Algebraic solution for the Natanzon hypergeometric potentials

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An algebraic method—based on a strategy that makes use of a realization of the algebra SO(2,1)—in terms of differential operators is used to solve the bound state problem for the most general Natanzon potentials for which the Schrödinger equation can be reduced to hypergeometric form (hence, *hypergeometric potentials*).

I. INTRODUCTION

The problem of finding the potentials for which the Schrödinger equation can be solved analytically using hypergeometric or confluent hypergeometric functions has been extensively studied. Natanzon[1] found what seems to be the most general potentials for which the Schrödinger equation can be reduced to confluent or general hypergeometric differential equations. These potentials will be called *Natanzon confluent potentials* and *Natanzon hypergeometric potentials*, respectively.

From the algebraic point of view, several potentials have been studied for quite some time. We first mention the *spectrum generating algebras* (SGA) using the SO(2,1) group. This technique has been quite successful in finding the bound state energy spectrum for the Natanzon confluent potentials such as the harmonic oscillator, the Coulomb potential, the Morse potential, and some relativistic cases as well.[2] Recently, it has been proven that the most general Natanzon confluent potential can be solved using the SGA technique by means of the SO(2,1) algebra.[3]

Other techniques have been applied to study these cases, for example, the potential group approach to deal with the bound and scattering sectors as in Ref. 4 and references therein. The algebraic treatment of the hypergeometric case has also been analyzed in Ref. 5. Other authors have dealt with the SO(2,2) group to study the bound states and scattering problems (Ref. 6 and references therein). See also Ref. 7.

On the other hand, supersymmetric quantum mechanics (SUSYQM) has been used as an algebraic method to find new solvable potentials as in Ref. 8. The SUSYQM techniques lead to purely algebraic solutions when the potentials are *shape invariant*.[9,10] Results of this case are also found in Ref. 11. More recent developments on algebraic and analytic methods are founds in Refs. 12–14.

An ingenious and rather different approach to tackle the hypergeometric case that will interest us in the present article was developed more than two decades ago by Ghirardi in Ref. 15. The method was successfully used in Ref. 16 to solve the bound state problem for several cases.

In the present work we prove that the Natazon hypergeometric potentials are completely solved using the techniques of Ref. 15 based on the SO(2,1) group. In this sense this article is a logical continuation of the SGA treatment of the confluent general case.[3] In Sec. II we analyze the SO(2,1) realization used in Ref. 15, to study hypergeometric potentials. In Sec. III the algebraic solution of the general Natanzon hypergeometric potential is presented. The wave functions are discussed in Sec. IV. In Sec. V an illustrative example is given and finally in Sec. VI some final comments are made.

It is worth mentioning that the parameters used in the present article can be directly related with the parameters occuring in the potential group approach[6,7] that makes use of the SO(2,2) algebra.

II. REALIZATION OF SO(2,1)

The way we are going to deal with the bound state problem associated to the Natazon hypergeometric potentials differs substantially from that of the confluent case. In the confluent case, the analysis is based on the fact that the Hamiltonian operator can be expressed as a *linear* combination of the generators of SO(2,1).[3]

The strategy designed by Ghiraradi[15] for the hypergeometric potentials on the other hand consists of (i) considering a particular realization of SO(2,1), (ii) assuming that the Hamiltonian is related to the (*quadratic*) Casimir operator of the algebra

$$(Q-q)\Psi(r,\phi)=\mathcal{G}(r)(E-H)\Psi(r,\phi), \tag{1}$$

where the Hamiltonian H (with $\hbar=1$ and the particle's mass $m=\frac{1}{2}$) is

$$H=-\frac{d^2}{dr^2}+V(r)+\frac{\ell(\ell+1)}{r^2}, \tag{2}$$

where q is the eigenvalue of the Casimir operator and $\mathcal{G}(r)$ is a function of r determined by self-consistency; and (iii) taking the eigenfunctions of the Casimir operator (or equivalently of the Hamiltonian) having the form

