Recent Developments in Cuts for MILP
1. Intersection cuts from lattice-free sets

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Mixed-Integer Linear Program (MILP)

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$$\begin{array}{lll} \max & c^T x & \\ \text{s.t.} & Ax = b & \\ & x_i \in \mathbb{Z} & \text{for } i \in I \\ & x_i \geq 0 & \text{for } i \in [n] \end{array}$$

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- Linear Relaxation: Consider all variables as continuous

Linear Relaxation: Optimal Simplex Tableau

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- A: $m \times n$ with m linearly independent rows

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& x_i \geq 0 & \text{for } i \in [n]
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- Basis B: $m \times m$ invertible submatrix of A
- B, N partition of A
- x^B, x^N: variables corresp. to B and N

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$f = B^{-1} b$

r^j : column of $-B^{-1} N$ corresponding to x_j

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Solution: $\bar{x} = (x^B, x^N)$

$$x^B = B^{-1}b \qquad\qquad x^N = 0$$

Gomory Mixed-Integer Cuts (GMI)

Row of the optimal tableau with basic variable x_i for $i \in I$:

$$x_i + \sum_{j \in I - i} \bar{a}_{ij} x_j + \sum_{j \in \bar{I}} \bar{a}_{ij} x_j = \bar{a}_{i0}$$

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Define

$$f_j = \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor \quad \forall j$$

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If $f_0 > 0$: Gomory Mixed-Integer Cut (GMI) [Gomory 60 [168]]:

$$\sum_{j \in I : f_j \leq f_0} f_j x_j + \sum_{j \in I : f_j > f_0} \frac{f_0(1 - f_j)}{1 - f_0} x_j + \sum_{j \in \bar{I} : \bar{a}_{ij} > 0} \bar{a}_{ij} x_j - \sum_{j \in \bar{I} : \bar{a}_{ij} < 0} \frac{f_0}{1 - f_0} \bar{a}_{ij} x_j \geq f_0$$

Split set

(π, π_0) such that

- $\pi \in \mathbb{Z}^n, \quad \pi_0 \in \mathbb{Z}$
- $\pi_j = 0$ for all $j \in \bar{I}$
- $gcd(\pi_1, \ldots, \pi_n) = 1$

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All points $x \in \mathbb{Z}^I \times \mathbb{R}^{\bar{I}}$ statisfy split (π, π_0):

$$\pi x \leq \pi_0 \qquad \text{or} \qquad \pi \mathrm{x} \geq \pi_0 + 1$$

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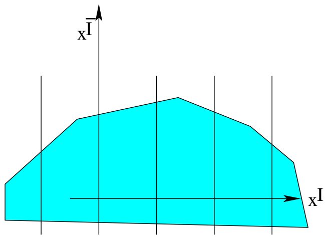
Boundary hyperplanes of (π, π_0):

- $H^1 = \{x \in \mathbb{R}^n \mid \pi x = \pi_0\}$
- $H^2 = \{x \in \mathbb{R}^n \mid \pi x = \pi_0 + 1\}$

Both H^1 and H^2 contain integer points

Split Cut

$Q \subseteq \mathbb{Z}^I \times \mathbb{R}^{\bar{I}}$: polyhedron



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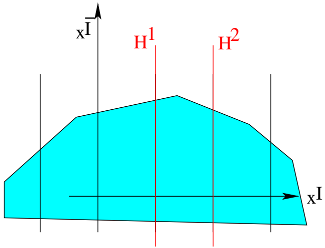

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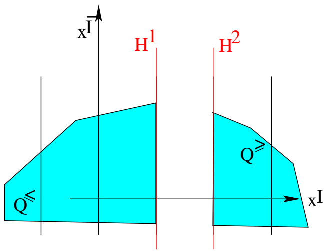
$$Q^{\leq} = Q \cap \{x \in \mathbb{R}^n \mid \pi x \leq \pi_0\} \quad Q^{\geq} = Q \cap \{x \in \mathbb{R}^n \mid \pi x \geq \pi_0 + 1\}$$



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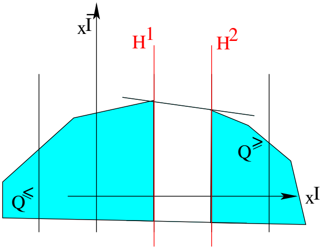
$Q(\pi, \pi_0) = \text{conv}(Q^{\leq} \cup Q^{\geq})$

Facets of $Q(\pi, \pi_0)$ that are not valid for Q are split cuts generated by (π, π_0)

[Cook, Kannan, Schrijver 1990 [88]]



