Extended formulations in combinatorial optimization: constructions and lower bounds

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VI IPCO Summer School
VIII Escuela de Verano en Matemáticas Discretas

Day 3

Today

1. Quick recapitulation
2. Geometry matters
3. Polytopes with high complexity
4. Approximate extended formulations
5. A look beyond: Positive semidefinite EFs
6. Conclusion

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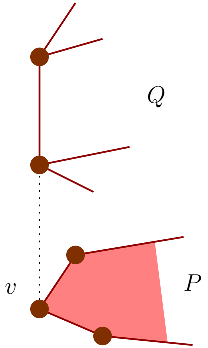
- $\mathrm{xc}(d\text{-cube}) = 2d$, $\mathrm{xc}(\text{regular } n\text{-gon}) = \Theta(\log n)$, ...

Today

Generic n-gons

Theorem (F, Rothvoß & Tiwary'12)

If P is a generic n-gon, then $\quad xc(P) \geqslant \sqrt{2n}$



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Theorem (**Rothvoß'11**)

There are 0/1-polytopes P in $\mathbb{R}^d$ such that every EF has size $2^{(1/2-o(1))d}$

History

Yannakakis's problem

From **Yannakakis**'11, repeating a problem in **Yannakakis**'91:

> I believe in fact that it should be possible to prove that there is no polynomial-size formulation for the TSP polytope or any other NP-hard problem, although of course showing this remains a challenging task.

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- $xc(CUT(n)) = 2^{\Omega(n)}$

$$\Downarrow$$

- $\exists\ (G_n)$ *s.t.* $xc(STAB(G_n)) = 2^{\Omega(n^{1/2})}$

Proof

Let $M = M(n)$ be the $2^n \times 2^n$ matrix with

$$M_{ab} := (1 - a^\intercal b)^2$$

for $a, b \in \{0,1\}^n$

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- M not a slack matrix, but embeds in a slack matrix
- suppmat(M) appears in **de Wolf**'03 for separating classical vs. quantum nondeterministic complexity

For $b \in \{0,1\}^n$:

$$1 - \langle 2\,\mathsf{diag}(a) - aa^\intercal, bb^\intercal \rangle \quad = \quad 1 - 2\langle \mathsf{diag}(a), bb^\intercal \rangle + \langle aa^\intercal, bb^\intercal \rangle$$

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$\mathsf{COR}(n) := \mathsf{conv}\{bb^\intercal \in \mathbb{R}^{n\times n} \mid b \in \{0,1\}^n\}$ correlation polytope

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Lemma (Key lemma)

For every $a \in \{0,1\}^n$, the inequality

$$(\star) \qquad \langle 2\,\mathit{diag}(a) - aa^\intercal, x \rangle \leqslant 1$$

is valid for COR(n). The slack of vertex $bb^\intercal$ w.r.t. $(\star)$ is M_{ab}.

Consider complete linear description for $\mathsf{COR}(n)$ starting with

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and corresponding slack matrix S

