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On the generalized Morse potential

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Abstract. The equivalence between the generalized Morse (GMP) and Eckart potentials is shown. The study of the hypergeometric Natanzon potentials using $SO(2,1)$ techniques is applied to compute the eigenfunctions and eigenvalues of the Eckart (GMP) potential. The action of the group generators is studied, with the result that a family of Eckart potentials is obtained which is different from the one obtained in SUSYQM.

1. Introduction

In [1] an interesting study of the solubility of generalized Morse potentials (GMP) was performed using the $SO(2,2)$ algebraic treatment for the hypergeometric Natanzon potentials [2] developed in [3]. The purpose of this paper is to analyse the same problem using the techniques given in [4] which are based on the $SO(2,1)$ algebra. This last group has been applied to the study of both the hypergeometric and confluent hypergeometric [5] Natanzon potentials. Also this approach has been used recently as a simple method to study a q-deformation of the Pöschl–Teller potentials [6].

Before analysing the GMP potential, a short summary of the results in [4] is presented to fix the notation and to exhibit the relevant results to be used. The hypergeometric Natanzon potentials V_N, those for which the Schrödinger equation can be transformed to an hypergeometric one, can be solved algebraically by means of the $SO(2,1)$ algebra as follows:

(a) a two-variable realization of $SO(2,1)$ is selected,
(b) the Schrödinger equation is written in terms of the Casimir operator of the algebra C, as $[H-E]\,\Psi(r,\phi) = G(r)[C-q]\,\Psi(r,\phi)$, where q is the eigenvalue of C, H is the Hamiltonian and E the corresponding eigenvalue. $G(r)$ is a function fixed by consistency, and
(c) the eigenfunctions of the Casimir have the form $\Psi(r,\phi) = \exp(\mathrm{i}m\phi)\Phi(r)$.

The hypergeometric Natanzon potentials are given by (we follow the notation of [7])

$$V_N = \frac{1}{R}(f z(r)^2 - (h_0 - h_1 + f)z(r) + h_0 + 1) + \frac{z(r)^2(1-z(r))^2}{R^2}\left[a + \frac{a + (c_1 - c_0)(2z(r) - 1)}{z(r)(z(r)-1)} - \frac{5\Delta}{4R}\right] \tag{1}$$

where

$$\Delta = \tau^2 - 4ac_0 \qquad \tau = c_1 - c_0 - a \qquad R = az(r)^2 + \tau z(r) + c_0.$$

The constants a, c_0, c_1, h_0, h_1 and f are called Natanzon parameters. The function $z(r)$ must satisfy

$$\frac{\mathrm{d}z(r)}{\mathrm{d}r} = \frac{2z(r)(1-z(r))}{\sqrt{R}}.$$

The generators of the $SO(2, 1)$ algebra: J_1, J_2 and J_0 satisfy the usual commutation relations: $[J_0, J_1] = \mathrm{i}J_2$, $[J_2, J_0] = \mathrm{i}J_1$, $[J_1, J_2] = -\mathrm{i}J_0$, as usual we define $J_\pm = J_1 \pm \mathrm{i}J_2$. The Casimir operator C is given by $C = J_0(J_0 \pm 1) - J_\mp J_\pm$. The two-variable realization of the $SO(2, 1)$ generators is taken to be

$$\exp(\mp\mathrm{i}\phi) J_\pm = \pm\left(\frac{z(r)^{1/2}(z(r)-1)}{z(r)'}\right)\frac{\partial}{\partial r} - \left(\frac{\mathrm{i}}{2}\frac{(z(r)+1)}{\sqrt{z(r)}}\right)\frac{\partial}{\partial y} + \frac{(z(r)-1)}{2}\left[\frac{(p \mp 1)}{\sqrt{z(r)}} \pm \frac{\sqrt{z(r)}z(r)''}{z(r)'^2}\right] \tag{2}$$

$$J_0 = -\mathrm{i}\frac{\partial}{\partial\phi} \tag{3}$$

where $z(r)' = \mathrm{d}z(r)/\mathrm{d}r$ and p is a function of the Natanzon parameters, independent of $z(r)$ and generally dependent on the energy of the system. The Casimir operator turns out to be