Corner Polyhedron

[Gomory, Johnson 1972 [172]]

For a basis B, relax constraints $x_i \geq 0$ for all $i \in B$:
corner(B):

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Relax $x_i \in \mathbb{Z}$ for all $i \in I \cap N$:

- lin. relax is a cone with a single vertex $(f, 0) \in \mathbb{R}^B \times \mathbb{R}^N$

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relaxed-corner(B) (RCP(B)):

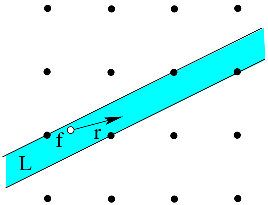
- $x_i \in \mathbb{Z}$ for all $i \in B$
- $x_i \in \mathbb{R}_+$ for all $i \in N$

[Andersen, Louveaux, Weismantel, Wolsey 2007 [13]]

Intersection cuts for corner(B)

S: closed convex set S with $\bar{x} \in int(S)$ and no point of $\mathbb{Z}^I \times \mathbb{R}^{\bar{I}}$ in $int(S)$

- r: direction



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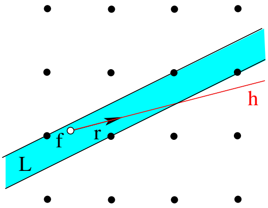

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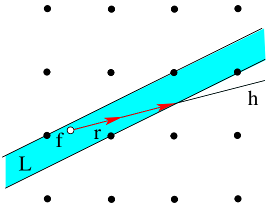
- $\psi(r) = \frac{1}{\lambda^*}$



$\lambda^* = 2 \qquad \psi(r) = \frac{1}{2}$

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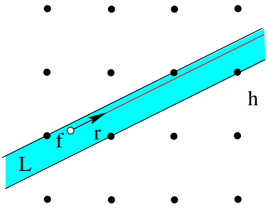

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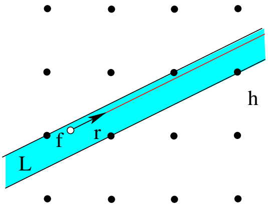
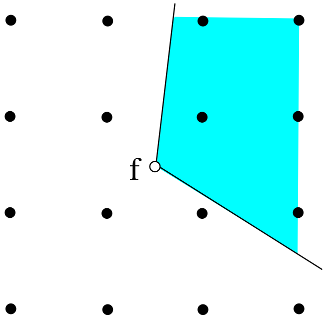


$\lambda^* = +\infty \qquad\qquad \psi(r) = 0$

Inequality for rays $\{r^j \mid j \in N\}$:

$$\sum_{j \in N} \psi(r^j)\, x_j \geq 1 \tag{1}$$

Example: $I = \{1, 2\} = B$ *and* $N = \{3, 4\}$



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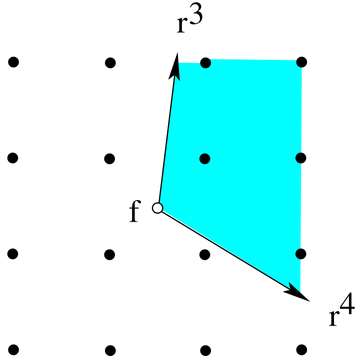


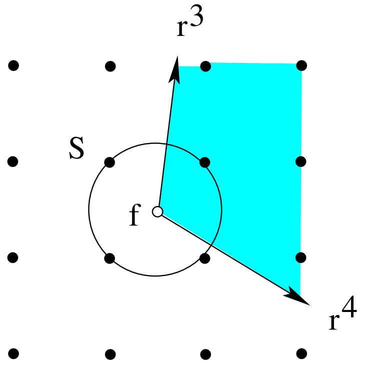
Feasible set contained in

$$\{(x_1, x_2)^T = f + r^3 x_3 + r^4 x_4 \mid x_3 \geq 0, x_4 \geq 0\}$$

Want

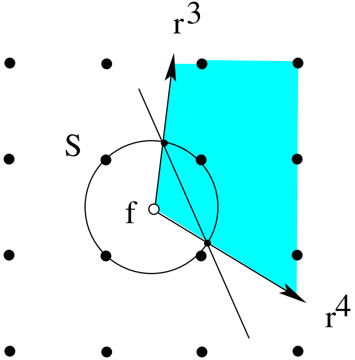
$$\{(x_3, x_4) \geq (0, 0) \mid f + r^3 x_3 + r^4 x_4 \text{ integer}\}$$

Example: $I = \{1, 2\} = B$ *and* $N = \{3, 4\}$ *(cont.)*



Any convex set $S \in \mathbb{R}^2$ with $f \in int(S)$ with no point of $\mathbb{Z}^2$ in $int(S)$

