VI IPCO SUMMER SCHOOL
VIII ESCUELA DE VERANO EN MATEMÁTICAS DISCRETAS
EXERCISES – DAY 2

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1. $\boxed{\star\star\star}$ Let M be the $n \times n$ matrix with $M_{ij} = (i-j)^2$. Find a $O(\log n)$-rank nonnegative factorization of M. (Hint: use reflections.)
2. $\boxed{\star\star\star\star}$ Find a linear description of the even polytope $\mathrm{EVEN}(n)$, in its original space.
3. $\boxed{\star\star}$ Let $\mathcal{L} \subseteq \{0,1\}^*$ be a regular language (that is, a language decided by a deterministic finite automaton), and let $P = P(n) := \mathrm{conv}\{x \in \{0,1\}^n \mid x \in \mathcal{L}\}$. Prove that P has a $O(n)$-size extended formulation.
4. $\boxed{\star}$ Let $P = (V, \leqslant_P)$ be a poset and $G = G(P)$ be the comparability graph of P. Verify that $\mathrm{STAB}(G(P)) = \{x \in \mathbb{R}^V \mid \exists\{(y_{v^-}, y_{v^+})\}_{v \in V} : x_v = y_{v^+} - y_{v^-}\ \forall v \in V,\ 0 \leqslant y_{v^-} \leqslant y_{v^+} \leqslant 1\ \forall v \in V,\ y_{v^+} \leqslant y_{w^-}\ \forall v \leqslant_P w\}$. (For an extra $\boxed{\star}$: Give a second proof of this.)
5. $\boxed{\star}$ Prove directly from Yannakakis' factorization theorem that if $P_1, \ldots, P_k$ are polytopes in $\mathbb{R}^d$ such that $\dim(P_i) \geqslant 1$ for all $i \in [k]$, and $P := \mathrm{conv}(P_1 \cup \cdots \cup P_k)$, then $\mathrm{xc}(P) \leqslant \mathrm{xc}(P_1) + \cdots + \mathrm{xc}(P_k)$. (Do you see why we need to exclude the case $\dim(P_i) \in \{-1, 0\}$?)
6. $\boxed{\star\star}$ For $a, b \in \{0,1\}^n$, let $\mathrm{EQ}(a,b) = 1$ if a and b are equal, and $\mathrm{EQ}(a,b) = 0$ if a and b are different. Prove that every deterministic protocol computing EQ has complexity at least n. Does this hold for randomized protocols computing EQ in expectation? (For an extra $\boxed{\star}$: Does this hold for randomized protocols with public randomness computing EQ with probability at least $1 - \varepsilon$?)
7. $\boxed{\star\star}$ Prove that if there exists a complexity-c randomized protocol that computes a given nonnegative matrix M in expectation, then $\log \mathrm{rk}_+(M) \leqslant c$.
8. $\boxed{\star\star}$ Give a $(3 \log n + O(1))$-complexity deterministic protocol computing the slack matrix of the stable set polytope of a claw-free graph. Deduce that the extension complexity of $\mathrm{STAB}(G)$ is $O(n^3)$, where G is a claw-free perfect graph with n vertices.
9. $\boxed{\star\star\star}$ Give a $(3 \log n + O(1))$-complexity randomized protocol computing the slack matrix of the spanning tree polytope of K_n in expectation. Deduce that the extension complexity of the spanning tree polytope of K_n is $O(n^3)$.
10. $\boxed{\star\star}$ Prove that if polytope Q is an extension of polytope P, the face lattice $\mathcal{L}(P)$ of P embeds into the face lattice $\mathcal{L}(Q)$, that is, there exists an injective mapping α from $\mathcal{F}(P)$ to $\mathcal{F}(Q)$ such that $F \subseteq G$ iff $\alpha(F) \subseteq \alpha(G)$. Deduce from this that $\mathrm{xc}(P) \geqslant \log_2(\#\text{ faces of } P)$.
11. $\boxed{\star}$ Prove that $\mathrm{xc}(P) = \Omega(\log n)$ if P is an n-gon and $\mathrm{xc}(P) = \Omega(n \log n)$ if P is the n-permutahedron, that is, $P = \mathrm{conv}\{(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n \mid \sigma \in \mathrm{Sym}(n)\}$ (for three extra $\boxed{\star\star\star}$: Prove that this lower bound is tight, using the existence of a $O(n \log n)$-size sorting network).
12. $\boxed{\star\star}$ Prove that $\mathrm{rk}_+(M) = \Omega(\log n)$ if M is the $n \times n$ matrix with $M_{ij} = (i-j)^2$.