$$\Psi(r,\phi)=\exp[im\phi]\Phi(r). \tag{3}$$

The realization of the SO(2,1) algebra to be used[15,17] is obtained as follows. The starting point is

$$J_\pm=\exp[\pm i\phi]\left(\pm A(r)\frac{\partial}{\partial r}-iB(r)\frac{\partial}{\partial\phi}+\frac{1}{2}(C_1(r)\pm C_2(r))\right), \tag{4}$$

$$J_0=-i\frac{\partial}{\partial\phi} \tag{5}$$

but after requesting that these operators obey the SO(2,1) commutation relations $[J_0,J_1]=iJ_2$, $[J_2,J_0]=iJ_1$, $[J_1,J_2]=-iJ_0$ where $J_\pm=J_1\pm iJ_2$, it is seen that it is necessary to satisfy

$$A(r)=\frac{1+B(r)^2}{B'(r)},\quad A(r)C_1'(r)=B(r)C_1(r),$$

where the primes denote derivative with respect to r.

It is straightforward to check that the above system is an identity if

$$C_1(r)=p\sqrt{1+B(r)^2}, \tag{6}$$

where $B(r)$ is an arbitrary function and p is an arbitrary integration constant. The fact that the algebra closure conditions leave free an arbitrary constant plays a central role in making this method powerful. This constant, plus the eigenvalue q of the Casimir operator Q, and m, the eigenvalue of the compact generator J_0, provides us precisely with the three parameters necessary to solve all the hypergeometric Natanzon potentials.

The rest is quite simple. The operator Q has in general both first and second derivatives with respect to r but from Eqs. (1) and (2) it is seen that we only need it to have second derivatives. This is achieved by requesting that[15]

$$C_2(r)=\frac{1+B^2}{B'^2}\,B''-B. \tag{7}$$

Finally, the change of function

$$B(r)=\frac{i}{2}\,\frac{1+z(r)}{\sqrt{z(r)}} \tag{8}$$

makes the generators take the form

$$\begin{aligned} J_\pm&=\exp[\pm i\phi]\left[\pm\frac{i\sqrt{z}}{z'}\,(z-1)\,\frac{\partial}{\partial r}+\frac{\sqrt{z}}{2z}\,(z+1)\,\frac{\partial}{\partial\phi}\mp\frac{1}{8}\,\frac{z''}{z'}\,(z-1)^2\pm\frac{z'}{16z^2}\,((z-2)^2-1)\right.\\ &\left.\pm\frac{ip}{2}\,\frac{\sqrt{z}}{z}\,(z-1)\mp\frac{i}{4z}\,\sqrt{z}(z+1)\right],\\ J_0&=-i\,\frac{\partial}{\partial\phi}, \end{aligned} \tag{9}$$

where—since $B(r)$ is an arbitrary function—$z(r)$ is an arbitrary function as well. The Casimir operator $Q=J_0^2-J_1^2-J_2^2$ that stems from the above realization is

$$\begin{aligned} Q&=\frac{z(z-1)^2}{z'^2}\,\frac{\partial^2}{\partial r^2}+\frac{1}{4}\,\frac{(z-1)^2}{z}\,\frac{\partial^2}{\partial\phi^2}+\frac{p}{2iz}\,(1-z^2)\,\frac{\partial}{\partial\phi}+\frac{z'''}{z'^3}\,\frac{z}{2}\,(1-z)^2-\frac{3}{4}\,\frac{z''^2}{z'^4}\,z(z-1)^2\\ &-\frac{p^2}{4}\,\frac{(1-z)^2}{z}+\frac{z}{4}+\frac{1}{4z}-\frac{1}{2}. \end{aligned} \tag{10}$$

In Sec. IV it will be seen that $z(r)$ introduced in Eq. (8) is the function that appears as the argument of the hypergeometric functions ${}_2F_1(a,b,c;z)$ of the carrier space of the representation.

