New Journal of Physics
The open-access journal for physics

A new perspective on stochastic resonance in monostable systems

H Calisto[1,3] **and M G Clerc**[2,3]
[1] Departamento de Física Facultad de Ciencias, Universidad de Tarapacá, Casilla 7-D, Arica, Chile
[2] Departamento de Física, Universidad de Chile, Casilla 487-3, Santiago, Chile
E-mail: hcalisto@uta.cl and marcel@galileo.dfi.uchile.cl

New Journal of Physics **12** (2010) 113027 (12pp)
Received 2 May 2010
Published 15 November 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/11/113027

Abstract. Stochastic resonance induced by multiplicative white noise is theoretically studied in forced damped monostable oscillators. A stochastic amplitude equation is derived for the oscillation envelope, which has a linear stochastic resonance. This phenomenon is persistent when nonlinearities are considered. We propose three simple systems—a horizontally driven pendulum, a forced electrical circuit and a laser with an injected signal—that display this stochastic resonance.

Contents

[3] Authors to whom any correspondence should be addressed.

1. Introduction

The description of macroscopic matter, i.e. matter composed of a large number of microscopic constituents, is usually done using a small number of coarse-grained or macroscopic variables. When spatial inhomogeneities are ignored, the evolution of these variables is described by deterministic ordinary differential equations. This reduction is possible due to a time scale separation, which allows a description in terms of the slowly varying macroscopic variables, which are in fact fluctuating variables due to the elimination of a large number of fast variables whose effect can be modelized by including suitable stochastic terms, *noise*, in the ordinary differential equations. The influence of noise in nonlinear systems has been the subject of intense experimental, numerical and theoretical investigations in past years [1, 2]. Far from being merely a perturbation to the idealized deterministic evolution or an undesirable source of randomness and disorganization, noise can induce specific and even counterintuitive dynamical behavior. The most well-known examples in stochastic dynamical systems are noise-induced transitions [3] and stochastic resonances ([4] and references therein). When a periodic signal embedded in a noisy background is applied as an input to a system, its response can be tuned to maximize the signal-to-noise ratio through stochastic resonance. It was initially proposed as a possible explanation for the cyclical glaciations on the Earth [5]–[8]. Moreover, this phenomenon has attracted enormous interest due to its possible technological applications in optimizing weak signals and by its connections with some biological and chemical mechanisms ([4] and references therein). Pioneering investigations of stochastic resonance were done in nonlinear dynamical systems driven by a combination of random and periodic forcing. For a bistable system, it was shown ([4] and references therein) that when the forcing frequency is double the Kramers escape rate [1], the transition induced by noise resonates with the forcing, and the response of the system is amplified. Hence, this stochastic resonance can be interpreted as the resonance of two oscillatory modes, which is analogous to 1:1 resonance in dynamical systems theory [9]. Although an external periodic forcing can be replaced by some internal source of periodic nature, nonlinearity seems to be the necessary ingredient for the occurrence of stochastic resonance ([4], and references therein; [10]–[13]). In this context, the stochastic resonance has been denominated *coherence resonance* [11, 12], because noise induces coherent motion as oscillations. Stochastic resonance has been mainly studied in bistable systems; however, it has been shown in [14] that stochastic resonance can occur in linear systems subject to a multiplicative dichotomic noise. Stochastic resonance has also been observed in monostable systems, such as a level crossing detector, with additive noise [15]. In past years, stochastic resonance in linear systems has been studied extensively [16]–[21]. In [22], the occurrence of the stochastic resonance phenomenon in a linear system driven by two correlated Gaussian white noises is studied and it is claimed that the problem of stochastic resonance in a linear system is closely related to the behavior of a generalized noisy logistic equation.

The aim of this paper is to study the persistence of the linear stochastic resonance induced by white noise when necessary nonlinear terms are considered and to identify the ingredients for observing this generic phenomenon. This result confirms that the interpretation of stochastic resonance by Gora [22] is adequate; that is, if the response is monotone and defined in an interval of the level of noise intensity, then the system can exhibit a stochastic resonance at the end of the interval. Hence, the nonlinear terms can generate a maximum in the response within the interval as we shall see later. We show that a multiplicative white noise supports a resonance in a forced under-damped monostable oscillator. A stochastic amplitude

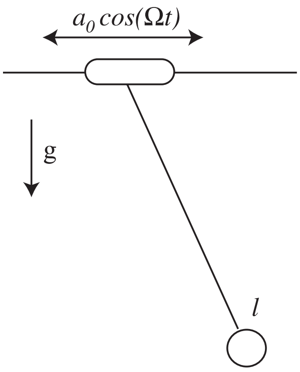

Figure 1. Schematic representation of a horizontally driven pendulum.

equation is derived for the oscillation envelope, which displays a linearly stochastic resonance. Numerically, this phenomenon is persistent when nonlinearities are considered. Hence, the system exhibits a stochastic resonance for the amplitude envelope. We describe three simple systems—a horizontally driven pendulum, a forced electrical circuit and a laser with an injected signal—that are used to show this stochastic resonance.

