Recent Developments in Cuts for MILP
2. Theoretical comparisons

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Mixed-Integer Linear Program (MILP)

$$
\begin{array}{lll}
\max & c^T x & \\
\text{s.t.} & Ax = b & \\
& x_i \in \mathbb{Z} & \text{for } i \in I \\
& x_i \geq 0 & \text{for } i \in [n]
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2-dimensional Relaxed Corner Polyhedron:

$$(RCP): \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \sum_{j \in N} \begin{pmatrix} r_1^j \\ r_2^j \end{pmatrix} x_j$$

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$R(f, \Gamma) =$ convex hull of (RCP)

(LRCP): Linear relaxation of (RCP)

Description of $R(f, \Gamma)$

Six relaxations of $R(f, \Gamma)$:

- $S(f, \Gamma) =$ (LRCP) $+$ all cuts from splits containing f in their interior
- $\triangle(f, \Gamma) =$ (LRCP) $+$ all cuts from triangles containing f in their interior
- $\triangle_i(f, \Gamma) =$ (LRCP) $+$ all cuts from triangles of type i containing f in their interior, for $i = 1, 2, 3$
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We have:

$$R(f, \Gamma) = S(f, \Gamma) \cap \triangle(f, \Gamma) \cap \square(f, \Gamma)$$

Comparison of split, triangle and quadrilateral cuts

Theoretical comparison:

- Upper and lower bounds on quality of relaxations vs. $R(f, \Gamma)$
- Upper and lower bounds on relaxations between themselves

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- Compare gap closed in practice

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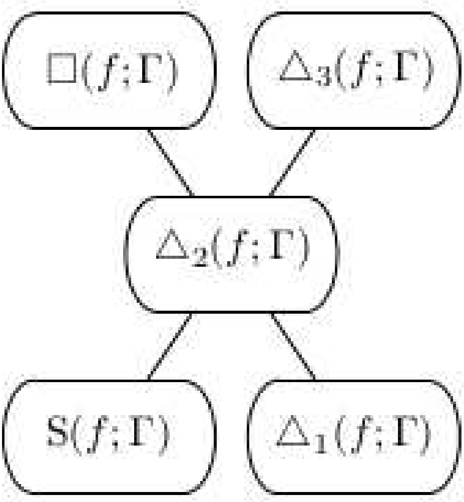
Probabilistic comparison:

- Over all choices of f and Γ in a set L:
  - Probability that L generates an inequality improving on the split closure
  - Prob. that some inequalities dominates others (coefficients or volume cut off)
- Select f and Γ uniformly; compare average and worst-case strength of split closure vs. triangle closure over all possible optimization directions

Comparing families



Domination between families

$L \subseteq \mathbb{R}^n$: lattice-free set; $\qquad \epsilon > 0$

$$relax(L, \epsilon) = \{x \in \mathbb{R}^n \mid ||x - \bar{x}|| \leq \epsilon, \text{ for some } \bar{x} \in L\}$$

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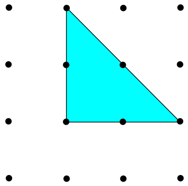
Proposition 2.1 [ACGT] Let $\mathcal{L}, \mathcal{L}'$ be families of lattice-free convex sets. Suppose that for every $\epsilon > 0$ and every $L \in \mathcal{L}$, there exists $L' \in \mathcal{L}'$ such that $L \subseteq relax(L', \epsilon)$. Then $\mathcal{L}'(f, \Gamma) \subseteq \mathcal{L}(f, \Gamma)$.



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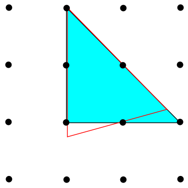

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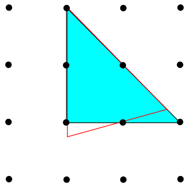

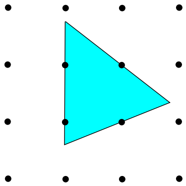

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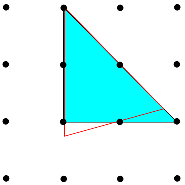

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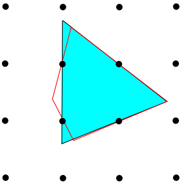

$\square(f, \Gamma) \subseteq \triangle_2(f, \Gamma)$

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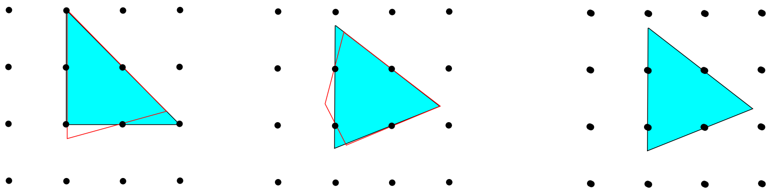

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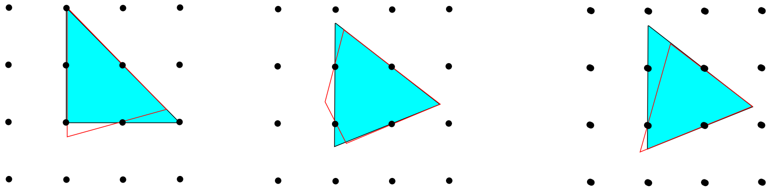

