Extended formulations in combinatorial optimization: constructions and lower bounds

Samuel Fiorini

Université libre de Bruxelles (ULB, Brussels)

VI IPCO Summer School
VIII Escuela de Verano en Matemáticas Discretas

Day 2

Today

1. Quick recapitulation
2. More examples and techniques
3. Communication complexity
4. Combinatorial lower bounds

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- Factorization theorem [**Yannakakis**'91]: $\text{xc}(P) = \text{rk}_+(S)$

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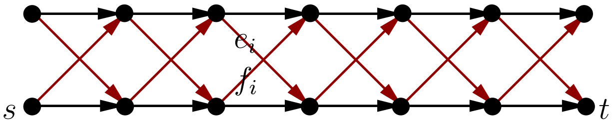

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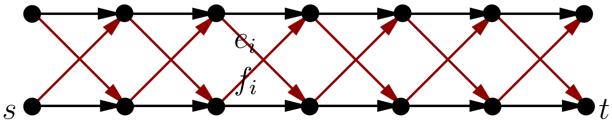

$$\text{EVEN}(n) = \{x \in \mathbb{R}^n \mid \exists y \ s\text{–}t \text{ flow of value } 1 : x_i = y_{e_i} + y_{f_i} \ \forall i\}$$

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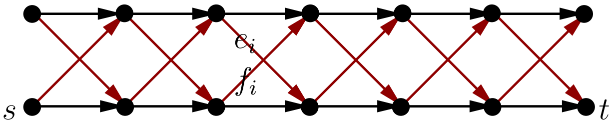

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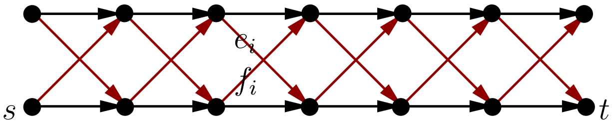

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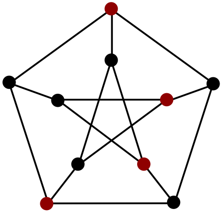
Exercise. Generalize this to all regular languages (that is, languages decided by a deterministic finite automaton).

Stable set polytope

Stable set polytope of graph $G = (V, E)$:

$$\text{STAB}(G) := \text{conv}\{\chi^S \in \{0,1\}^V \mid S \text{ stable set of } G\}$$



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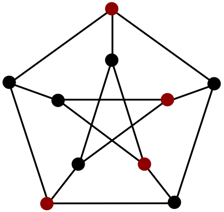

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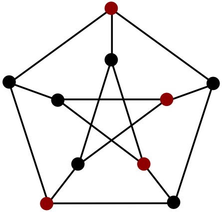

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Theorem (**Chvátal**'75)

STAB$(G) =$ *QSTAB*(G) *if and only if* G *is perfect*

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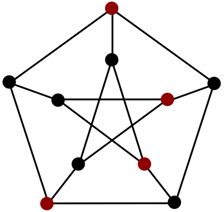

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Theorem (**Chvátal**'75)

$STAB(G) = QSTAB(G)$ if and only if G is perfect, that is,

$\chi(H) = \omega(H)$ for all $H \overset{ind}{\subseteq} G$ (χ = chromatic #; ω = clique #)

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. . . of a comparability graph

$P = (V, \leqslant_P)$ partially ordered set (poset)

$G = G(P)$ comparability graph of P (always perfect)

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- $\forall v \in V$: (y_{v^-}, y_{v^+}) open interval $\subseteq (0, 1)$
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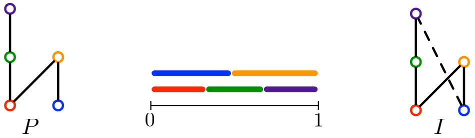

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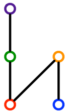

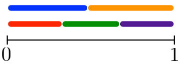

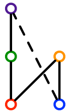

$$\text{STAB}(G(P)) = \Big\{ x \in \mathbb{R}^V \;|\; \exists \{(y_{v^-}, y_{v^+})\}_{v \in V} \text{ consistent with } P \text{ s.t. } x_v = y_{v^+} - y_{v^-} \quad \forall v \in V \Big\}$$

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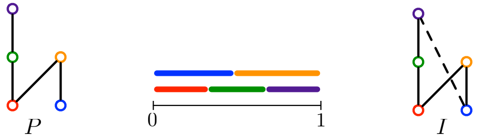

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Exercise. Prove this.