Example: $I = \{1, 2\} = B$ *and* $N = \{3, 4\}$ *(cont.)*

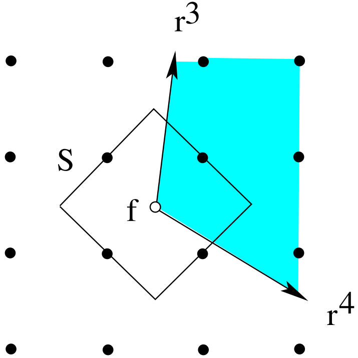


Compute intersection of the rays with the boundary of S

Cut defined by these points is valid: $a_3x_3 + a_4x_4 \geq 1$

[Balas 1971 [22]]

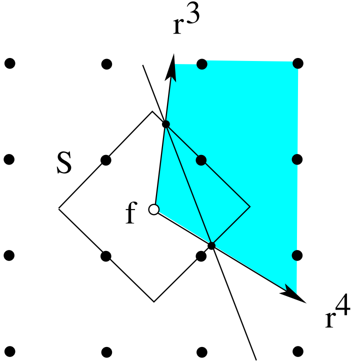
Using a Different Convex Set



Octahedron S in $\mathbb{R}^2$

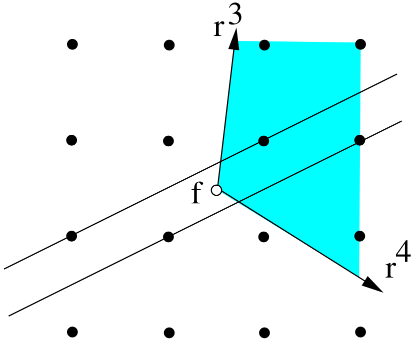
$\bar{f} \in int(S)$ with no point of $\mathbb{Z}^2$ in $int(S)$

Using a Different Convex Set



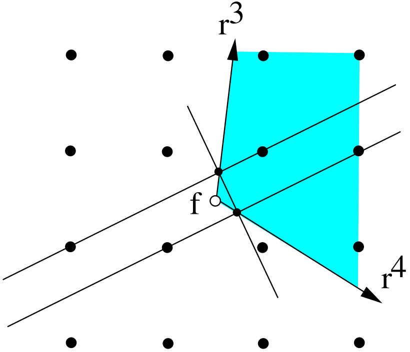
Compute intersection of the rays with the boundary of S

Split Cuts



- Strip S in $\mathbb{R}^2$
- $f \in \mathit{int}(S)$ with no point of $\mathbb{Z}^2$ in $\mathit{int}(S)$

Split Cuts



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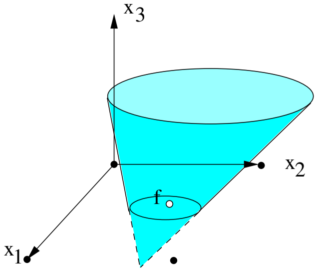
Intersection cuts for corner(B)

$I = \{1, 2\}$ $\bar{I} = \{3\}$



S': closed convex set with $(f, 0) \in int(S')$

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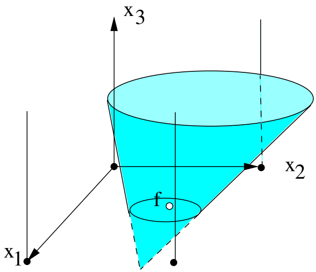

$int(S') \cap (\mathbb{Z}^I \times \mathbb{R}^{\bar{I}}) = \emptyset$

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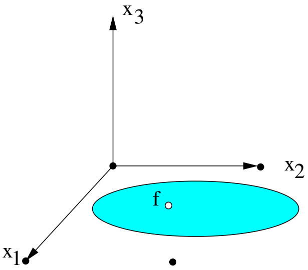

S: projection of S' onto $\mathbb{R}^I$

Note: $int(S) = proj(int(S'))$ [289]

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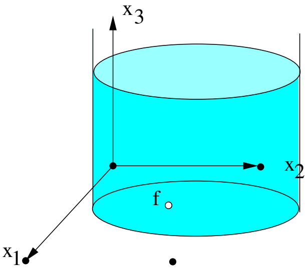

$S'' = S \times \mathbb{R}^{\bar{I}}$

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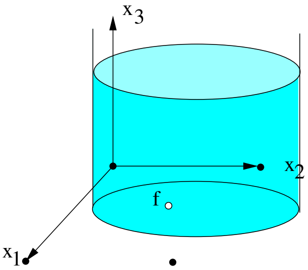