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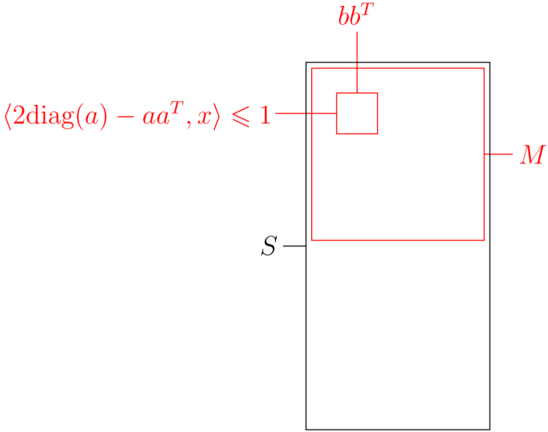
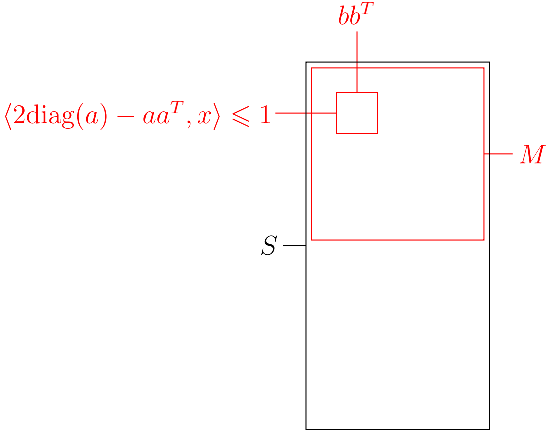
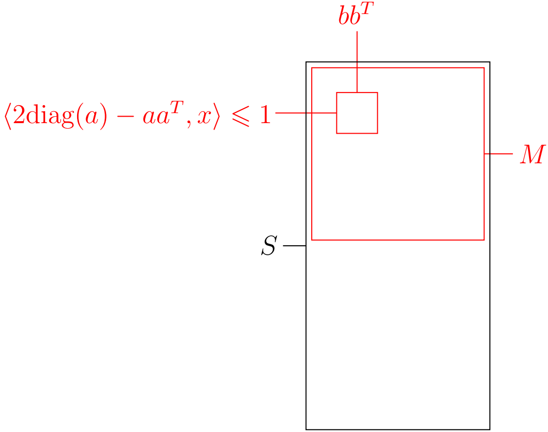
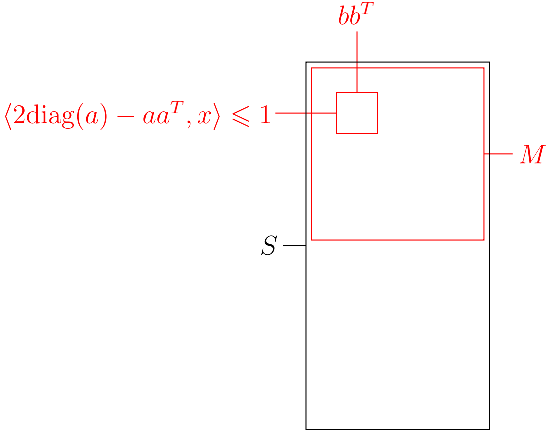




$\mathsf{xc}(\mathsf{COR}(n))$



$$\mathsf{xc}(\mathsf{COR}(n)) \quad = \quad \mathsf{rk}_+(S)$$



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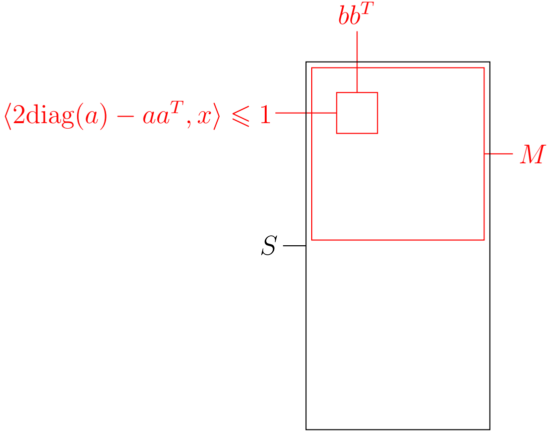

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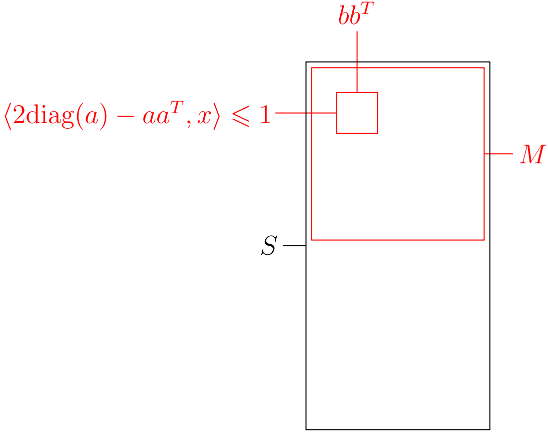

$$
\begin{aligned}
\mathsf{xc}(\mathsf{COR}(n)) &= \mathsf{rk}_+(S) \\
&\geqslant \mathsf{rk}_+(M) \\
&\geqslant \mathsf{rc}(M) \\
&= 2^{\Omega(n)} \qquad \text{(\textbf{de Wolf}'03, building on \textbf{Razborov}'92)}
\end{aligned}
$$

Cut polytope. $\boxed{\mathsf{xc}(\mathsf{CUT}(n)) = \mathsf{xc}(\mathsf{COR}(n-1)) = 2^{\Omega(n)}}$

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Lemma (monotonicity)

- Q *is an extension of* $P \implies xc(Q) \geqslant xc(P)$
- P *contains* F *as a face* $\implies xc(P) \geqslant xc(F)$

Stable set polytope. $\forall\ k\ \exists\ H_k$ with $O(k^2)$ vertices s.t. STAB(H_k) has a face $F = F(k)$ that is an extension of COR(k).