2. Stochastic resonance in a linear system

To understand the physical mechanism of linear stochastic resonance, we consider a classical mechanics example—an horizontally driven pendulum with stochastic fluctuations in the damping coefficient, which is depicted in figure 1. The corresponding equation of motion is [23]

$$\ddot{\theta} = -\omega_0^2 \sin(\theta) + \frac{a_0 \Omega^2}{\ell} \cos(\Omega t) \cos(\theta) - 2\mu(t)\dot{\theta}, \tag{1}$$

where θ stands for the angle between the pendulum and the vertical axis, $\omega_0 \equiv \sqrt{g/\ell}$ is the natural frequency of the pendulum, $\mu(t)$ is a stochastic function that accounts for damping, and $\{a_0, \Omega\}$ are, respectively, the displacement amplitude and frequency of the horizontal oscillations of the pendulum support. To model the stochastic damping we consider

$$\mu(t) = \mu_0 + \mu_1 \xi(t),$$

where μ_0 and μ_1 are free parameters and $\xi(t)$ is a stochastic process; more precisely μ_0 is the mean value of the damping, μ_1^2 the intensity of the noise and $\xi(t)$ a Gaussian white noise defined by its mean value $\langle \xi(t) \rangle = 0$ and correlation function $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$. It is noteworthy that experimental observations of fluctuations in physical systems driven out of equilibrium generically do not display this type of statistical property described by their correlation distributions or power spectrum density (for instance see [24]). Then, the fluctuations are not accurately described by white noise. However, this type of noise allows a qualitative description of the dynamics and mechanisms of many phenomena, such as *noise-induced transition* [3], *stochastic resonance* ([4] and references therein), *noise-induced front propagation* [25], *imperfect bifurcations* [26] and so on. For simplicity, we have considered white noise in the present paper.

In the linear regime equation (1) becomes

$$\ddot{\theta} + 2\left(\mu_0 + \mu_1 \xi(t)\right)\, \dot{\theta} + \omega_0^2 \theta = \frac{a_0 \Omega^2}{\ell} \cos\left(\Omega t\right), \tag{2}$$

where the multiplicative noise term is interpreted in the Stratonovich prescription [1]. The above model is just the equation for a harmonic oscillator with a fluctuating damping parameter used in several problems of hydrodynamics, chemical waves, dendritic growth and motion of vortices (see [19] and references therein) to mention a few.

Close to the resonance $\Omega \approx \omega_0$, let us introduce the detuning parameter $\nu \equiv \Omega - \omega_0$, with $\nu \ll \omega_0$, and consider small deterministic damping and forcing terms—$\mu \ll 1$ and $a_0 \ll 1$—one can write the solution for θ as

$$\theta(t) = a(t) \mathrm{e}^{\mathrm{i}\omega_0 t} + a^*(t) \mathrm{e}^{-\mathrm{i}\omega_0 t}, \tag{3}$$

where a is a small varying amplitude ($a(t) \ll 1$ and $\ddot{a}(t) \ll \dot{a}(t) \ll 1$), which accounts for the oscillation of the pendulum. Introducing the above solution into equation (2), one obtains the following equation for the complex amplitudes $a(t)$:

$$\dot{a} = -\left(\mu_0 + \mu_1 \xi(t)\right) a - \mathrm{i}\gamma \mathrm{e}^{\mathrm{i}\nu t}, \tag{4}$$

where $\gamma \equiv a_0 \omega_0 (1 + (\nu/\omega_0))^2 / 4\ell$ is the intensity of forcing for the amplitude equation. Separating a into its real and imaginary parts, $a = x + \mathrm{i}y$, we obtain

$$\begin{aligned} \dot{x} &= -\left(\mu_0 + \mu_1 \xi(t)\right) x + \gamma \sin\left(\nu t\right), \\ \dot{y} &= -\left(\mu_0 + \mu_1 \xi(t)\right) y - \gamma \cos\left(\nu t\right). \end{aligned} \tag{5}$$

These equations have been studied by several authors in the literature with the aim to observe stochastic resonance for various types of noise [16]–[21]. To take the mean value, we follow the same procedure as in [20], where the occurrence of stochastic resonance is studied in a linear system subjected to a linear and quadratic colored noise and driven by a periodic sinusoidal signal.

