Extended formulations in combinatorial optimization: constructions and lower bounds

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VI IPCO Summer School
VIII Escuela de Verano en Matemáticas Discretas

Day 1

Today

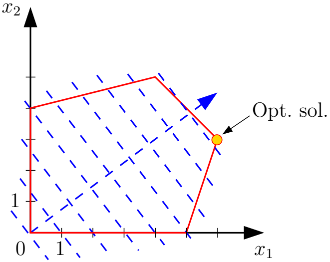
1. Introduction
2. Basics on polytopes and polyhedra
3. Factorization theorem

Today

Linear programming

Example

$$
\begin{aligned}
\max \quad & 4x_1 + 3x_2 \\
\text{subject to} \quad & -x_1 + 4x_2 \leqslant 16 \\
& x_1 + x_2 \leqslant 9 \\
& 3x_1 - x_2 \leqslant 15 \\
& x_1 \geqslant 0 \\
& x_2 \geqslant 0
\end{aligned}
$$



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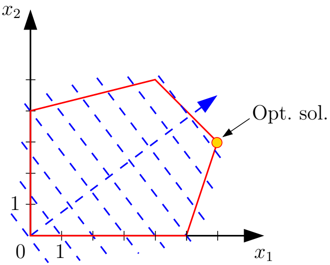


LP in general form:

$$\begin{aligned} \max \quad & \sum_{i=1}^{d} w_i x_i = w^\mathsf{T} x \\ \text{subject to} \quad & Ax \leqslant b \end{aligned}$$

Linear programming

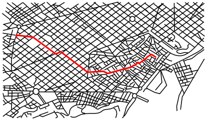
A successful tool

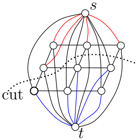
Fact

Linear programming is **very successful** both in practice and theory

3 reasons for this success:

1. Many **problems** can be expressed as LPs





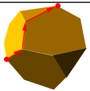
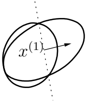
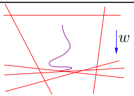
2. Rich **theory**: LP duality can certify the quality of solutions

$$\begin{array}{rl} \max & w^\intercal x \\ \text{s.t.} & Ax \leqslant b \end{array} \quad = \quad \begin{array}{rl} \min & b^\intercal y \\ \text{s.t.} & A^\intercal y = w \\ & y \geqslant 0 \end{array}$$

3. There exist powerful **algorithms** for solving LPs

Linear programming

Algorithms

Algorithm / method	Poly-time?	In practice?	Generalizes to convex?
 (simplex)	No	**Very fast**	No
 (ellipsoid)	**Yes**	Slow	Yes
 (int. point)	Yes	Fast	**Yes**

Linear programming
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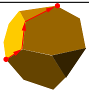
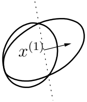
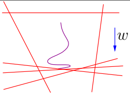
 (simplex)	No	**Very fast**	No
 (ellipsoid)	**Yes**	Slow	Yes
 (int. point)	Yes	Fast	**Yes**

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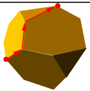
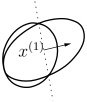
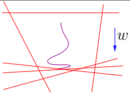
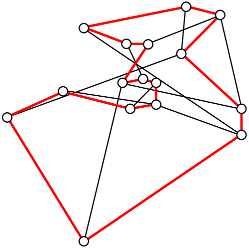
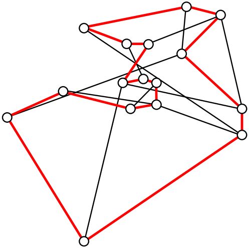
 (simplex)	No	**Very fast**	No
 (ellipsoid)	**Yes**	Slow	Yes
 (int. point)	Yes	Fast	**Yes**

All practical algorithms have sensitivity to **size** of input LP

$\Longrightarrow$ try to minimize size: variables, constraints are **resources**



TSP: Given a graph $G = (V, E)$ and distances d_{ij} for each $ij \in E$, find tour of minimum length.