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Observe:

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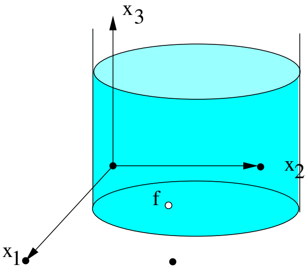

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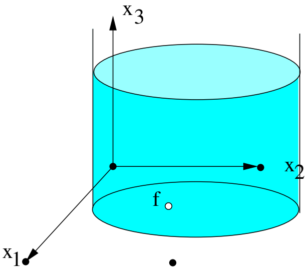

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In pictures, will assume that sets are of the form S''
$\Rightarrow$ can draw pictures in $\mathbb{R}^I$

Intersection cut: validity

Theorem 6.5: Let C be a closed convex set whose interior contains the point $\bar{x}$ but no point of $\mathbb{Z}^{I} \times \mathbb{R}^{\bar{I}}$. The intersection cut (1) is a valid inequality for corner(B).

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- split cuts are intersection cuts
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For corner(B), intersection cuts are strongest

Intersection cut for corner(B)

Remark 6.6: Let C_1, C_2 be two closed convex sets whose interiors contain $\bar{x}$ but no point of $\mathbb{Z}^I \times \mathbb{R}^{\bar{I}}$. If $C_1 \subseteq C_2$, then the intersection cut (1) relative to C_2 dominates the intersection cut (1) relative to C_1.

Theorem 6.12: Every nontrivial facet of corner(B) is an intersection cut.

Maximal $\mathbb{Z}^I \times \mathbb{R}^{\bar{I}}$*-free sets*

Lemma 6.17: Let C be a full-dimensional maximal $\mathbb{Z}^I \times \mathbb{R}^{\bar{I}}$-free convex set and let K be its projection onto $\mathbb{R}^I$. Then K is a maximal $\mathbb{Z}^I$-free convex set and $C = K \times \mathbb{R}^{\bar{I}}$.

Theorem 6.18: Let $K \subseteq \mathbb{R}^I$ be a full-dimensional set. Then K is a maximal $\mathbb{Z}^I$-free convex set if and only if K is a polyhedron that does not contain any point of $\mathbb{Z}^I$ in its interior but there is a point of $\mathbb{Z}^I$ in the interior of each of its facets.

Theorem 6.19: Any full-dimensional maximal $\mathbb{Z}^I$-free convex set K is a polyhedron with at most $2^{|I|}$ facets.

"Proof" of Theorem 6.18

Assume that K is bounded (general proof is more technical)

- Show that K is a polytope
- Show that all facets of K contain an integer point in their interior:

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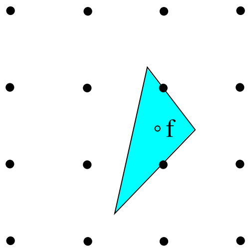

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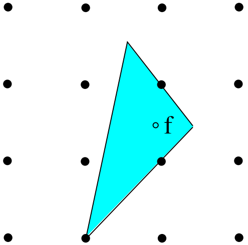

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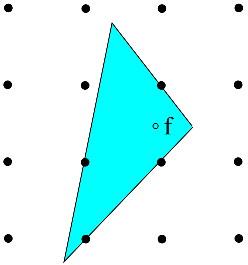

Gauge function

Let L be a lattice-free convex set in $\mathbb{R}^n$ contaning $(f, 0)$ in its interior.

The function ψ used in (1) to generate the coefficient of the intersection cut generated by L is the gauge function of L.

Gauge function

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The function ψ used in (1) to generate the coefficient of the intersection cut generated by L is the gauge function of L.

Theorem [Borozan, Cornuéjols 2009 [BC]]
Let ψ be the gauge function of a maximal lattice-free set containing f in its interior. Then ψ is nonnegative, positively homogeneous, piecewise linear and convex

Intersection cuts from two rows of the tableau

Optimal tableau for basis B:

$$x^B = B^{-1}b - B^{-1}N\, x^N = f + \sum_{j \in N} r^j x_j$$

$$x_j \geq 0 \text{ for all } j \in B \cup N$$

- Relax $x_i \geq 0$ for all $i \in B \Rightarrow$ corner(B)

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- Relax $x_i \geq 0$ for all $i \in B \Rightarrow$ corner(B)
- Select indices of two basic variables $i_1, i_2 \in B \cap I$
- Erase from system all equalities except those defining x_{i_1} and $x_{i_2} \Rightarrow$ relaxation: :

$$x_{i_1} = f_{i_1} + \sum_{j \in N} r^j_{i_1} x_j$$
$$x_{i_2} = f_{i_2} + \sum_{j \in N} r^j_{i_2} x_j$$
$$x_j \geq 0 \text{ for all } j \in N$$
$$x_{i_1}, x_{i_2} \in \mathbb{Z}$$