To study the statistical features of the above model (5), we consider the equation of the mean value of this model, which has the form

$$\langle \dot{x}(t) \rangle_{\mathrm{st}} = -\left(\mu_0 - \frac{\mu_1^2}{2}\right) \langle x(t) \rangle_{\mathrm{st}} + \gamma \sin\left(\nu t\right), \tag{6}$$

where the term proportional to μ_1^2 is an effective force generated by a stochastic term as a result of the Stratonovich prescription, which is an energy injection term. It is important to note that in the Ito prescription [1], the equation for the mean value has $\mu_1 = 0$. The above model is a simple linear damped forcing system. The general solution of model (6) has the form

$$\langle x(t) \rangle = x_0 \mathrm{e}^{-(\mu_0 - (\mu_1^2/2))t} - \mathcal{A} \cos\left(\nu t + \phi\right), \tag{7}$$

where x_0 is a parameter related to the initial condition. It is important to note that the average value exists only if $\mu_0 - \mu_1^2/2 > 0$. Hence, the forcing induces a periodic oscillation. Usually the amplitude of this periodic solution is termed *response* ($\mathcal{A}$), where the amplitude $\mathcal{A}$ and the phase ϕ are given by

$$\mathcal{A} = \frac{a_0 (\omega_0 + \nu)^2 / \left(4\omega_0 \ell\right)}{\sqrt{\left(\mu_0 - \mu_1^2/2\right)^2 + \nu^2}} \tag{8}$$

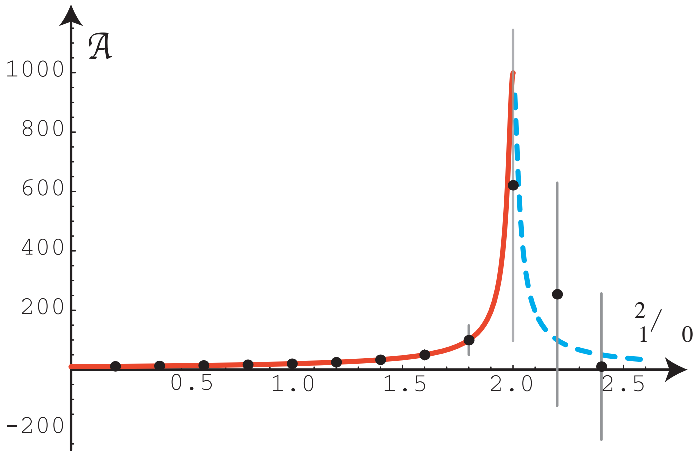

Figure 2. Response $\mathcal{A}$ versus noise intensity (μ_1^2/μ_0). The circles are the mean value of the response of model (4) for $\mu_0 = 0.1$, $\gamma = 1$ and $\nu = 0.001$ after 10^4 time iterations. The error bars stand for the dispersion of the ensemble of systems after the same time iterations. The continuous and dashed curves are, respectively, the amplitude $\mathcal{A}$ of formula (11) for the same values of parameters, in the intervals $0 \leqslant \mu_1^2/\mu_0 < 2$ and $\mu_1^2/\mu_0 \geqslant 2$. The maximum is obtained at $\mu_1^2/\mu_0 = 2$ and the value of the amplitude for the respective parameter is 1000.

and

$$\tan(\phi) = \frac{\mu_0 - \mu_1^2/2}{\nu}. \tag{9}$$

A similar result for the response has been found in [18]. We emphasized that the response is stable—the mean value converges to the periodic solution—only for $\mu_0 > \mu_1^2/2$. When this inequality is not satisfied the steady state does not exist; that is, the mean value and higher order cumulants are not defined.

The homogeneous solutions of the above equation as a function of time converge to zero only when $\mu_0 - \mu_1^2/2 > 0$ and diverge to infinity when $\mu_0 - \mu_1^2/2 < 0$. Hence, the response curve $\mathcal{A}(\mu_1^2/\mu_0)$ exists only for the interval $0 \leqslant \mu_1^2/\mu_0 \leqslant 2$ and has a maximum at $\mu_1 = \mu_c \equiv \sqrt{2\mu_0}$. For $\mu_1 > \sqrt{2\mu_0}$, the mean value of the steady state solution does not exist. Therefore, the stationary probability is not defined in this parameter region. Note that the maximum is at the end of the interval of validity of formula (8). Hence, in the interval $0 \leqslant \mu_1^2/\mu_0 \leqslant 2$ the system exhibits a stochastic resonance at $\mu_1^2/\mu_0 = 2$ [22]. In figure 2, the continuous and dashed curves depict the response curves in the existence region of the stationary state solution, respectively. Outside this region the stationary state does not exist.

To verify the above results, we have studied equation (4) numerically using the Stratonovich prescription and a four-order Runge–Kutta numerical integration method [27]. From numerical integration of equation (4), we have computed the mean value of the response at a given time. The circles illustrated in figure 2 are the mean values of the response after iterations of 10^4 time periods ($\mathrm{d}t = 0.05$). In order to study the dispersion of the mean value of the response at finite time, we have considered an ensemble of 100 systems. In figure 2, the error bars are the dispersion obtained numerically after 10^4 iterations. The dispersion close to the resonance value ($\mu^2 = 2\mu_0$) increases significantly, in the parameter region $\mu_0 - \mu_1^2/2 > 0$, which is represented by the dashed curve in figure 2. The dispersion at finite time in the ensemble increases systematically with time iterations. Hence, in this parameter region the system does not have a stationary state.