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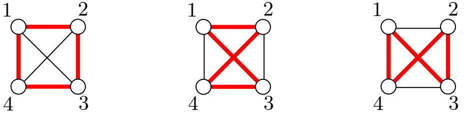
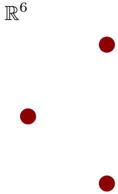
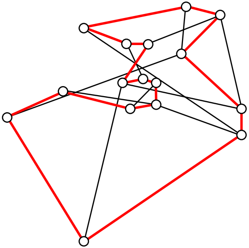


$(1, 0, 1, 1, 0, 1)$ $(1, 1, 0, 0, 1, 1)$ $(0, 1, 1, 1, 1, 0)$



Combinatorial optimization

LP formulation of TSP



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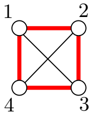

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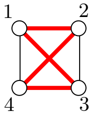

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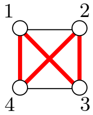

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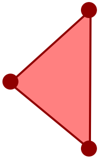
$\mathbb{R}^6$



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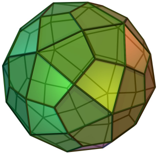
$$\mathrm{TSP}(G) := \mathrm{conv}\{x \in \{0,1\}^{|E|} \mid x \text{ encodes tour of } G\}$$



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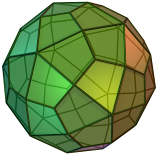

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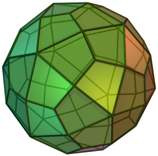

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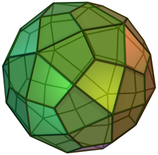

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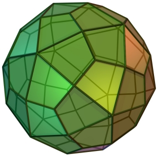

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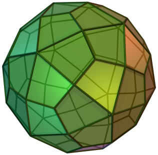

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In such a way that: $Ax \geqslant b \iff \exists y : Ex + Fy = g,\ y \geqslant \mathbf{0}$

Extended formulations

Reformulating an LP

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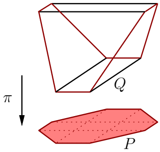

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Definition (Extension)

Polytope Q in $\mathbb{R}^e$ is an extension of P if $\exists$ linear π with $\pi(Q) = P$

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Size of EF $:=$ #inequalities in EF

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Extended formulations

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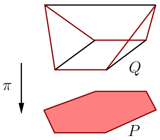

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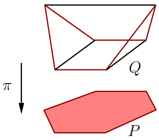

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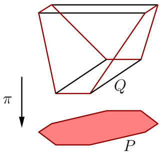

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If P is a regular n-gon, then xc$(P) = O(\log n)$!!!

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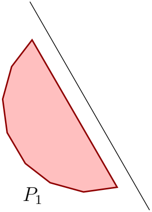

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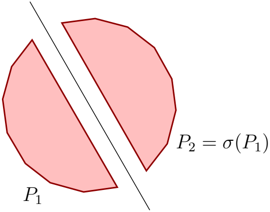

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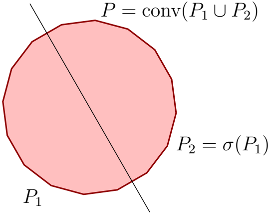

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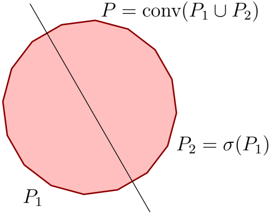

$$\mathsf{xc}(P) = \mathsf{xc}(\mathsf{conv}(P_1 \cup \sigma(P_1))) \leqslant \mathsf{xc}(P_1) + 2$$

Fundamental question and dictionary

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Regarding (variables and) constraints as resources, what problems can be solved via LP with a given amount of resources?

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Dictionary:

Algorithmic	Geometric
Problem	Polytope
Algorithm	EF
Complexity	Size
Hardness result	Lower bound

Course outline

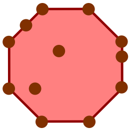
In this course on extended formulations (EFs), we will see:

1. many examples of EFs
2. techniques for constructing EFs
3. techniques for proving lower bounds on the size of EFs
4. a proof that TSP(K_n) has no polynomial-size EF

Today

(Convex) polytopes

1. A V-polytope is the convex hull of a finite set of points



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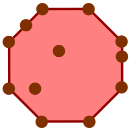