Of interest is the fact that in Ref. 15 there are a few misprints for the expressions of $J_\pm$ and Q. We use the representation D^+ which is bounded below, therefore the operator J_0 has the eigenvalues m as can be seen from Eqs. (3) and (9). It is well known[18] that these eigenvalues are given by

$$m=\nu+\tfrac{1}{2}+\sqrt{q+\tfrac{1}{4}}, \tag{11}$$

with $\nu=0,1,2,\ldots$

Spherically symmetric Hamiltonians (1)–(10) imply

$$E-V(r)-\frac{\ell(\ell+1)}{r^2}=\frac{z'''}{2z'}-\frac{3z''^2}{4z'^2}-\frac{qz'^2}{z(z-1)^2}-\frac{(m^2+p^2)z'^2}{4z^2}+\frac{mpz'^2(1+z)}{2z^2(1-z)}+\frac{z'^2}{4z^2}, \tag{12}$$

$$\mathscr{G}(r)=\frac{z(z-1)^2}{z'^2}. \tag{13}$$

In the following we take $\ell=0$.

III. NATANZON HYPERGEOMETRIC POTENTIALS. ALGEBRAIC SOLUTION

The Natanzon hypergeometric potentials[1] depend on six parameter: f, h_0, h_1, a, c_0, c_1, and on the function $z(r)$

$$V(r)=\frac{fz^2-(h_0-h_1+f)z+h_0+1}{R}+\left[a+\frac{a+(c_1-c_0)(2z-1)}{z(z-1)}-\frac{5\Delta}{4R}\right]\left[\frac{z(1-z)}{R}\right]^2, \quad (14)$$

where τ and Δ are constants while R is a quadratic polynomial on z

$$R=az^2+\tau z+c_0, \quad \tau=c_1-c_0-a, \quad \Delta=\tau^2-4ac_0 \quad (15)$$

(we are closely following the notation of Ref. 11). Furthermore $z(r)$ satisfies

$$\frac{dz}{dr}=\frac{2z(1-z)}{\sqrt{R}}. \quad (16)$$

The function $z(r)$ is the same one used in Sec. II.

To go on with the algebraic method it is convenient to rewrite the right hand side of Eq. (12) in the form of the Natanzon hypergeometric potential (14). Using Eqs. (15) and (16) and after some algebraic manipulations the right hand side of Eq. (12) can be written as

$$\begin{aligned}\text{rhs}=&-\frac{((m+p)^2-1)z^2-2(m^2+p^2-2q-1)z+(m-p)^2}{R}\\&-\left[a+\frac{a+(c_1-c_0)(2z-1)}{z(z-1)}-\frac{5\Delta}{4R}\right]\left[\frac{z(1-z)}{R}\right]^2.\end{aligned}$$

Equating this expression with the left hand side of Eq. (12) and using explicitly the potential (14) one obtains

$$ER=z^2(f+1-(m+p)^2)+z(2(m^2+p^2-2q-1)-(h_0-h_1+f))+h_0+1-(m-p)^2. \quad (17)$$

Both sides of this equation are second degree polynomials in z. Equating the coefficients of the powers of z on both sides yields

$$\begin{aligned}aE&=f+1-(m+p)^2,\\ \tau E&=2(m^2+p^2-2q-1)-(h_0-h_1+f),\\ c_0E&=h_0+1-(m-p)^2.\end{aligned} \quad (18)$$

This is a linear system for $(m+p)^2$, $(m-p)^2$, and q giving

$$(p+m)^2=-aE+f+1, \quad (p-m)^2=-c_0E+h_0+1, \quad 4q+1=-c_1E+h_1+1. \quad (19)$$

Subtracting the square root of the second equation from the first one yields an expression for m which we equate with Eq. (11) to obtain the energy spectrum equation

$$2\nu+1=\alpha_\nu-\beta_\nu-\delta_\nu, \quad (20)$$

where

$$\alpha_\nu=\sqrt{-aE+f+1}=p+m,$$

$$\beta_\nu=\sqrt{-c_0E+h_0+1}=p-m, \tag{21}$$

$$\delta_\nu=\sqrt{-c_1E+h_1+1}=\sqrt{4q+1}.$$

Equation (20) gives the energy spectrum for the most general Natanzon hypergeometric potential. It coincides with the result given in Ref. 11. It is important to realize that α_ν and β_ν are square roots and the specific sign they take varies in different examples. In Sec. V these signs are determined in one particular case.