Returning to the initial problem of the horizontally driven pendulum, $\theta(t)$ is approached by formula (3), where $a = x + \mathrm{i}y$. Then, taking the mean value of the above expression

$$\langle\theta(t)\rangle_{\mathrm{st}} = 2\,\langle x(t)\rangle_{\mathrm{st}} \cos(\omega_0 t) - 2\,\langle y(t)\rangle_{\mathrm{st}} \sin(\omega_0 t) \tag{10}$$

and replacing the corresponding expression for the stationary mean values of x and y, we obtain

$$\langle\theta(t)\rangle_{\mathrm{st}} = -2\mathcal{A}\,\cos(\Omega t + \phi) \tag{11}$$

for $\mu_0 - \mu_1^2/2 > 0$. In the other case, the average value is not defined.

Therefore, up to linear order, the pendulum exhibits a linear stochastic resonance. However, this picture can be changed when nonlinear terms are taken into account. In the next section, we shall study the persistence of this resonance in the presence of nonlinear terms.

3. Nonlinear stochastic resonance

The nonlinear dynamics of a periodically forced damped oscillator subjected to a small perturbation is described by

$$\dot{a} = -\mu a + \mathrm{i}\alpha|a|^2 a + \gamma \mathrm{e}^{\mathrm{i}\nu t} + \mathrm{h.o.t.}, \tag{12}$$

where $a(t)$ is the complex amplitude of oscillation, μ accounts for dissipation, α stands for nonlinear response in the frequency, γ is the amplitude of the forcing (without loss of generality this parameter can be chosen to be real), and ν is the detuning of the forcing. The higher order terms (h.o.t.) stand for the high-order nonlinear response in frequency, nonlinear forcing and dissipation terms. For instance, close to the first resonance of the deterministic horizontally driven pendulum, equation (1), we can deduce the above model by introducing the following change in variables:

$$\theta(t) = a(t)\mathrm{e}^{\mathrm{i}\omega_0 t} + a^*(t)\mathrm{e}^{-\mathrm{i}\omega_0 t} + \frac{\omega_0}{48}\left(a^3 \mathrm{e}^{\mathrm{i}3\omega_0 t} + a^{*3}\mathrm{e}^{-\mathrm{i}3\omega_0 t}\right) + \cdots, \tag{13}$$

where

$$\begin{aligned} \alpha &= -\frac{\omega_0}{4}, \\ \gamma &= \frac{a_0\omega_0}{4\ell}\left(1 + \frac{\nu}{\omega_0}\right)^2. \end{aligned} \tag{14}$$

Using the scaling $\mu \ll 1$, $a \sim \mu^{1/2}$, $\partial_t \sim \mu^{1/2}$, α of the order of 1, $\gamma \sim \mu^{3/2}$, $\nu \sim \sqrt{\mu}$ and the rotation $A \equiv a\mathrm{e}^{-\mathrm{i}\nu t}$, the dominating dynamics in equation (12) is

$$\dot{A} = -(\mu - \mathrm{i}\nu)A + \mathrm{i}\alpha|A|^2 A + \gamma. \tag{15}$$

If we take into account stochastic fluctuations in the parameters, the above equation reads

$$\dot{A} = -\left[\mu + \mu_1\xi_1(t) - \mathrm{i}\left(\nu + \nu_1\xi_2(t)\right)\right]A + \mathrm{i}\alpha|A|^2 A + \gamma + \gamma_1\xi_3(t), \tag{16}$$

where $\{\xi_1, \xi_2, \xi_3\}$ are Gaussian white noises with zero mean values and correlation functions $\langle\xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t - t')$. $\{\mu_1^2, \nu_1^2, \gamma_1^2\}$ represent, respectively, the level of noise intensity of the dissipation, detuning and forcing. For the sake of simplicity, we shall concentrate on the effect

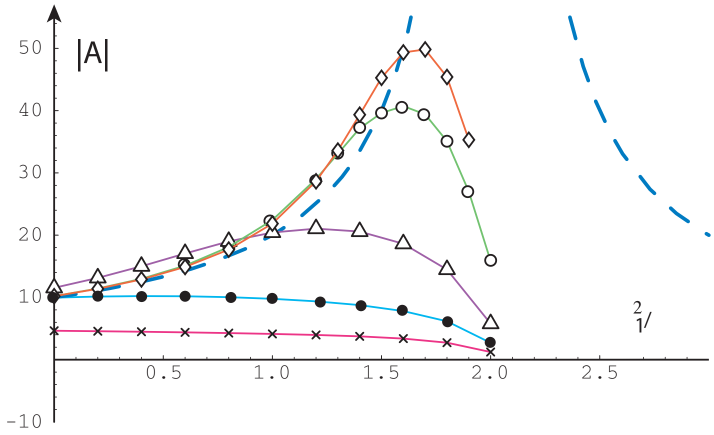

Figure 3. Amplitude $|A|$ versus noise intensity (μ_1^2/μ) for nonlinear amplitude equation (17) with $\mu = 0.1$, $\gamma = 1$, $\nu = 0.001$, ($\diamond$) $\alpha = 0.000\,005$, ($\circ$) $\alpha = 0.000\,01$, ($\triangle$) $\alpha = 0.0001$, ($\bullet$) $\alpha = 0.001$ and ($\times$) $\alpha = 0.01$. The dashed curve is the formula $\mathcal{A}(\mu = 0.1, \gamma = 1, \nu = 0.01)$.