Union of polyhedra

Theorem (Balas'98, 85)

Consider k polyhedra $P^i := \{x \in \mathbb{R}^d \mid A^i x \leqslant b^i\}$. For each i:

- *rec*$(P^i) := \{x \mid A^i x \leqslant 0\} =$ *cone*$\{r^i_1, \ldots, r^i_{q_i}\}$
- $P^i =$ *conv*$\{v^i_1, \ldots, v^i_{p_i}\} +$ *cone*$\{r^i_1, \ldots, r^i_{q_i}\}$ *(for $P^i \neq \varnothing$)*

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Then

$$\begin{aligned} A^i x^i &\leqslant \delta^i b^i \quad && \forall i \\ \sum_{i \in K} x^i &= x && \\ \sum_{i \in K} \delta^i &= 1 && \\ \delta^i &\geqslant 0 && \forall i \end{aligned}$$

is an EF of

$$P := \textit{conv}\left(\bigcup_{i:P_i \neq \varnothing} \{v^i_1, \dots, v^i_{p_i}\}\right) + \textit{cone}\left(\bigcup_i \{r^i_1, \dots, r^i_{q_i}\}\right)$$

Theorem (Union of polyhedra, restated)

If $P_1, \ldots, P_k$ are polyhedra in $\mathbb{R}^d$ such that $\dim(P_i) \geqslant 1$ for all i, and $P :=$ conv$(P_1 \cup \cdots \cup P_k)$:

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Union of polyhedra

...an application to vertex interdiction

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Theorem (**Ahmed, Angulo, Dey & Kaibel**'13)

Let $V \subseteq \{0,1\}^d$ and $P =$ conv(V) be a 0/1-polytope. For a point $v \in V$, let $P - v :=$ conv$(V \setminus \{v\})$. Then:

$$\mathit{xc}(P - v) \leqslant d\, \mathit{xc}(P)$$

and more generally for $v_1, \ldots, v_k \in V$:

$$\mathit{xc}(P - v_1 - v_2 - \cdots - v_k) \leqslant \mathit{poly}(\mathit{xc}(P), k)$$

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Deterministic communication protocols

$f : A \times B \to \{0,1\}$ Boolean function ($\equiv$ binary matrix)

Two players:

- Alice knows $a \in A$
- Bob knows $b \in B$

want to **compute** $f(a,b)$ **by exchanging bits**

Goal: Minimize complexity := #bits exchanged

Deterministic communication protocols

Example

	b_1	b_2	b_3	b_4
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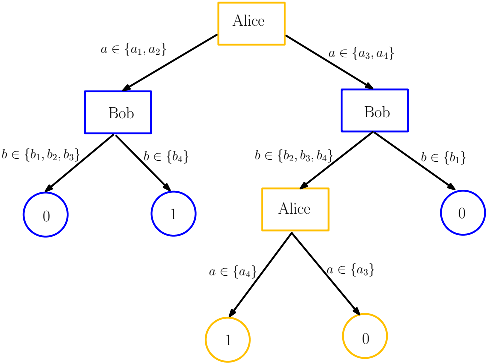
a_2	0	0	0	1
a_3	0	0	0	0
a_4	0	1	1	1



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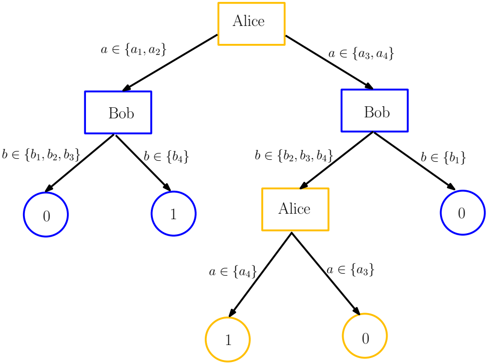

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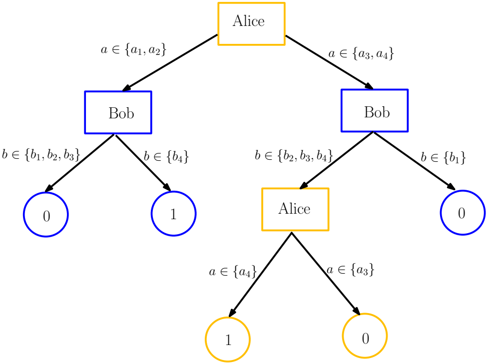

Observation

$\exists$ complexity c protocol for computing $f \implies \mathrm{rk}_+(f) \leqslant 2^c$

G graph with n vertices

$A = \{a \in \{0,1\}^n \mid a \text{ encodes a clique in } G\}$

$B = \{b \in \{0,1\}^n \mid b \text{ encodes a stable set in } G\}$

$$f(a,b) = \begin{cases} 1 & \text{if } a, b \text{ are disjoint} \\ 0 & \text{if } a, b \text{ intersect} \end{cases} = 1 - a^\intercal b = (1 - a^\intercal b)^2$$

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Theorem (Yannakakis'91)

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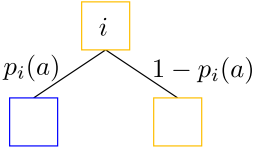
Corollary (Yannakakis'91)