$$C = (z(r)-1)^2\left[\frac{z(r)}{z(r)'^2}\frac{\partial^2}{\partial r^2} + \frac{\mathrm{i}}{4z(r)}\frac{\partial^2}{\partial\phi^2} + \frac{\mathrm{i}p(z(r)+1)}{2(z(r)-1)z(r)}\frac{\partial}{\partial\phi}\right] + (z(r)-1)^2\left[\frac{z(r)z(r)'''}{2z(r)'^3} - \frac{3z(r)z(r)''^2}{4z(r)'^4} - \frac{(p^2-1)}{4z(r)}\right]. \tag{4}$$

Since the representation D^+ is used, the eigenvalues of the compact generator J_0 are known to be

$$m(\nu) = \nu + \tfrac{1}{2} + \sqrt{q(\nu) + \tfrac{1}{4}} \qquad \nu = 0, 1, \ldots \tag{5}$$

and the energy spectrum is given by

$$2\nu + 1 = \alpha(\nu) - \beta(\nu) - \delta(\nu) \tag{6}$$

where

$$\begin{aligned} \alpha(\nu) &= \sqrt{-aE(\nu) + f + 1} = p(\nu) + m(\nu) \\ \beta(\nu) &= \sqrt{-c_0E(\nu) + h_0 + 1} = p(\nu) - m(\nu) \\ \delta(\nu) &= \sqrt{-c_1E(\nu) + h_1 + 1} = \sqrt{4q(\nu) + 1}. \end{aligned} \tag{7}$$

The carrier space of the representation is found to be [4]

$$\Psi_{p(\nu)q(\nu)m(\nu)}(r) \propto \exp(\mathrm{i}m(\nu)\phi)z(r)^{(p(\nu)-m(\nu))/2}(1-z(r))^{\sqrt{4q(\nu)+1}/2}R^{1/4} \times {}_2F_1(-\nu, p(\nu) + m(\nu) - \nu, p(\nu) - m(\nu) + 1, z(r)) \tag{8}$$

where the subindices are the eigenvalues of the Casimir $q(\nu)$, the eigenvalues of the compact generator $m(\nu)$, and the parameter $p(\nu)$. These are the group parameters that characterize the Natanzon potentials.

2. GMP and Eckart potentials

The GMP (V_{gmp}) and Eckart (V_E) potentials are given by

$$V_{gmp} = A_1\left(1 - \frac{B_1}{\exp(\omega r) - 1}\right)^2 + C_1 \tag{9}$$

$$V_E = K_1 + K_2 \coth(\alpha r) + K_3 \operatorname{csch}(\alpha r)^2. \tag{10}$$

The constants C_1 and K_1 allow us to fix the minimum of the energy spectrum. It is an easy matter to check that both expressions coincide if

$$A_1 = \frac{(K_2 + 2K_3)^2}{4K_3} \qquad B_1 = -\frac{4K_3}{K_2 + 2K_3}$$
$$C_1 = \frac{4K_1K_3 - 4K_3^2 - K_2^2}{4K_3} \qquad \omega = 2\alpha \tag{11}$$

or equivalently

$$K_1 = (1 + B_1 + \tfrac{1}{2}B_1^2)A_1 + C_1 \qquad K_2 = -\tfrac{1}{2}A_1B_1(B_1 + 2) \qquad K_3 = \tfrac{1}{4}A_1B_1^2 \tag{12}$$

which show that V_{gmp} and V_E are, in fact, the same function. From now on the notation in [8] is followed for V_E, namely

$$V_E = A^2 + \frac{B^2}{A^2} - 2B\coth(\alpha r) + A(A - \alpha)\operatorname{csch}(\alpha r)^2. \tag{13}$$

The next step is to analyse algebraically the Eckart potential. The GMP is obtained by relating (A_1, B_1, C_1) with (A, B, α).