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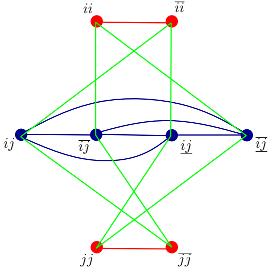

xc(STAB(H_k))

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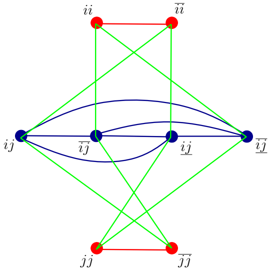

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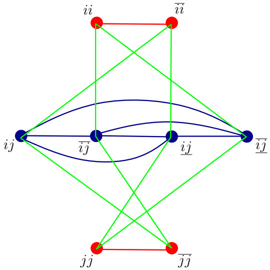

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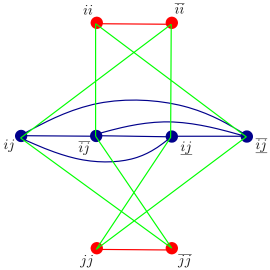

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\begin{aligned}
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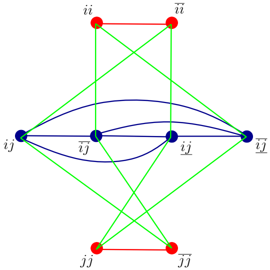

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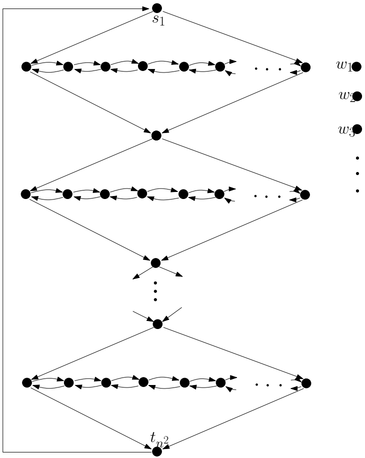
$\Longrightarrow \forall\ n\ \exists\ n$-vertex G_n s.t. $\boxed{\text{xc}(\text{STAB}(G_n)) = 2^{\Omega(n^{1/2})}}$



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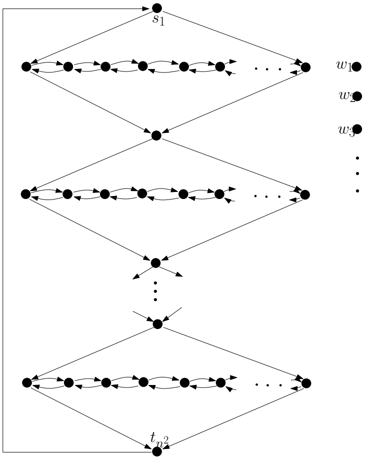

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Theorem (FMPTW'12)

$xc(TSP(n)) = 2^{\Omega(n^{1/2})}$

Extension/extended formulation of a pair

- $P \subseteq Q \subseteq \mathbb{R}^d$ with P polytope, Q polyhedron
- $L \subseteq \mathbb{R}^e$ polytope

Definition (extension/EF of a pair)

L is an extension of (P, Q) if $\exists$ linear π with $P \subseteq \pi(L) \subseteq Q$

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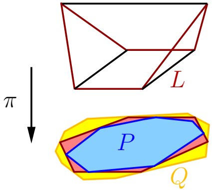

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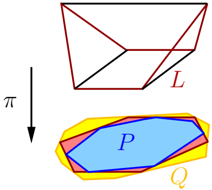


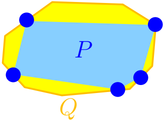
Definition (extension complexity of a pair)

$\text{xc}(P, Q) = \min\{\#\text{facets}(L) \mid L \text{ is an extension of } (P, Q)\}$