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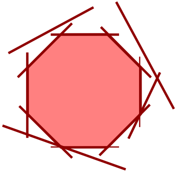
2. A *H-polytope* is the intersection of a finite # of halfspaces (provided this intersection is bounded)



$$P = \{x \in \mathbb{R}^d \mid h_1(x) \leqslant 0, \ldots, h_m(x) \leqslant 0\}$$

where

$$h_i(x) = A_i x - b_i \qquad \forall i$$

Fundamental theorem

Theorem

For all $P \subseteq \mathbb{R}^d$:

$$P \text{ is a } V\text{-polytope} \iff P \text{ is a } H \text{ polytope}$$

Proof ($\Longrightarrow$)

$$\begin{aligned} P &= \text{conv}\{v_1, \ldots, v_n\} \\ &= \left\{ x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^n : x = \sum_{j=1}^{n} y_j v_j,\ \sum_{j=1}^{n} y_j = 1,\ y_j \geqslant 0\ \forall j \right\} \\ &= \left\{ x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^n : Ex + Fy = g,\ y \geqslant 0 \right\} \end{aligned}$$

So P is the projection into x-space of a H-polytope! This is our 2nd example of an EF. (**Exercise.** What kind of polytope is Q?)

$$P = \text{proj}_x(Q) \quad \text{where} \quad Q = \left\{ (x, y) \in \mathbb{R}^{d+n} \mid Ex + Fy = g,\ y \geqslant 0 \right\}$$

Eliminating one variable (Fourier-Motzkin elimination)

To conclude the proof, it suffices to show that eliminating a single variable from a system of linear constraints "produces" a new system of linear constraints (changing notations):

$$
\begin{array}{rcl}
Q & = & \{(x,y) \in \mathbb{R}^{d+1} \mid A_i x + b_i y \leqslant c_i \ \forall i\} \\
 & = & \left\{ (x,y) \in \mathbb{R}^{d+1} \mid \begin{array}{rcll} A_{i_0} x & \leqslant & c_{i_0} & \forall i_0 \in I_0 \\ y & \leqslant & b_{i_+}^{-1} c_{i_+} - b_{i_+}^{-1} A_{i_+} x & \forall i_+ \in I_+ \\ y & \geqslant & b_{i_-}^{-1} c_{i_-} - b_{i_-}^{-1} A_{i_-} x & \forall i_- \in I_- \end{array} \right\}
\end{array}
$$

where $I_0 = \{i \mid b_i = 0\}, \quad I_+ = \{i \mid b_i > 0\}, \quad I_- = \{i \mid b_i < 0\}$

Then $\mathrm{proj}_x(Q)$ is defined by:

$$
\begin{array}{rcll}
A_{i_0} x & \leqslant & c_{i_0} & \forall i_0 \in I_0 \\
b_{i_-}^{-1} c_{i_-} - b_{i_-}^{-1} A_{i_-} x & \leqslant & b_{i_+}^{-1} c_{i_+} - b_{i_+}^{-1} A_{i_+} x & \forall i_+ \in I_+, i_- \in I_-
\end{array}
$$

$\square$

Projections

From proof, see that if $Q = \{(x, y) \in \mathbb{R}^{d+k} \mid Ax + By \leqslant c\}$ then

$$\text{proj}_x(Q) = \{x \in \mathbb{R}^d \mid u^\intercal Ax \leqslant u^\intercal c \text{ for finite \# of } u \in C\},$$

where $C := \{u \in \mathbb{R}^m \mid u^\intercal B = 0, u \geqslant 0\}$ is the projection cone

Farkas' lemma

Lemma (Farkas' lemma)

If $P = \{x \in \mathbb{R}^d \mid Ax \leqslant b\}$ is a H-polyhedron and $c^\intercal x \leqslant \delta$ is valid for P, then either

- $P = \varnothing$ *or*
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that is, either

- $\exists u \geqslant 0 : u^\intercal A = 0,\ u^\intercal b = -1$ *or*
- $\exists u \geqslant 0 : u^\intercal A = c^\intercal,\ u^\intercal b = \delta' \leqslant \delta$

Polar of a projection is an intersection

To prove other direction of the fundamental theorem, we polarize

If P is (V- or H-)polytope in $\mathbb{R}^d$ with $0 \in \text{int}(P)$, polar of P is

$$P^{\Delta} := \{z \in \mathbb{R}^d \mid \forall x \in P : x^{\intercal} z \leqslant 1\}$$