$\forall$ *perfect graphs* G*:* $\quad xc(STAB(G)) = 2^{O(\log^2 n)} = n^{O(\log n)}$

Computing a function in expectation

The main differences:

- Alice and Bob can use (private) random bits to make choices



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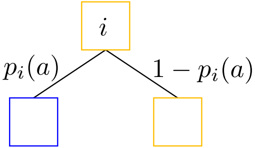

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Theorem (Faenza, F, Grappe & Tiwary'11)

If $c = c(f)$ is the minimum complexity of a randomized communication protocol with nonnegative outputs computing f in expectation, then

$$rk_+(f) = \Theta(2^c)$$

Write $M = TU$, where

- $T \in \mathbb{R}_+^{m \times r}$ row-stochastic (w.l.o.g.)
- $U \in \mathbb{R}_+^{r \times n}$
- $r \leqslant \mathsf{rk}_+(M) + 1$

Write $M = TU$, where
$$\left| \begin{array}{l} T \in \mathbb{R}_+^{m\times r} \text{ row-stochastic (w.l.o.g.)} \\ U \in \mathbb{R}_+^{r\times n} \\ r \leqslant \mathrm{rk}_+(M) + 1 \end{array} \right.$$

Protocol:

- Alice gets row index i, Bob gets column index j
- Alice picks random column index $k \in [r]$ w.p. T_{ik}, sends it to Bob
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Expected value on input (i, j): $$\sum_{k=1}^{r} T_{ik}U_{kj} = M_{ij}$$

Complexity: $\log \mathsf{rk}_+(M) + O(1)$

The threefold way

Three equivalent ways to look at EFs:

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3. A $\log r$-complexity randomized protocol computing S in expectation

Rectangle covering bound

Taking supports

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\begin{aligned}
S &= TU \qquad \text{rank-}r\text{ non-negative factorization} \\
&= \sum_{k=1}^{r} T^k U_k \qquad \text{sum of } r \text{ nonnegative rank-1 matrices}
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Theorem (Yannakakis'91)

$$rk_+(S) \geqslant rc(S)$$

Rectangle covering bound

. . . also known as

$$\begin{matrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{matrix}$$

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Observation

rc(S) is the best lower bound that only depends on supp(M)!

Rectangle covering bound

Summary

$$
\begin{tikzcd}
\text{Polytope} \arrow[r] \arrow[d] & \text{Slack matrix} \arrow[r] & \begin{array}{l}\text{Support matrix}\\ \text{of slack matrix}\end{array} \arrow[dd] \\
\begin{array}{l}\text{Extended}\\ \text{formulation}\end{array} \arrow[d] & & \\
\#\ \text{ieqs} & \geq & \begin{array}{l}\text{smallest } \#\ \text{1-rectangles}\\ \text{needed to cover all 1s}\end{array}
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Number of faces

Proposition (Goemans'09)

$\mathsf{xc}(P) \geqslant \log_2\big(\#\text{ faces of } P\big)$

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Exercise. Use this to give lower bounds on the extension complexities of n-gons and permutahedra.

The rectangle graph

Definition

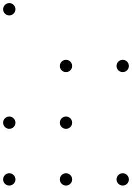
M 0/1-matrix

G_M := *rectangle graph*:

- $V(G_M)$:= 1-entries of M

Example

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$



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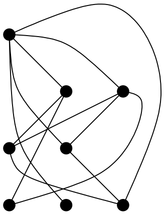

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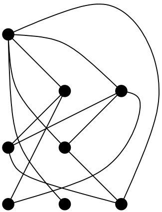

Lemma (F, Kaibel, Pashkovich, Theis'13)

$\text{rc}(M) = \chi(G_M)$

Clique number — aka fooling sets

Chromatic number $\geqslant$ maximum size of a clique

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Cliques $\equiv$ "Fooling sets"

In the matrix, clique $\equiv$ fooling set :=

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Example: n-gon

rectangle covering number $\approx \log n$

max size of fooling set $= 4$

Example: Cubes

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Every extension of a d-cube has at least $2d$ facets.

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Support of the slack matrix:

- Columns: $\xi \in \{0,1\}^d$
- Rows: d for $x_i \geqslant 0$ (upper part) and d for $x_i \leqslant 1$ (lower part)
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\begin{array}{ccccccc|ccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
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\end{array}
$$

What about $\log_2(\#\text{faces})$?

- $\#\text{faces} \approx 3^d$
- $\#$ facets of an extension
  $\geqslant \text{rc} \geqslant d\log_2 3$ $(\log_2 3 = 1.585\ldots)$

Limits of fooling sets

Proposition (F, Kaibel, Pashkovich, Theis'13)

For M support matrix of slack matrix of d-polytope,

$$\omega(G_M) \leqslant (d+1)^2$$