Slack matrix of a pair

Let $V = \{v_1, \dots, v_\ell\} \subseteq \mathbb{R}^d$ s.t. $P = \mathrm{conv}(V)$

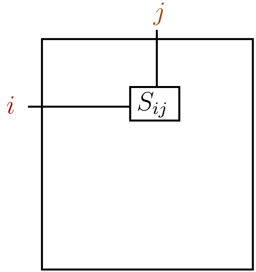
Let $A \in \mathbb{R}^{k \times d}$, $b \in \mathbb{R}^k$ s.t. $Q = \{x \in \mathbb{R}^d \mid Ax \leqslant b\}$

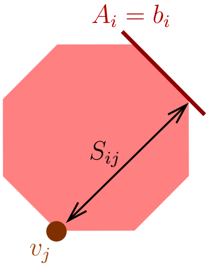


Definition (slack matrix)

Slack matrix $S = S^{P,Q} \in \mathbb{R}_+^{k \times \ell}$ of (P, Q) (w.r.t. $Ax \leqslant b$ and V):

$$S_{ij}^{P,Q} := b_i - A_i v_j$$





Remark. Every nneg matrix is the slack matrix of some pair!

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Theorem (Factorization theorem for pairs)

For every slack matrix $S^{P,Q}$ of (P, Q): $\quad$ $\mathit{xc}(P, Q) = \mathit{rk}_+(S^{P,Q})$

Linear encodings of combinatorial optimization problems

Definition (linear encoding)

A linear encoding of a comb. opt. problem is a pair $(\mathcal{L}, \mathcal{O})$ where

- $\mathcal{L} \subseteq \{0,1\}^*$ feasible solutions
- $\mathcal{O} \subseteq \mathbb{R}^*$ admissible objective functions

An instance is a pair (d, w) where $d \geqslant 1$ and $w \in \mathcal{O} \cap \mathbb{R}^d$

Given (d, w)*, find* $x \in \mathcal{L} \cap \{0,1\}^d$ *such that* $w^\intercal x$ *is max/min*

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"Faithfulness" condition:

instances of the problem $\longrightarrow$ instances of the linear encoding

"Constraints do not depend on instance, only on d."
"Instances are encoded in the objective function."

Approximate extended formulations

For a maximization problem

Linear encoding $(\mathcal{L}, \mathcal{O}) \rightsquigarrow$ pair of nested polyhedra $P \subseteq Q$:

- $P := \mathrm{conv}(\{x \in \{0,1\}^d \mid x \in \mathcal{L}\})$
- $Q := \{x \in \mathbb{R}^d \mid \forall w \in \mathcal{O} \cap \mathbb{R}^d : w^\intercal x \leqslant \max\{w^\intercal y \mid y \in P\}\}$

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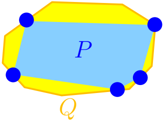
Geometrically: $P \subseteq \{x \mid \exists y : Ex + Fy = g,\ y \geqslant \mathbf{0}\} \subseteq \rho Q$

Sizes of approximate extended formulations

- $\mathcal{L} \rightsquigarrow P = \text{conv}(V)$
- $\mathcal{O} \rightsquigarrow Q = \{x \in \mathbb{R}^d \mid Ax \leqslant b\}$



Observation:

1. $\rho Q = \{x \in \mathbb{R}^d \mid Ax \leqslant \rho b\}$
2. $S_{ij}^{P,\rho Q} = \rho b_i - A_i v_j = S_{ij}^{P,Q} + (\rho - 1) b_i$

Corollary

Minimum size of a ρ-approximate EF $= \mathrm{rk}_+(S^{P,\rho Q})$

Polyhedral inapproximability of CLIQUE

Theorem

W.r.t. natural (faithful) linear encoding, CLIQUE has

- *a (trivial) poly-size n-approximate EF*
- *no $2^{o(n^{2\epsilon})}$-size $n^{1/2-\epsilon}$-approximate EF, for all $\epsilon \in (0, 1/2)$*

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The encoding:

- $d = n^2$
- $x \in \{0,1\}^{n \times n}$ is feasible if $x_{ij} = b_i b_j$ for $b \in \{0,1\}^n$
- $w \in \mathbb{R}^{n \times n}$ is admissible if
  - $w_{ii} \in \{0,1\}$ for all i
  - $w_{ij} = w_{ji} \in \{-1, 0\}$ for all i, j
- G graph with $V(G) \subseteq [n] \longrightarrow w^G := I(G) - A(\bar{G})$