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Then (two last eqs. from Farkas' lemma):

$$(\mathrm{conv}\{v_1, \dots, v_n\})^\Delta = \{x \in \mathbb{R}^d : v_1^\intercal x \leqslant 1, \dots, v_n^\intercal x \leqslant 1\}$$

$$\Big(\{x \in \mathbb{R}^d : A_1 x \leqslant 1, \dots, A_m x \leqslant 1\}\Big)^\Delta = \mathrm{conv}\{A_1^\intercal, \dots, A_m^\intercal\}$$

$$(P^\Delta)^\Delta = P$$

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$$\Big(\{x \in \mathbb{R}^d : A_1 x \leqslant 1, \ldots, A_m x \leqslant 1\}\Big)^{\Delta} = \mathrm{conv}\{A_1^{\mathsf{T}}, \ldots, A_m^{\mathsf{T}}\}$$

$$(P^{\Delta})^{\Delta} = P$$

Exercise. Prove that the polar of projecting polytope Q into x-space is intersecting polytope Q^{Δ} with x-space, and use this to prove other direction of fundamental theorem.

Elimination blows up the number of inequalities

If Q is defined by m constraints

Then $\text{proj}_x(Q)$ is defined by at most $\frac{m^2}{4}$ inequalities

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EFs try to exploit this phenomenon: by adding few variables, decrease much the number of inequalities

Faces of a polytope

Definition (Face)

If P is a polytope in $\mathbb{R}^d$ and $c^\intercal x \leqslant \delta$ is valid for P, then

$$F := P \cap \{x \in \mathbb{R}^d \mid c^\intercal x = \delta\} \qquad \text{is a face of } P$$

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Particular cases:

- $\varnothing$ is a face, of dimension -1 (use $0^\intercal x \leqslant 1$)
- a face of dimension 0 is a vertex
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$\mathcal{L}(P) = (\mathcal{F}(P), \subseteq)$ face lattice of P

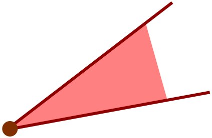
Cones

1. A *V-cone* is the nonnegative hull of a finite set of vectors

$$C = \text{cone}\{g_1, \ldots, g_k\}$$

2. A *H-cone* is the intersection of a finite # of *linear* halfspaces

$$C = \{x \in \mathbb{R}^d \mid Ax \geqslant 0\}$$



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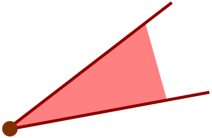

$$\forall C \subseteq \mathbb{R}^d : C \text{ is a } V\text{-cone} \iff C \text{ is a } H\text{-cone}$$

General polyhedra

1. A *V*-polyhedron is the sum of a polytope and a cone

$$P = \text{conv}\{v_1, \ldots, v_n\} + \text{cone}\{g_1, \ldots, g_k\}$$

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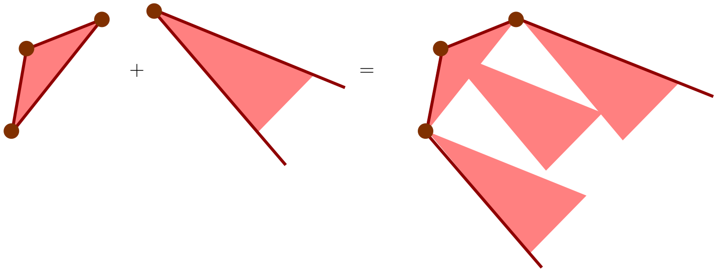

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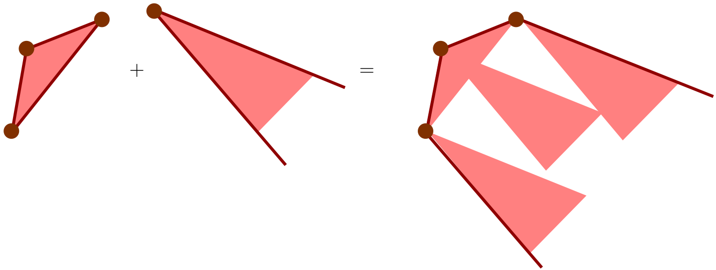

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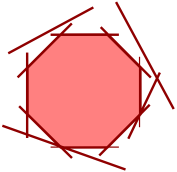
Today