of fluctuations in the dissipation; that is, $\nu_1 = \gamma_1 = 0$. Hence, the amplitude equation under study reads

$$\dot{A} = -\left[\mu + \mu_1 \xi_1(t) - \mathrm{i}\nu\right] A + \mathrm{i}\alpha |A|^2 A + \gamma. \tag{17}$$

In the previous section, we have shown that equation (17) has a stochastic resonance when $\alpha = 0$. Then, for small α we expect that stochastic resonance persists; that is, in the parameter range where the steady state exists the response is not monotonic and has a maximum as a function of the level of noise intensity. Figure 3 shows the average amplitude of the response after 10^4 time iterations versus the level of noise intensity obtained from numerical simulation of equation (17). The dashed curve stands for the response of the linear system ($\alpha = 0$), formula (8) for the amplitude. The ($\diamond$), ($\circ$), ($\triangle$), ($\bullet$) and ($\times$) symbols are the average amplitudes obtained after 10^4 time iterations when α is increased systematically. Hence, stochastic resonance persists when nonlinearities are considered and the nonlinear terms allow the system to have a maximum inside the interval of existence of the steady state ($0 \leqslant \mu_1 \leqslant \sqrt{2\mu}$). The maximum of the resonance curve moves to a small noise intensity and decreases when the nonlinearity is increased. There is a critical value of the nonlinearity α for which the maximum is reached at zero noise intensity. The ($\bullet$) and ($\times$) symbols in figure 3 have a maximum at zero noise intensity. Hence, the inclusion of the nonlinearities provides evidence that in the linear limit there is a stochastic resonance at $\mu_1^2/\mu = 2$.

From the above scenarios, we deduce that the intensity of the nonlinearity α changes the frequency of oscillation and destroys the stochastic resonance; that is, the maximum decreases and disappears when nonlinearity is considered. Then, nonlinearity plays a similar role as dissipation in the deterministic resonance [23].

To study the dispersion of the signal, we have considered an ensemble of 100 identical systems with the same level of noise intensity. At a given finite time, we compute the mean value and the dispersion—standard deviation—in the amplitude. Figure 4 depicts the resonance curve obtained from the ensemble and the error bars represent the dispersion in the amplitude. For noise intensities close to the critical value ($\mu_1 < \mu_c = 2\mu$), we observe numerically that the

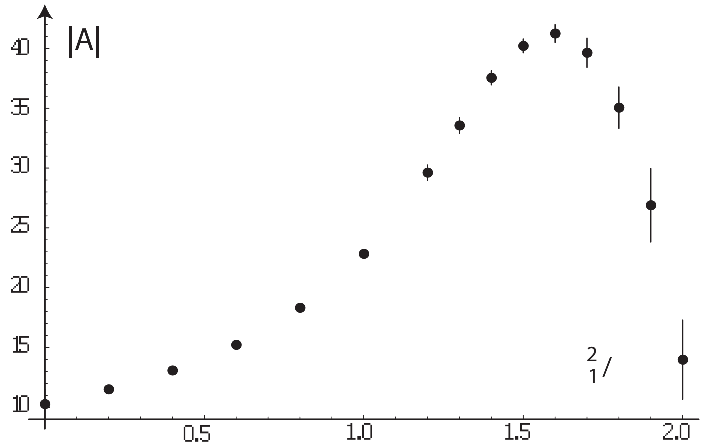

Figure 4. Average of amplitude $|A|$ after 10 000 time iterations versus noise intensity (μ_1^2/μ) obtained by equation (17) with $\mu = 0.1$, $\gamma = 1$, $\nu = 0.001$ and $\alpha = 0.000\,01$. The error bars represent the dispersion obtained from an ensemble of 100 identical systems.

system exhibits large fluctuations and these increase when the noise intensity reaches the critical value μ_c. This dynamical behavior is as expected, because when the noise intensity reaches the critical value the average of a real linear part of equation (17) goes to zero. Hence, the realizations of the dynamical system exhibit large excursions and these are responsible for the increment of the dispersion. A study of the different stochastic resonances of the full stochastic equation (16) is in progress.

4. Stochastic forcing and damping monostable oscillators

In this section, we shall present different idealized non-multiple stable physical systems that could exhibit a stochastic resonance induced by multiplicative white noise.