$\triangle_2(f, \Gamma) \subseteq \triangle_1(f, \Gamma)$ $\qquad$ $\square(f, \Gamma) \subseteq \triangle_2(f, \Gamma)$ $\qquad$ $\triangle_3(f, \Gamma) \subseteq \triangle_2(f, \Gamma)$

Comparing relaxations

$C \subseteq \mathbb{R}^n_+$: convex set is monotone when

$$x \in C \text{ and } y \geq x \qquad \Rightarrow \qquad y \in C$$

Example: $R(f,\Gamma)$, $S(f,\Gamma)$, $\triangle(f,\Gamma), \square(f,\Gamma)$

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C_1, C_2: monotone convex sets
How much to inflate C_2 to contain C_1:

$$\rho(C_1, C_2) = \inf\{\frac{1}{\alpha} \mid C_1 \subseteq \alpha C_2\}$$

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$\mathcal{L}$: family of convex lattice-free sets

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Computing an upper bound on $\rho(\mathcal{L}_1, \mathcal{L}_2)$

We have

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\begin{aligned}
\frac{1}{\rho(\mathcal{L}_1,\mathcal{L}_2)} = \inf_{\Gamma, f \in L \in \mathcal{L}_2} \quad & \sum_{j \in \Gamma} \psi_L(r^j)\; x_j \\
& \sum_{j \in \Gamma} \psi_C(r^j)\; x_j \quad \geq 1 \qquad \text{for all} \;\; C \in \mathcal{L}_1 \text{ containing } f \\
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$\Rightarrow \{r^j \mid j \in \Gamma\}$ is exactly $\Gamma(L) :=$ set of corner rays for L

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Difficulties:

- Need to check all $L \in \mathcal{L}_2$
- For each $L \in \mathcal{L}_2$ one inequality for each $C \in \mathcal{L}_1$

Unimodular transformation

Unimodular transformation: $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with

$$\phi(x) = v + Mx$$

where
$v \in \mathbb{Z}^n$

$M \in \mathbb{Z}^{n \times n}$ with $\det(M) = \pm 1$

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$v \in \mathbb{Z}^n$

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Need to check all $L \in \mathcal{L}_2$ only up to unimodular transformation

Computing an upper bound on $\rho(S, \triangle_1)$

All $L \in \triangle_1$ are identical up to unimodular transformation.
We can just pick one.
T: vertices $(0,0),(0,2),(2,0)$
T_I: vertices $(1,0),(0,1),(1,1)$

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$$
\begin{array}{rlll}
z = & \min & \displaystyle\sum_{j=1}^{3} x_j & \\
& & \displaystyle\sum_{j=1}^{3} \psi_S(r^j)\, x_j \ \geq 1 & \text{for all split } S \text{ containing } f \\
& & x_j \in \mathbb{R}_+ & \text{for all } j = 1,2,3
\end{array}
$$

Then $\frac{1}{z} = \rho(S, \triangle_1)$

Computing an upper bound on $\rho(S, \triangle_1)$ (cont.)

Replace the infinite number of inequalities in the LP by a small number:

S_1: split $0 \leq x_1 \leq 1$

S_2: split $0 \leq x_2 \leq 1$

S_3: split $1 \leq x_1 + x_2 \leq 2$

$$
\begin{array}{rlllll}
z = & \min & \displaystyle\sum_{j=1}^{3} x_j & & & \\
& & \displaystyle\sum_{j=1}^{3} \psi_{S_t}(r^j)\; x_j & \geq 1 & & \text{for} \quad t = 1, 2, 3 \\
& & \tilde{x}_j \in \mathbb{R}_+ & & \text{for all} \quad j = 1, 2, 3 &
\end{array}
$$

Then $\frac{1}{z} \geq \rho(S, \triangle_1)$

Computing an upper bound on $\rho(S, \triangle_1)$ *(cont.)*

Assume that $f \in T_I$

$$\begin{array}{rrrrrl} z = & \min & x_1 & +x_2 & +x_3 & \\ & & \frac{f_1+f_2}{f_1+f_2-1}x_1 & +x_2 & +x_3 & \geq 1 \\ & & x_1 & +\frac{2-f_1}{1-f_1}x_2 & +x_3 & \geq 1 \\ & & x_1 & +x_2 & +\frac{2-f_2}{1-f_2}x_3 & \geq 1 \\ & & & x \geq 0 & & \end{array}$$

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Optimal solution:

$$x_1^* = \frac{f_1+f_2-1}{2} \qquad x_2^* = \frac{1-f_1}{2} \qquad x_3^* = \frac{1-f_2}{2}$$

with value $z = x_1^* + x_2^* + x_3^* = \frac{1}{2}$

Computing a lower bound on $\rho(S, \triangle_1)$

Lemma 6.3 [43]: If f is in the interior of triangle T_I then the split closure is defined by S_1, S_2, S_3.