Maximal lattice-free sets in $\mathbb{R}^2 \times \mathbb{R}^N$

Select maximal $\mathbb{Z}^2 \times \mathbb{R}^N$-free convex set $C \in \mathbb{R}^2 \times \mathbb{R}^N$:

- Lemma 6.17: $C = K \times \mathbb{R}^{\bar{I}}$ with K full dim. $\mathbb{Z}^2$-free set

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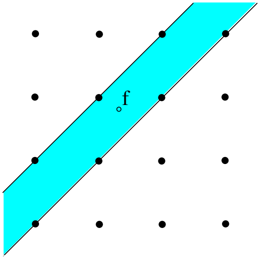

Split

Maximal lattice-free sets in $\mathbb{R}^2 \times \mathbb{R}^N$

Select maximal $\mathbb{Z}^2 \times \mathbb{R}^N$-free convex set $C \in \mathbb{R}^2 \times \mathbb{R}^N$:

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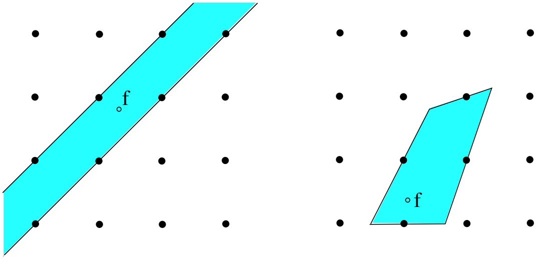
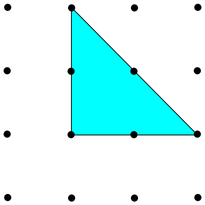


Split Quadrilateral

Maximal Lattice-free triangles

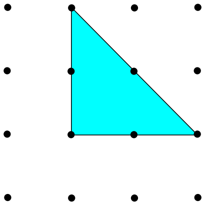
- Type 1:
  - Three integral vertices
  - Exactly one integral point in the interior of each edge



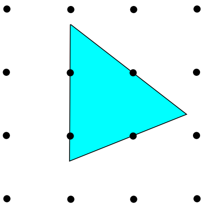
Type 1

Maximal Lattice-free triangles

- Type 1:
  - Three integral vertices
  - Exactly one integral point in the interior of each edge
- Type 2:
  - At least one fractional vertex v
  - Exactly one integ. point in interior of both edges adjacent to v
  - At least two integral points on the third edge



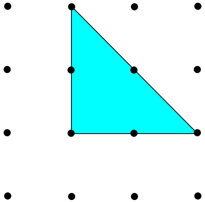
Type 1



Type 2

Maximal Lattice-free triangles

- Type 1:
  - Three integral vertices
  - Exactly one integral point in the interior of each edge
- Type 2:
  - At least one fractional vertex v
  - Exactly one integ. point in interior of both edges adjacent to v
  - At least two integral points on the third edge
- Type 3:
  - Exactly one integ. point in the interior of each edge, no others



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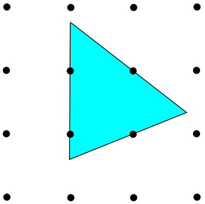
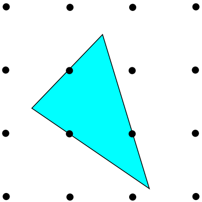


Type 2



Type 3

Facet defining sets in $\mathbb{R}^2 \times \mathbb{R}^N$

The facets are

- split inequalities with infinite direction r^j for some $j \in N$ or when a ray condition holds
- triangle inequalities with corners on half-lines $f + \lambda r^j$ for some $j \in N$, $\lambda > 0$, or satisfying another ray condition,
- quadrilateral inequalities with corners on half-lines $f + \lambda r^j$ for some $j \in N$, $\lambda > 0$ and satisfying a ratio condition

Separation of 2-dimensional intersection cuts

Ignoring "ray conditions":

- Splits
- Type 1 triangles with corner rays
- Type 2 triangles with corner rays
- Type 3 triangle with corner rays
- Quadrilaterals with corner rays

Separation of 2-dimensional intersection cuts

Ignoring "ray conditions":

- Splits
- Type 1 triangles with corner rays
- Type 2 triangles with corner rays
- Type 3 triangle with corner rays
- Quadrilaterals with corner rays

Questions:

- Should we try?
- Theoretical justification (Friday)
- Empirical evidence (Saturday)