4.1. A horizontally driven pendulum

Owing to the generical nature of the phenomenon under study in this paper, a potential candidate for the realization of this stochastic resonance in non-multistable systems is a horizontally driven pendulum with stochastic damping (cf figure 1, which considers a simple planar pendulum with oscillatory support in the horizontal direction). Thus, the description of this system reads

$$\ddot{\theta} = -\omega_0^2 \sin(\theta) + \gamma \cos((\omega_0 + \nu)t) \cos(\theta) - (\mu + \mu_1 \xi(t))\, \dot{\theta}, \tag{18}$$

where ν is the detuning parameter. The deterministic limit of the above model (18) has an attractive periodical solution (limit cycle), which in general does not have a simple analytical expression. We term this solution $\theta_0(t)$, which is obtained numerically. To study the effect of stochastic fluctuations, for each period $2\pi/\omega_0$, we have computed the amplitude of oscillation

$$A_\theta(t) = \frac{\omega_0}{\pi} \int_t^{t+2\pi/\omega_0} \theta(t')\theta_0(t')\, \mathrm{d}t',$$

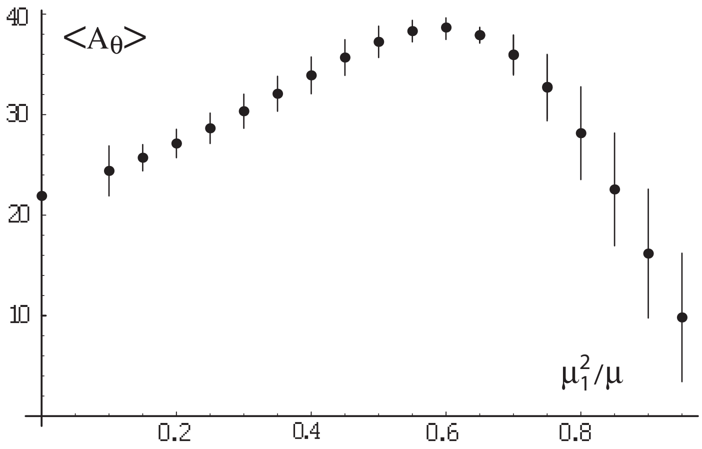

Figure 5. The resonance curve for a horizontally driven pendulum with stochastic damping equation (18) by $\omega_0 = 1.0$, $\nu = 0.02$ and $\mu = 0.5$. The mean value of amplitude ($\langle A_\theta \rangle$) in degrees as a function of the noise intensity.

which is a stochastic variable. The above amplitude corresponds to a projection of the function $\theta(t)$ in $\theta_0(t)$; that is, A_θ accounts for the amplitude in the mode $\theta_0(t)$ of the stochastic signal $\theta(t)$. For a large given time ($t \gg 2\pi/\omega_0$), we have computed the mean value ($\langle A_\theta \rangle$) and the dispersion of amplitude. Figure 5 shows a typical resonance curve obtained from a numerical simulation of equation (18). Hence, there is a critical value of noise intensity for which the amplitude of the periodical signal has a maximum. The error bars represent the dispersion in the amplitude. It is important to remark that for small forcing and damping terms the shape of the response curve is similar to that shown by the amplitude equation (17). However, this is valid for small angles, which are of the order of 1°. Figure 5 provides evidence for the persistence of stochastic resonance far from the region of validity of the stochastic amplitude equation (17). Therefore, the linear stochastic resonance exhibited by this system persists when nonlinearity is considered.

One of the main objections to a physical realization of the horizontally driven pendulum with stochastic damping is that stochastic damping must be small experimentally. In the next section, we shall present another potential candidate for the realization of this stochastic resonance.

4.2. A van der Pol circuit with a periodic voltage source

A tetrode multivibrator circuit with a periodic voltage source is described by (see chapter 4 of [29])

$$L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \left(R - \beta I^2\right)\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{I}{C} = \frac{\mathrm{d}V(t)}{\mathrm{d}t},$$

where I is the electrical current, L is the inductance, C is the capacitance, R is the resistance to low current, β accounts for the nonlinear response of resistance as a function of current and $V(t)$ is the periodic voltage source, $V(t) = V_0 \sin(\omega t)$. The natural frequency of this circuit is $1/\sqrt{LC}$. The above model is well known as the van der Pol oscillator. We assume that the voltage source can oscillate harmonically close to the natural frequency $\omega = 1/\sqrt{LC} + \nu$ ($\nu \ll 1$). Then we can introduce the ansatz

$$I = a\mathrm{e}^{\mathrm{i}t/\sqrt{LC}} + a^{*}\mathrm{e}^{\mathrm{i}t/\sqrt{LC}} - \mathrm{i}\frac{\beta\sqrt{LC}}{8}a^{3}\,\mathrm{e}^{\mathrm{i}3t/\sqrt{LC}} + \mathrm{i}\frac{\beta\sqrt{LC}}{8}a^{*3}\,\mathrm{e}^{-\mathrm{i}3t/\sqrt{LC}},$$

and we obtain equation (12), where

$$\mu = \frac{R}{2L}, \qquad \alpha = -\mathrm{i}\beta, \qquad \gamma = \frac{V_0}{4L}, \qquad \nu = w - \omega_0.$$