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In general, to prove a lower bound on $\rho(\mathcal{L}_1, \mathcal{L}_2)$:

- Select $L \in \mathcal{L}_2$, f
- Find a point $\bar{x}$ in $\mathcal{L}_1(f, \Gamma(L))$
- $z = \sum_{j \in \Gamma(L)} \psi_L(r^j)\, \bar{x}_j$
- $\frac{1}{z}$ is a lower bound on $\rho(\mathcal{L}_1, \mathcal{L}_2)$

Lower, upper bounds on ρ

Entry (i,j): lower bound, upper bound on $\rho(i,j)$

	S	$\triangle_1$	$\triangle_2$	$\triangle_3$	$\square$	R
S	$-$	2	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$\triangle_1$	$+\infty$	$-$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$\triangle_2$	1	1	$-$	1.125, 1.5	1.125, 1.5	1.125, 1.5
$\triangle_3$	1	1	1	$-$	1.125, 1.5	1.125, 1.5
$\square$	1	1	1	1.125, 1.5	$-$	1.125, 1.5

[Awate, Cornuéjols, Guenin, Tuncel 2013 [ACGT]]

Probabilistic comparisons (model (i))

[Del Pia, Wagner, Weismantel 2011 [DPP]]

- Fix a lattice-free convex set L
- Let f vary uniformly in $int(L)$ and use $\Gamma(L)$ as rays.
- For any $z > 1$, compute $P^L(z) := Prob(\rho(S, S \cup L) \leq z)$.

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Findings:

- $P^L(z)$ tends to 1 when lattice-width of $L \in \triangle_2$ tends to 1

$$P^L(z) \geq \begin{cases} 0 & \text{if } 1 < z \leq w \\ \frac{(z-w)(2zw-w-z)}{w^2(z-1)^2} & \text{if } w < z \leq \frac{w}{w-1} \\ \frac{(z-w)(2zw-w-z)+(w-1)^2(z-1)^2-1}{w^2(z-1)^2} & \text{if } \frac{w}{w-1} < z < +\infty \end{cases}$$

(similar result for $L \in \triangle_3$)

Probabilistic comparisons (model (ii))

[Basu, Cornuéjols, Molinaro 2010 [BCM]]

- Let f and Γ be selected uniformly
- Compare average (avg) and worst-case (wc) gap between closures over all possible cost vector.
- For a cost vector $c \in \mathbb{R}^N_+$

$$gap(C_1, C_2, c) = \frac{\{\min c\ x \mid x \in C_1\}}{\{\min c\ x \mid x \in C_2\}}$$

with value $+\infty$ if $C_1 = \emptyset$ or $\{\min c\ x \mid x \in C_2\} = 0$.

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- Compare average (avg) and worst-case (wc) gap between closures over all possible cost vector.
- For a cost vector $c \in \mathbb{R}_+^N$

$$gap(C_1, C_2, c) = \frac{\{\min c\ x \mid x \in C_1\}}{\{\min c\ x \mid x \in C_2\}}$$

with value $+\infty$ if $C_1 = \emptyset$ or $\{\min c\ x \mid x \in C_2\} = 0$.

Findings:

- For $\alpha \geq 1$ and $|\Gamma| \to +\infty$: $\quad Prob(wc(\triangle, S) \geq \alpha) \approx \frac{1}{\alpha} - \frac{1}{4\alpha^2}$

Probabilistic comparisons (model (ii))

[Basu, Cornuéjols, Molinaro 2010 [BCM]]

- Let f and Γ be selected uniformly
- Compare average (avg) and worst-case (wc) gap between closures over all possible cost vector.
- For a cost vector $c \in \mathbb{R}_+^N$

$$gap(C_1, C_2, c) = \frac{\{\min c\ x \mid x \in C_1\}}{\{\min c\ x \mid x \in C_2\}}$$

with value $+\infty$ if $C_1 = \emptyset$ or $\{\min c\ x \mid x \in C_2\} = 0$.

Findings:

- For $\alpha \geq 1$ and $|\Gamma| \to +\infty$: $\quad Prob(wc(\triangle, S) \geq \alpha) \approx \frac{1}{\alpha} - \frac{1}{4\alpha^2}$
- For $\alpha > 1$ and $\epsilon > 0$ (lower bound on entries in c): $Prob(avg(R(f,\Gamma), S(f,\Gamma)) \leq \alpha) \geq 1 - \frac{1}{|N|}$

Probabilistic comparisons (model (iii))

[He, Ahmed, Nemhauser 2011 [HAN]]

- Let f vary uniformly in the unit square U with uniformly distributed rays.
- Compare the two splits containing U vs. the four Type 1 triangles containing U;
- Comparison based on coefficients on rays or volume cut off. Probability that one type dominates the other.

Probabilistic comparisons (model (iii))

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- Comparison based on coefficients on rays or volume cut off. Probability that one type dominates the other.

Findings:

- One of the two splits is more likely to dominate a Type 1 triangle than the opposite. Probability tends fast to 0 as number of rays increases.
- Same conclusion for volume cut off. Probability tends to 1 as number of rays increases.