VI IPCO SUMMER SCHOOL
VIII ESCUELA DE VERANO EN MATEMÁTICAS DISCRETAS
EXERCISES – DAY 1

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1. $\boxed{\star\star}$ Let Q be a polytope in $\mathbb{R}^d$ and $a^T x \leqslant \beta$ a valid inequality for Q such that $||a||_2 = 1$. Consider the reflection σ with respect to the hyperplane $a^T x = \beta$. Letting $P := \operatorname{conv}(Q \cup \sigma(Q))$, prove that $\operatorname{xc}(P) \leqslant \operatorname{xc}(Q) + 2$. Deduce from this that the extension complexity of a regular n-gon is $O(\log n)$.
2. $\boxed{\star}$ In the fundamental theorem, we described P as the projection into x-space of Q, where $Q = \{(x, y) \in \mathbb{R}^{d+n} \mid x = \sum_{j=1}^n y_j v_j, \sum_{j=1}^n y_j = 1, y_j \geqslant 0\ \forall j\}$. What kind of polytope is Q?
3. $\boxed{\star}$ Prove the following version of Farkas' lemma, directly from Fourier-Motzkin elimination. If $Ax \leqslant b$ denotes a system of m linear inequalities over d variables, then exactly one of the two following assertions hold:
   (i) $\exists x \in \mathbb{R}^d$ such that $Ax \leqslant b$.
   (ii) $\exists u \in \mathbb{R}^m$ such that $u^\intercal A = 0$, $u^\intercal b = -1$, $u \geqslant 0$.
4. $\boxed{\star\star\star}$ Let Q be a polytope and P be any projection of Q. Show that for each integer k, the number of k-faces of P is at most the number of k-faces of Q.
5. $\boxed{\star\star}$ Suppose that $Q = \{(x, y) \in \mathbb{R}^{d+k} \mid Ax + By \leqslant 1\}$ is a polytope such that $0 \in \operatorname{int}(Q)$, and let $P = \operatorname{proj}_x(Q)$. Prove that $0 \in \operatorname{int}(P)$ and
$$P^\Delta = Q^\Delta \cap x\text{-space}$$
   Prove the other direction of the fundamental theorem using this.
6. $\boxed{\star}$ Find a 3-polytope P with 6 vertices in $\mathbb{R}^3$ such that $P \cap \{x \in \mathbb{R}^3 \mid x_3 = 0\}$ is a regular 8-gon.
7. $\boxed{\star\star\star\star}$ Suppose that Q is a polytope with m facets. Prove that projecting out 1 variable can create a polytope P with $\lceil \frac{m}{2} \rceil \lfloor \frac{m}{2} \rfloor \sim \frac{m^2}{4}$ facets. (Hint: use cyclic polytopes, which have the property that every two vertices are contained in an edge.) Find a general upper bound in terms of m for the case where 2 variables are projected out. How tight is this bound?
8. $\boxed{\star\star}$ Prove the following factorization theorem for pairs of polyhedra: if P and Q is a pair of polyhedra such that $P \subseteq Q$, P is pointed and the affine hull of P is not contained in Q, then the minimum number of facets of a polyhedron that projects to a polyhedron K with $P \subseteq K \subseteq Q$ equals the nonnegative rank of the slack matrix of the pair P, Q. (First, you have to find a suitable definition of the slack matrix of a pair of polyhedra.)
9. $\boxed{\star\star\star}$ Let M be the $n \times n$ matrix with $M_{ij} = (i - j)^2$. Find a $O(\log n)$-rank nonnegative factorization of M.