{dt} = \frac{\pi D_2^2}{4}\frac{dh_{II}}{dt}$$

$$\Rightarrow \frac{d\forall_{II}}{dt} = \boxed{\frac{\pi D_2^2}{4}\frac{dh_{II}}{dt} = \frac{\pi D_1^2}{4} \cdot V - \frac{\pi d^2}{4}\sqrt{2 g h_{II}}}$$

a) $\frac{dh_{II}}{dt} = 0 \quad ; \quad h_{II} = h_0$

$$\Rightarrow \frac{\cancel{\pi} D_1^2}{\cancel{4}} \cdot V = \frac{\cancel{\pi} d^2}{\cancel{4}}\sqrt{2 g h_0} \quad \Rightarrow \quad d = \sqrt{\frac{D_1^2 \cdot V}{\sqrt{2 g h_0}}}$$

b) Basado en la misma ecuación diferencial

$$\frac{\pi D_2^2}{4}\frac{dh_{II}}{dt} = -\frac{\pi D_1^2}{4}\frac{dh_I}{dt} - \frac{\pi d^2}{4}\sqrt{2g h_{II}}$$

Ahora $\frac{dh_{II}}{dt} = 0$, $\frac{dh_I}{dt} = -V'$ y $h_{II} = h_f$

$$\Rightarrow \frac{\cancel{\pi} D_1^2}{\cancel{4}}\cdot V' = \frac{\cancel{\pi} d^2}{\cancel{4}}\sqrt{2g h_f}$$

$$\Rightarrow \sqrt{h_f} = \frac{D_1^2 \cdot V'}{d^2\sqrt{2g}} \quad \Rightarrow \quad h_f = \frac{D_1^4 \cdot V'^2}{2g\, d^4}$$

pero $d = \sqrt{\frac{D_1^2 \cdot V}{\sqrt{2g h_0}}}$ $\Rightarrow$ $h_f = \frac{D_1^4 \cdot V'^2}{2g \frac{D_1^4 \cdot V^2}{2g h_0}}$

$$\Rightarrow \boxed{h_f = \left(\frac{V'}{V}\right)^2 h_0}$$ como $V' > V \Rightarrow h_f > h_0$

c)

$$\frac{\cancel{\pi} D_2^2}{\cancel{4}}\frac{dh_{II}}{dt} = \frac{\cancel{\pi} D_1^2}{\cancel{4}}\cdot V' - \frac{\cancel{\pi} d^2}{\cancel{4}}\sqrt{2g h_{II}}$$

$$\frac{D_2^2\, dh_{II}}{D_1^2 \cdot V' - d^2\sqrt{2g h_{II}}} = dt$$

Llamando $A = \frac{D_1^2 \cdot V'}{D_2^2}$; $B = -\frac{d^2\sqrt{2g}}{D_2^2}$

$$\Rightarrow \frac{dh_{II}}{A + B\sqrt{h_{II}}} = dt$$

cambio de variable

$$u = A + B\sqrt{h_{II}} \Rightarrow \sqrt{h_{II}} = \frac{u - A}{B}$$

$$du = \frac{-B}{2}\frac{1}{\sqrt{h_{II}}}\, dh_{II}$$

$$\Rightarrow dh_{II} = -du\,(u - A)\frac{2}{B^2}$$

$$\Rightarrow \frac{du\,(A - u)}{u}\frac{2}{B^2} = dt$$

$$\frac{2}{B^2}\left(\frac{A}{u} - 1\right) du = dt \qquad \frac{2}{B^2}\int_{u_0 = A + B\sqrt{h_0}}^{u_1 = A + B\sqrt{\frac{h_0 + h_f}{2}}}\left(\frac{A}{u} - 1\right) du = \int_0^{t^*} dt$$

$$\frac{2}{B^2}\Big[A \ln u - u\Big]_{u_0}^{u_1} = t^*$$

$$\Rightarrow t^* = \frac{2}{B^2}\left[A \ln\left(A + B\sqrt{\frac{h_0 + h_f}{2}}\right) - \left(\cancel{A} + B\sqrt{\frac{h_0 + h_f}{2}}\right) - A \ln\left(A + B\sqrt{h_0}\right) + \cancel{A} + B\sqrt{h_0}\right]$$

$$t^* = \frac{2}{B^2}\left[A \ln\left(\frac{A + B\sqrt{(h_0 + h_f)/2}}{A + B\sqrt{h_0}}\right) + B\left(\sqrt{h_0} - \sqrt{\frac{h_0 + h_f}{2}}\right)\right]$$

$$t^* = \frac{D_2^2}{d^4 g}\left[D_1^2 \cdot V'\left(\frac{D_1^2 V' - d^2\sqrt{2g(h_0 + h_f)/2}}{D_1^2 \cdot V' - d^2\sqrt{2g h_0}}\right) + d^2\left(\sqrt{2g\left(\frac{h_0 + h_f}{2}\right)} - \sqrt{2g h_0}\right)\right]$$