IV. THE WAVE FUNCTION

Our next task is to relate the group parameters m, p, and q with the parameters that appear in the hypergeometric differential equation related to the Natanzon potential given in Eq. (14). The carrier space of the representation of the algebra defined in Eq. (9) is[15]

$$\Phi(r)=\frac{z^{c/2}(1-z)^{(a+b+1-c)/2}}{\sqrt{z'}}\ {}_2F_1(a,b,c;z), \tag{22}$$

where $z=z(r)$ is the same function used in Sec. II. The parameters a, b, and c are those that appear in ${}_2F_1(a,b,c;z)$ and a should not be confused with the parameter that appears in the potential (14). Making a detailed comparison of Eq. (12) with the hypergeometric differential equation for ${}_2F_1(a,b,c,z(r))$ (as in Appendix A of Ref. 15) it follows that the parameters a, b, and c are constrained to satisfy

$$(m+p)^2=(b-a)^2, \quad (p-m)^2=(c-1)^2, \quad 4q+1=(a+b-c)^2. \tag{23}$$

One possible solution for this set of equations is

$$b-a=p+m=a_\nu, \quad c-1=p-m=\beta_\nu, \quad a+b-c=\sqrt{4q+1}=\delta_\nu. \tag{24}$$

With all the previous results, the wave function is

$$\Phi_{qmp}(r)\propto z^{\beta_\nu/2}(1-z)^{\delta_\nu/2}R^{1/4}{}_2F_1(-\nu,\alpha_\nu-\nu,1+\beta_\nu,z), \tag{25}$$

which coincides with the expression given in Ref. 11.

Instead of labeling the states with a, b, and c we would rather use q, m, and p which are the natural parameters which occur in the algebraic realization (9) and (10).

V. EXAMPLE

A simple case comes about by choosing the potential (14) with $f=4A(A+D)/D^2$, $h_0=(B-A^2+DA)(B-A^2-DA)/(DA)^2$, $h_1=(B+A^2+DA)(B+A^2-DA)/(DA)^2$, $c_0=1/D^2$, $c_1=c_0$, $a=0$ and $R(z)=c_0$ and $z(r)=\frac{1}{2}(1+\tanh(Dx))$. The potential simplifies to the Rosen–Morse potential

$$V(r)=A^2+\frac{B^2}{A^2}+2B\tanh(Dx)-A(A+D)\operatorname{sech}^2(Dx). \tag{26}$$

The possible ambiguities related to Eq. (21) disappear when it is required that the energy $E(\nu)$ grows with ν, the ground state is larger than the minimum value of the potential, and the maximum eigenvalue is less than the asymptotic value of the potential. From this it is seen that $\partial\leqslant(A-\sqrt{B})/D$ and the values of the parameters to solve this problem are seen from Eq. (21) to be

$$\alpha = p + m = \frac{2A + D}{D},$$

$$\beta_\nu = p - m = \frac{1}{AD}\sqrt{(A^2 - B)^2 - A^2 E_\nu}, \tag{27}$$

$$\delta_\nu = \sqrt{4q + 1} = \frac{1}{AD}\sqrt{(A^2 + B)^2 - A^2 E_\nu}.$$

Since α as defined in Eq. (21) in the present example does not depend on ν, we have omitted that subindex. The energy spectrum then is

$$E_\nu = A^2 + B^2/A^2 - (A - \nu D)^2 - \frac{B^2}{(A - \nu D)^2}. \tag{28}$$

From this example it is seen that the states of the system—labeled $\{q,m,p\}$—are in a different representation of the SO(2,1) algebra. This should be clear because the Casimir operator Q is directly connected to the Hamiltonian H [eqs. (1), (2)].