The Natanzon parameters for the Eckart potential are

$$a = c_0 = \frac{1}{\alpha^2} \qquad c_1 = 0 \qquad h_0 = \frac{(A^2 + B)^2}{A^2\alpha^2} - 1$$
$$h_1 = 4\frac{A(A - \alpha)}{\alpha^2} \qquad f = \frac{(A^2 - B)^2}{A^2\alpha^2} - 1 \tag{14}$$

and the function $z(r)$

$$z(r) = \exp(2\alpha r) \tag{15}$$

as is easily checked. The determination of the energy spectrum is obtained from (6), (7) and (14) after requiring that it increase with ν and $E(\nu = 0) = 0$, the result is

$$E(\nu) = A^2 + \frac{B^2}{A^2} - (A + \alpha\nu)^2 - \frac{B^2}{(A + \alpha\nu)^2} \qquad \nu = 0 \ldots \nu_{\max}. \tag{16}$$

The maximum value for ν is obtained as follows. First we notice that the upper bound of $E(\nu)$ is $E(\nu) \leqslant V_E(r = \infty) = (B - A^2)^2/A^2$. With this result and using (16) we obtain $\nu_{\max} = [(\sqrt{B} - A)/\alpha]$ where $[x]$ is the integer part of x; this also leads to $B > A^2$.

From (16) together with (7) and (14) the values of $q(\nu)$, $m(\nu)$ and $p(\nu)$ are

$$q(\nu) = \frac{A(A - \alpha)}{\alpha^2} \qquad m(\nu) = \frac{A}{\alpha} + \nu \qquad p(\nu) = -\frac{B}{\alpha(A + \alpha\nu)}. \tag{17}$$

The expression for $z(r)$ given in (15) replaced in (2)–(4) give for the $SO(2, 1)$ generators and the Casimir operator

$$
\begin{aligned}
J_{\mp} &= \exp(\mp \mathrm{i}\phi)\left[-\mathrm{i}\cosh(\alpha r)\frac{\partial}{\partial\phi} \mp \frac{\sinh(\alpha r)}{\alpha}\frac{\partial}{\partial r} + p(\nu)\sinh(\alpha r)\right] \\
J_0 &= -\mathrm{i}\frac{\partial}{\partial\phi} \\
C &= \sinh(\alpha r)^2\left[\frac{1}{\alpha^2}\frac{\partial^2}{\partial r^2} + 2\mathrm{i}p(\nu)\coth(\alpha r)\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi^2} - p(\nu)^2\right].
\end{aligned}
\tag{18}
$$

The results (16), (17) and (8) solve completely the Eckart or equivalently the GMP potentials, while (18) displays the generators and Casimir operators.

Remark. Operating with the Casimir on a function $\Phi(r, \phi) = \mathrm{e}^{\mathrm{i}m\phi} f(r)$ it is easily found

$$
\begin{aligned}
[C - q]\Phi(r, \phi) = \frac{\sinh(\alpha r)^2}{\alpha^2}\mathrm{e}^{\mathrm{i}m\phi}\bigg[\frac{-\partial^2}{\partial r^2} - 2B\coth(\alpha r) + \frac{A(A-\alpha)}{\sinh(\alpha r)^2} \\
+ (A+\alpha\nu)^2 + \frac{B^2}{(A+\alpha\nu)^2}\bigg] f(r) = 0
\end{aligned}
$$

so that the radial part of $\Phi(r, \phi)$ is an eigenfunction of the Hamiltonian with an Eckart potential (13) and energy eigenvalue (16) if $\Phi(r, \phi)$ is an eigenfunction of the Casimir (18). This illustrates the relation given in the introduction between C and H in (b).

Next the action of the $SO(2, 1)$ generators on the carrier space is going to be considered. A state labelled by $\{p(\nu), q(\nu), m(\nu)\}$ is given by (17)

$$
\begin{aligned}
\Psi_{p(\nu)q(\nu)m(\nu)} &= S\exp(\mathrm{i}m(\nu)\phi) z(r)^{\frac{1}{2}(p(\nu)-m(\nu))}(1 - z(r))^{\frac{1}{2}(\delta(\nu)+1)} \\
&\quad \times {}_2F_1(-\nu, p(\nu) + m(\nu) - \nu, 1 + p(\nu) - m(\nu), z(r))
\end{aligned}
\tag{19}
$$

where S is a normalization constant.