Theorem (nonnegative rank of shifted UDISJ)

Let M *be any* $2^n \times 2^n$ *matrix such that*

- $M_{ab} = 1 + (\rho - 1)$ *if* $|a \cap b| = 0$
- $M_{ab} = 0 + (\rho - 1)$ *if* $|a \cap b| = 1$

Then

1. *rk*$_+(M) = 2^{\Omega(n)}$ *if* ρ *is a fixed constant*
2. *rk*$_+(M) = 2^{\Omega(n^{1-2\beta})}$ *if* $\rho = O(n^\beta)$ *for some constant* $\beta < 1/2$

Remark: This is the Unique DISJointness (partial) matrix when $\rho = 1$.

The corruption bound in general

Consider any matrix $S \in \mathbb{R}_{+}^{k \times \ell}$

Assume that **weights** $\mu \in \mathbb{R}^{k \times \ell}$ ($-$'s allowed) satisfy:

$$\langle \mu, X \rangle \leqslant ||X||_{\infty} \quad \forall X \in \mathbb{R}_{+}^{k \times \ell} \text{ that is rank-1}$$

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$$\implies \boxed{r \geqslant \frac{\langle \mu, S \rangle}{||S||_\infty}}$$

Core tool: Razborov's corruption lemma

Let $1 \leqslant \ell \leqslant (n+1)/4$

Distribution μ **on pairs** $(a, b) \in 2^{[n]} \times 2^{[n]}$ **with** $|a| = |b| = \ell$:

- on $A := \{(a, b) \mid |a \cap b| = 0\}$: $\frac{1}{4} \times$ uniform
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Lemma (Razborov's corruption lemma, improved)

Then for every $0 < \epsilon < 1$*:*

$$2^{\frac{\epsilon^2}{4 \ln 2}\ell + O(\log \ell)} \left((1-\epsilon)\,\mathbb{E}\left[X \mid A\right] - \mathbb{E}\left[X \mid B\right]\right) \leqslant \|X\|_\infty$$

Today

1. Quick recapitulation
2. Geometry matters
3. Polytopes with high complexity
4. Approximate extended formulations
5. A look beyond: Positive semidefinite EFs
6. Conclusion

The threefold way

The following are equivalent [**Yannakakis**'88/91, **FFGT**'11]:

1. A linear system $Ex + Fy = g$, $y \geqslant \mathbf{0}$ with $y \in \mathbb{R}^r$ s.t.

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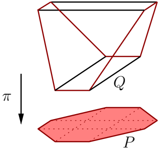

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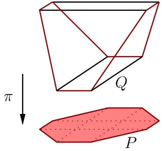
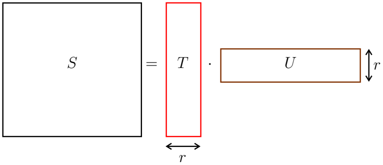


2. A rank-r nonnegative factorization $S = TU$ of slack matrix S



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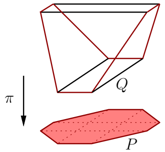

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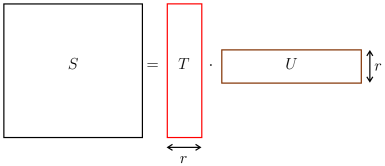

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The following are equivalent [**FMPTW**'12; **Gouveia, Parillo & Thomas**'11]:

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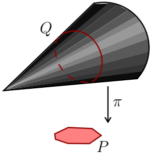

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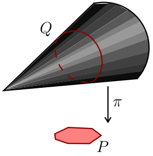

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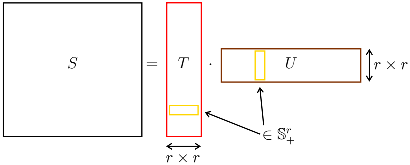

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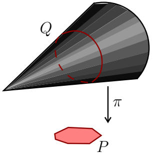

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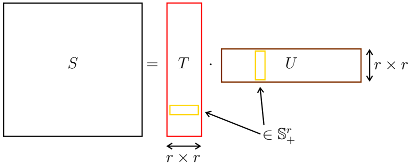

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Thank You! :)