Hence the van der Pol circuit with a periodic voltage source oscillating at a frequency close to the circuit's natural frequency is described by model (12). If we consider the temperature fluctuations' effects on resistance, the effective resistance reads (see [30] and references therein)

$$R(t) = R_0 + R_1\xi(t),$$

where R_0 is the mean value of resistance, R_1^2 stands for resistance fluctuation (noise intensity) and $\xi(t)$ is a Gaussian white noise with zero mean value and the correlation $\langle\xi(t)\xi(t')\rangle = \delta(t-t')$. Note that R_1^2 could be related to temperature or external sources of fluctuations and that in the previous model, we are considering passive resistance. This kind of resistance does not exhibit negative values and $\xi(t)$ is a Nyquist noise. However, one can consider active elements and get equivalent behavior, but for simplicity we consider the resistance with a white noise, which corresponds to an approximation of the Nyquist noise. Hence, we expect to observe the stochastic resonance induced by multiplicative white noise in this monostable setup by means of temperature control. In contrast to the previous mechanical example, the fluctuations in resistance (damping) are relevant and manageable. An experimental study of this system is in progress.

4.3. An optical cavity

We consider an optical cavity with an injected signal. This dynamical system is described in terms of the well-known Maxwell–Bloch equations for a collection of two-level atoms in the slowly varying approach and including the spatial effects by [31, 32]

$$\begin{aligned}
\partial_t E &= -\kappa\left[\left(1+\mathrm{i}\frac{\omega_{\mathrm{c}}}{\kappa}\right)E - P - \mathrm{i}a\nabla_{\perp}^{2}E\right] + f\mathrm{e}^{\mathrm{i}\delta t},\\
\partial_t P &= -\gamma_{\perp}\left[\left(1-\mathrm{i}\frac{\omega_{\mathrm{A}}}{\gamma_{\perp}}\right)P - ED\right],\\
\partial_t D &= -\gamma_{\parallel}\left[D - \Lambda + \frac{(E^{*}P + EP^{*})}{2}\right],
\end{aligned} \tag{19}$$

where E, P and D are the envelope of the electromagnetic field, the atomic polarization and the population inversion, respectively, ω_{c} and ω_{A} are the atomic and cavity resonance frequencies, $\{\kappa, \gamma_{\perp}, \gamma_{\parallel}\}$ are their respective loss rates, Λ is the pumping parameter, f is the amplitude of the external electric field and a is the diffraction coefficient. δ is the detuning of the external signal with respect to the laser frequency without an external field $\omega \equiv (\kappa\omega_{\mathrm{c}} + \gamma_{\perp}\omega_{\mathrm{A}})/(\kappa + \gamma_{\perp})$.

The detuning of the cavity frequency and atomic frequency is $\theta \equiv (\omega_c - \omega_A)/(\kappa + \gamma_\perp)$. Note that the above model considers only the transversal variation perpendicular to the direction of propagation of the electromagnetic field. The spatial dependence on the direction of propagation was eliminated by considering as valid the uniform field approximation and a single longitudinal mode laser, which are common assumptions in laser theory. Model (19) accounts for a unified description of two coupled oscillators with a neutral mode close to 1:1 resonance [33].

Near the laser instability, model (19) can be approximated by model (12). Introducing the bifurcation parameter $\varepsilon^2 = \Lambda - \Lambda_c$, where Λ_c stands for the critical pumping value. In addition, one assumes that δ and f are small quantities that scale as $\delta \sim \varepsilon^2$ and $f \sim \varepsilon^3$ and considering the ansatz [34]

$$\begin{pmatrix} E \\ P \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \Lambda_c \end{pmatrix} + \varepsilon A(T = \varepsilon^2 t) \mathrm{e}^{-\mathrm{i}\omega t} \begin{pmatrix} 1 \\ 1 + \mathrm{i}\theta \\ 0 \end{pmatrix} - \varepsilon^2 |A|^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{20}$$

in equation (19), one obtains the following solvability condition:

$$\frac{\mathrm{d}A}{\mathrm{d}T} = \left[\frac{\kappa + \gamma_\perp}{\gamma_\perp} (\Lambda - \Lambda_c) + \mathrm{i}\theta \frac{\gamma_\perp - \kappa}{\kappa + \gamma_\perp} \right] \left(A - |A|^2 A \right) + \frac{\left(1 - \mathrm{i}(2\theta\kappa/(\kappa + \gamma_\perp + \theta^2 (\gamma_\perp - \kappa/\kappa + \gamma_\perp)^2))\right)}{\kappa} f \, \mathrm{e}^{\mathrm{i}\delta t}. \tag{21}$$

Therefore, one obtains model (12) again. Considering the optical cavity with an injected signal below the threshold ($\Lambda - \Lambda_c < 0$) and adding thermal fluctuations to loss rates, we deduce model (17). Hence, we expect to observe the stochastic resonance induced by multiplicative white noise in an optical cavity with an injected signal below the threshold by means of temperature control.