It is important to notice that there is a set of $SO(2, 1)$ algebras that are labelled by the parameter $p(\nu)$ as is seen from (2); these will be denoted by $SO(2, 1)^{p(\nu)}$. The number of allowed values of $p(\nu)$ is given by $\nu_{\max}$ for a given Eckart potential with (A, B, α) fixed. For each value of $p(\nu)$ there is a single state that belongs to the physical system being treated; all these states have the same label $q(\nu)$ given in (17).

Operating with J_+ on the state (19) leads to a state labelled by $\{p(\nu), q(\nu), m(\nu) + 1\}$, notice that $p(\nu)$ and $q(\nu)$ are fixed since they label a specific representation. From (17) the parameters that characterize the potential must change: invariance of $q(\nu)$ implies that A and α are unchanged, while invariance of $p(\nu)$ requires B to be modified.

From (17) it is seen that $m(\nu) \to m(\nu)+1 = m(\nu+1)$ and since $p(\nu) \to p(\nu+1) = p(\nu)$ it implies

$$
p(\nu) = -\frac{B}{\alpha(A+\alpha\nu)} = -\frac{B_1}{\alpha(A+\alpha(\nu+1))}
\tag{20}
$$

where B_1 is the new value of the parameter B. It is convenient to write this relation in such a way that the ν dependence is exhibited explicitly

$$
B(\nu+1) = B(\nu)\frac{A+\alpha(\nu+1)}{A+\alpha\nu} = B(\nu)\left(1 + \frac{1}{m(\nu)}\right)
\tag{21}
$$

solving this recursion relation leads to

$$
B(\nu) = B_0 m(\nu)
\tag{22}
$$

where B_0 is a constant independent of ν. This ν dependence of $B(\nu)$ amounts to a scaling of B_0 and in this sense this situation is a particular case of the one reported in [9]. The result is that a new Eckart potential has been found in such a way that $(A, B(\nu), \alpha) \to (A, B(\nu+1), \alpha)$. From now on the fixed value of $p(\nu)$ is denoted p_0.

For the representation labelled by $\{q(\nu), p_0\}$ a family of Eckart potentials with parameters $(A, B(k), \alpha)$ has been found with $B(k)$ given by (22) where k labels the states in the representation $\{q(\nu), p_0\}$; B_0 is the parameter for $k = 0$. Next it is asked whether there is an upper bound for the value of k. The answer comes from the observation that $k_{\max}$ grows as $\sqrt{B(k)}$ (due to the comment after (16)), while the label $m(k)$ grows linearly with k; the maximum value $k = K$ is obtained from

$$\sqrt{B_0} = \alpha\sqrt{m(K)} \tag{23}$$

in other words, there is a finite number of Eckart potentials associated to p_0.

Next the explicit result of acting with the $SO(2, 1)$ generators on the state (19) is exhibited. The result for J_+ is

$$\begin{aligned} J_+\Psi_{p(\nu)q(\nu)m(\nu)} &= -S(m(\nu) - p(\nu)) \exp(\mathrm{i}(m(\nu) + 1)\phi) z(r)^{\frac{1}{2}(p(\nu)-m(\nu)-1)} \\ &\quad \times (1 - z(r))^{\frac{1}{2}(\delta(\nu)+1)}\, {}_2F_1(-\nu - 1, p(\nu) + m(\nu) - \nu, p(\nu) - m(\nu), z(r)). \end{aligned} \tag{24}$$

The following identity has been used [10]:

$$\begin{aligned} {}_2F_1(a+1, b+1, c+1, z(r)) &= \frac{-c}{abz(r)(1 - z(r))}((c - 1)\, {}_2F_1(a - 1, b, c - 1, z(r)) \\ &\quad + (z(r)b - c + 1)\, {}_2F_1(a, b, c(r))). \end{aligned}$$

The normalization of (19) is obtained by noting that after acting once with J_+ a factor $p(\nu) - m(\nu)$ appears so that starting from $\nu = 0$ the action of J_+^ν reproduces (19). The value $|S|^2 = \int_0^\infty |\Psi_{p(\nu)q(\nu)m(\nu)}|^2\, \mathrm{d}r$ with $\nu = 0$ is a beta function and therefore, normalization of $\Psi_{p(\nu)q(\nu)m(\nu)}$ follows directly using the method presented in [1].