VI. FINAL COMMENTS

In this article we have shown that a particular realization of the SO(2,1) algebra completely solves all the class of hypergeometric Natanzon potentials. It is also shown that an integration constant—that we call p—together with the eigenvalue q of the Casimir operator Q and the eigenvalue m of the compact generator J_0, provide us with the three necessary parameters to solve the above potentials. The solution gives the bound state energy spectra and the corresponding wave functions.

As a conclusion, we can say that a complete algebraic description of the Natanzon confluent and hypergeometric potentials can be achieved using SO(2,1) algebras.

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[1] G. A. Natanzon, Teor. Mat. Fiz. **38**, 146 (1979).
[2] P. Cordero and S. Hojman, Lett. Nuovo Cimento **4**, 1123 (1970); P. Cordero and G. C. Ghirardi, Nuovo Cimento A **2**, 217 (1971); P. Cordero, S. Hojman, P. Furlan, and G. C. Ghirardi, *ibid.* **3**, 807 (1971); P. Cordero and G. C. Ghirardi, Fortschr. Phys. **20**, 105 (1972); P. Cordero and S. Salamó, Int. J. Theor. Phys. **13**, 265 (1975).
[3] P. Cordero and S. Salamó, J. Phys. A **24**, 5299 (1991); P. Cordero and S. Salamó, Found. Phys. **23**, 675 (1993).
[4] Y. Alhassid, F. Gürsey, and F. Iachello, Phys. Rev. Lett. **50**, 873 (1983); Ann. Phys. **148**, 346 (1983); Y. Alhassid, F. Iachello, and J. Wu, Phys. Rev. Lett. **56**, 271 (1986).
[5] A. O. Barut, A. Inomata, and R. Wilson, J. Phys. A **20**, 4083 (1987).
[6] J. Wu, Y. Alhassid, and F. Gürsey, Ann. Phys. **196**, 163 (1989).
[7] J. Wu and Y. Alhassid, J. Math. Phys. **31**, 557 (1990).
[8] F. Cooper, J. N. Ginocchio, and A. Wipf, J. Phys. A **22**, 3707 (1989); G. Levai, *ibid.* **22**, 689 (1991); P. Cordero and S. Salamó, in *Condensed Matter Theories*, edited by A. N. Proto and J. Aliaga (Plenum, New York, 1992), Vol. 7; F. Cooper, J. N. Ginocchio, and A. Wipf, Phys. Rev. Lett. A **129**, 145 (1988).
[9] L. Gendeshtein, JETP. Lett. **38**, 356 (1983).
[10] R. Dutt, A. Khare, and U. P. Sukhatme, Phys. Rev. Lett. B **181**, 295 (1986); J. W. Dabrowska, A. Khare, and U. P. Sukhatme, J. Phys. A **21**, L125 (1988).
[11] F. Cooper, J. Ginocchio, and A. Khare, Phys. Rev. D **36**, 2458 (1987).
[12] S. Brajamani and C. A. Singh, J. Phys. A **23**, 3421 (1990).
[13] M. J. Englefield and C. Quesne, J. Phys. A **24**, 3557 (1991).
[14] L. Salem and R. Montemayor, Phys. Rev. A **43**, 1169 (1991); L. Chetouani, L. Guechi, A. Lecheheb, and T. F. Harmann, J. Math. Phys. **34**, 1257 (1993).

[15] G. C. Ghirardi, Nuovo Cimento A **10**, 97 (1972); Fortschr. Phys. **21**, 653 (1973).
[16] M. Villasante, MSc thesis, Universidad de Chile, 1980.
[17] W. Miller, *Lie Theory and Special Functions* (Academic, New York, 1968).
[18] A. O. Barut and C. Fronsdal, Proc. R. Soc. London, Ser. A **287**, 532 (1965); B. G. Wybourne, *Classical Groups for Physicist* (Wiley, New York, 1974).