Slack matrices

. . . Of polytopes



$$P = \{x \mid A_1 x \leqslant b_1, \ldots, A_m x \leqslant b_m\}$$

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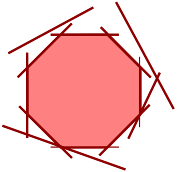

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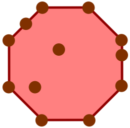

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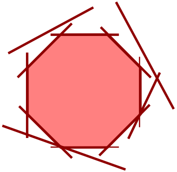
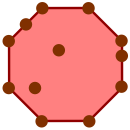
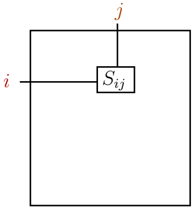


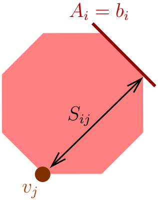


$P = \{x \mid A_1 x \leqslant b_1, \ldots, A_m x \leqslant b_m\}$ $\qquad$ $P = \text{conv}\{v_1, \ldots, v_n\}$

Definition

Slack matrix $S \in \mathbb{R}_+^{m \times n}$ of polytope P: $\qquad S_{ij} := b_i - A_i v_j$





Nonnegative factorizations

Definition

A rank-r nonnegative factorization of $S \in \mathbb{R}^{m \times n}$ is

$$S = TU \quad \text{where} \quad T \in \mathbb{R}_{+}^{m \times r} \quad \text{and} \quad U \in \mathbb{R}_{+}^{r \times n}$$

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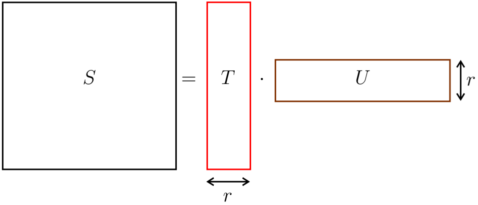

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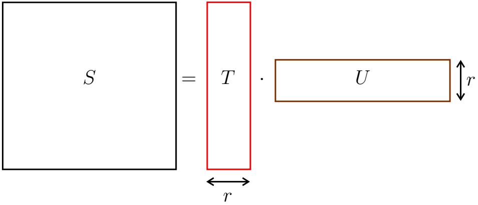

Definition (nonnegative rank of S)

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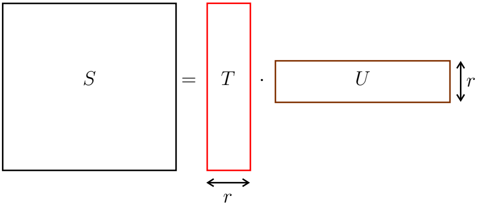

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$$\begin{aligned}\mathrm{rk}_+(S) &:= \min\{r \mid \exists \text{ rank-}r \text{ nonnegative factorization of } S\}\\ &= \min\{r \mid S \text{ is sum of } r \text{ nonnegative rank-1 matrices}\}\end{aligned}$$

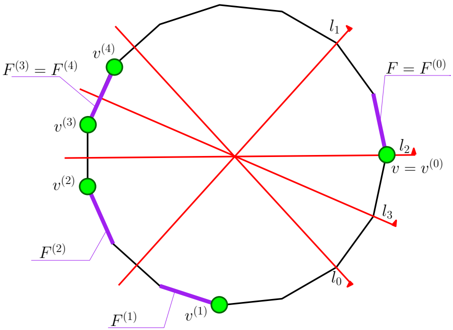
Factorization theorem

Theorem (Yannakakis' factorization theorem)

Let P be a polytope with $\dim(P) \geqslant 1$.
For every slack matrix S of P: $\mathit{xc}(P) = \mathit{rk}_+(S)$

Factorization theorem

Example: regular polygon



Halfspaces: ℓ_1^+, ℓ_2^+, $\ldots$
Folding sequence of vertex v: $v^{(0)} = v$, $v^{(1)}$, $v^{(2)}$, $\ldots$
Folding sequence of facet F: $F^{(0)} = F$, $F^{(1)}$, $F^{(2)}$, $\ldots$
Slack ($v^{(i)}$ w.r.t. $F^{(i)}$) = slack ($v^{(i+1)}$ w.r.t. $F^{(i+1)}$) + correction