Similarly, for J_- acting on the state (19), it is found

$$\begin{aligned} J_-\Psi_{p(\nu)q(\nu)m(\nu)} &= S\nu(1 - 2m(\nu) + \nu) \exp(\mathrm{i}(m(\nu) - 1)\phi)\, z(r)^{\frac{1}{2}(p(\nu)-m(\nu)+1)} \\ &\quad (1 - z(r))^{\frac{1}{2}(\delta(\nu)+1)}\, {}_2F_1(-\nu, p(\nu) + m(\nu) - \nu - 1, 2 + p(\nu) - m(\nu), z(r)) \end{aligned} \tag{25}$$

after using [10]

$$\begin{aligned} {}_2F_1(a - 1, b, c - 1, z(r)) &= \frac{1}{1 - c}((1 - c + b)\, {}_2F_1(a, b, c, z(r)) \\ &\quad + (1 - z(r))b\, {}_2F_1(a, b + 1, c, z(r)). \end{aligned}$$

Let us examine the result given in (24). We have proved that this resulting state corresponds to an Eckart potential with parameters $(A, B(\nu+1) = B(m(\nu)+1)/m(\nu), \alpha)$. Therefore, the Natanzon parameters for this system are those given in (14) with $B \to B(\nu+1)$, obviously the corresponding $z(r)$ is the same as given in (15). For the energy spectra we have the expression given in (16), where $B \to B(\nu + 1)$, we then have

$$E(\lambda) = A^2 + \frac{B(\nu+1)^2}{A^2} - (A + \alpha\lambda)^2 - \frac{B(\nu+1)^2}{(A + \alpha\lambda)^2} \qquad \lambda = 0 \ldots \lambda_{\max} \tag{26}$$

where now $\lambda_{max} = [(\sqrt{B(\nu+1)} - A)/\alpha]$. The remaining question that we must answer regarding the state under consideration is which eigenvalue λ corresponds to it. This can be done easily if we look, for example, at the first relation of (7), we have

$$\alpha(\nu) + 1 = \sqrt{-aE(\lambda) + ff + 1} \tag{27}$$

where ff is given by

$$ff = \frac{(A^2 - B(\nu+1))^2}{A^2\alpha^2} - 1$$

as is seen from (14). Using the fact that $\alpha(\nu)$ is obtained from (7) and (17) as

$$\alpha(\nu) = \frac{-B + (A + \nu\alpha)^2}{\alpha(A + \nu\alpha)}$$

than relation (27) is satisfied for $\lambda = \nu + 1$.

3. Final comments

We have shown that the solvability of the GMP is due to the fact that it belongs to the class of the Eckart potential, a member of the hypergeometric Natanzon potentials which is solved algebraically by means of $SO(2, 1)$ algebra. In the carrier space of each $SO(2, 1)^{p(\nu)}$ representation, $CSO(2, 1)^{p(\nu)}$, there are eigenstates of Hamiltonians with different Eckart potentials. It has been shown that a finite number of such potentials appears. The states arise from the applications of the generators of the algebra on states belonging to a particular $CSO(2, 1)^{p(\nu)}$. In other words, in the space S defined as $S = \{CSO(2, 1)^{p(\nu)}; \nu = 0 \ldots \nu_{max}\}$, the states occurring in S are those corresponding to eigenstates of Eckart's potentials in such a way that they have the same parameter A with the parameters B varying according to (22).

In the algebraic SUSYQM [11] treatment of the Eckart potential, the supersymmetric operators connects states as follows: $(A, B, \alpha) \rightarrow (A - \alpha, B, \alpha)$ [8, 12]. Then the supersymmetric partner of (A, B, α) clearly are not in S defined above, since all the states in S share the same Casimir eigenvalues $q(\nu)$ which depend on A as is seen from (17). The result obtained here is a natural extension of the chain of potentials generated by SUSYQM.

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