5. Conclusion and comments

We have shown that multiplicative Gaussian white noise induces stochastic resonance in forced damped monostable oscillators. We expect similar behavior for other types of fluctuations such as colored noise, shot noise and so on. We have identified the ingredients for observing this generic phenomenon. From the theory of the amplitude equation, we have deduced the mechanism of this resonance, which is related to a balance of the decreasing of the damping and nonlinear saturation. This phenomenon is common to a wide class of forcing oscillators. We have proposed simple systems that must exhibit the stochastic resonance induced by multiplicative white noise: a horizontally driven pendulum, a forced electrical circuit and a laser with an injected signal. Numerically, we have studied the horizontally driven damped pendulum, which displays the stochastic resonance, even far from the validity region of the amplitude equation.

Acknowledgments

We are grateful to C Falcon for many interesting discussions. The simulation software *DimX*, developed at INLN, has been used for all the numerical simulations presented in this paper.

We acknowledge support from the ring program ACT15 of *Programa Bicentenario* of the Chilean Government. MGC acknowledges financial support from FONDECYT project no. 1090045. HC acknowledges partial support from *Dirección de Investigacion* UTA (project no. 4720-10).

References

[1] van Kampen N G 1992 *Stochastic Processes in Physics and Chemistry* (Amsterdam: North-Holland)
[2] Gardiner C W 1990 *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Berlin: Springer)
[3] Horsthemke W and Lefever R 1984 *Noise-Induced Transition* (Berlin: Springer)
[4] Gammaitoni L, Hanggi P, Jung P and Marchesoni F 1998 *Rev. Mod. Phys.* **70** 223
[5] Benzi R *et al* 1981 *J. Phys. A: Math. Gen.* **14** L453
[6] Nicolis C and Nicolis G 1981 *Tellus* **33** 225
[7] Nicolis C 1981 *Solid Phys.* **74** 473
[8] Benzi R *et al* 1982 *Tellus* **34** 10
[9] Clerc M G and Marsden J E 2001 *Phys. Rev.* E **64** 067603
[10] Stocks N G, Stein N D and McClintock P V E 1993 *J. Phys. A: Math. Gen.* **26** L385
[11] Hu Gang *et al* 1993 *Phys. Rev. Lett.* **71** 807
[12] Sigeti D and Horsthemke W 1989 *J. Stat. Phys.* **54** 1217
[13] Tessone C J, Mirasso C R, Toral R and Gunton J D 2006 *Phys. Rev. Lett.* **97** 194101
[14] Fulinski A 1995 *Phys. Rev.* E **52** 4523
[15] Gingl Z, Kiss L B and Moss F 1995 *Europhys. Lett.* **29** 191
[16] Berdichevsky V and Gitterman M 1996 *Europhys. Lett.* **36** 161
[17] Barzykin A V and Seki K 1997 *Europhys. Lett.* **40** 117
[18] Fulinski A and Góra P 2000 *J. Stat. Phys.* **101** 483
[19] Gitterman M 2005 *Physica* A **352** 309
[20] Calisto H, Mora F and Tirapegui E 2006 *Phys. Rev.* E **74** 022102
[21] Li J H and Han Y X 2006 *Phys. Rev.* E **74** 051115
[22] Gora P F 2004 *Acta Phys. Pol.* B **35** 1583
[23] Landau L D and Lifchitz E M 2003 *Mechanics* (Burlington: Pergamon)
[24] Falcon C and Fauve S 2009 *Phys. Rev.* E **80** 056213
[25] Clerc M G, Falcon C and Tirapegui E 2005 *Phys. Rev. Lett.* **94** 148302
Clerc M G, Falcon C and Tirapegui E 2006 *Phys. Rev.* E **74** 011303
[26] Ortega I, Clerc M G, Falcon C and Mujica N 2010 *Phys. Rev.* E **81** 046208
[27] San Miguel M and Toral R 2000 *Instabilities and Nonequilibrium Structures* vol 6 ed E Tirapegui, J Martínez and R Tiemann (Dordrecht: Kluwer)
[28] Arnold V I 1983 *Geometrical Methods in the Theory of Ordinary Differential Equations (Fundamental Principles of Mathematical Science* vol 250*)* (New York: Springer)
[29] Stratonovich R L 1967 *Topics in the Theory of Random Noise* vol 2 (New York: Gordon and Breach)
[30] Gomila G, Pennetta C, Reggiani L, Sampietro M, Ferrari G and Bertuccio G 2004 *Phys. Rev. Lett.* **92** 226601
[31] Newell A and Moloney J 1992 *Nonlinear Optics* (Redwood, CA: Addison-Wesley)
[32] Coullet P, Aboussy D and Tredicce J 1998 *Phys. Rev.* E **58** 5347
[33] Clerc M, Coullet P and Tirapegui E 1999 *Opt. Commun.* **167** 159
[34] Coullet P, Gil L and Rocca F 1994 *Opt. Commun.* **111** 173