Comment on "Asymptotics of Large Bound States of Localized Structures"

In a recent Letter by Kozyreff *et al.* [1], the authors analyze stationary fronts connecting uniform and spatially periodic states. They claim that the size of the resulting localized patterns and their properties cannot be predicted by weakly nonlinear methods since they have their origin in exponentially small terms which do not appear in any polynomial order of the spatial unfolding of the usual normal forms [2]. They propose a different expansion and derive the snaking bifurcation curve and other stationary properties, repeating our previous results [3,4], where we used weakly nonlinear analysis with some slight modifications. The argument in [1] is that it is a separation of spatial scales which leads to the elimination of the scale of the pattern in the amplitude equation and hence cannot explain a situation where this scale plays a role, which is what happens around the core of the front and it would be nonsense to study the core using usual normal forms unless we make modifications incorporating the lost scale. These modifications appear in [3,4] as an external periodic forcing that explains the core dynamics and the interaction of two cores, which we show is at the origin of the existence of a whole family of stationary localized structures. It is this last stationary property which has been reobtained in [1], but our old work clearly shows that weakly nonlinear methods suitably modified can completely solve the problem contrarily to the claim in [1].

We can summarize our approach as follows: the original physical equation for $\vec{u}(x,t)$ has the stable solution $\vec{u}=0$ and we write $\vec{u}=A(X,T)e^{iqx}\hat{u}+\text{c.c.}+\text{h.o.t.}$, where X and T are suitable scaled variables, in the coexistence region near the bifurcation point where the spatially periodic solution of wave vector q appears. The modifications to the normal form is the incorporation of nonresonant terms which are $\Sigma_{n,m}a_{mn}A^m\bar{A}^n e^{iqx(m-n-1)}$, where a_{mn} are calculable coefficients and we call the result the amended normal form [4]. These extra terms break the parasite constant phase invariance of the normal form leaving the real invariance $A\to Ae^{i\alpha}$, $x\to x-\alpha/q$. From the amended equation, we derive dynamical equations for the core of the front and for their interactions, which give us the sizes and properties of all these localized structures near a bifurcation point. Using dynamical systems theory the previous stationary results were first obtained in Refs. [5] in 1D.

Finally, we remark that we can avoid the use of the amended normal forms which are an intermediate step to obtain the dynamics of the cores. To illustrate this, we consider the supercritical Swift-Hohenberg equation close to the Maxwell point, $\partial_t u=(-3/16+\gamma)9\nu^2u/10+\nu u^3-u^5-(\partial_{xx}+q^2)u$, where q is the typical wave number of the pattern exhibited by this model [$q\sim O(1)$], γ controls the shift at the Maxwell point and $\gamma\sim 2^8\exp(-q2^4\pi/\sqrt{3})/3\ll 1$. In order to describe the localized patterns we introduce the following ansatz: $u=\nu(3^5/2^9 5^3)^{1/4}\{(1+e^{-9(x+\Delta(\tau=\gamma t)/2)/16\sqrt{10q}})^{-1/2}+(1+e^{-9(x-\Delta(\tau=\gamma t)/2)/16\sqrt{10q}})^{-1/2}-1+\gamma w_1(x,t,\Delta)\}e^{i(qx+\theta)}+\text{c.c.}$, where $\Delta(\tau)$ stands for the distance between the cores of the fronts, that is, the size of the localized patterns. We assume that $\Delta\sim-\sqrt{4/3}\ln(\gamma)\gg 1$. θ is an arbitrary phase and $w_1(x,t,\Delta)$ is a small correction function. We obtain at first order in γ the solvability condition

$$\dot{\Delta}(t)=-\alpha\exp(-\sqrt{3/4}\Delta)+\beta\cos(2q\Delta/\sqrt{\epsilon})+2\gamma, \tag{1}$$

where $\alpha=27\sqrt{3}/64$ and $\beta=64\sqrt{3}q^2\exp(-q4\pi/\sqrt{\epsilon})/3\epsilon$. The oscillatory form of the right-hand side shows that the system has several equilibria—the localized patterns—obtained from $\dot{\Delta}=0$ (an analogous expression has been derived in [1]), and one can determine the size, existence, and stability of all of them. When γ is changed the equilibria states change and one trivially deduces the snaking bifurcation curve [3]. It is important to note that a similar expression to $\dot{\Delta}(t)$ has been deduced to localized peaks nucleating over a pattern of lower amplitude [6]. In summary, all stationary and dynamical properties of localized patterns can be predicted by means of weakly nonlinear analysis.

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[1] G. Kozyreff and S. J. Chapman, Phys. Rev. Lett. **97**, 044502 (2006).
[2] C. Elphick, G. Iooss, and E. Tirapegui, Phys. Lett. A **120**, 459 (1987).
[3] M. G. Clerc and C. Falcon, Physica (Amsterdam) **356A**, 48 (2005).
[4] M. G. Clerc, C. Falcon, and E. Tirapegui, Phys. Rev. Lett. **94**, 148302 (2005); Phys. Rev. E **74**, 011303 (2006).
[5] P. Coullet, C. Riera, and C. Tresser, Phys. Rev. Lett. **84**, 3069 (2000); A. R. Champneys, Physica (Amsterdam) **112D**, 158 (1998).
[6] U. Bortolozzo, M. G. Clerc, C. Falcon, S. Residori, and R. Rojas, Phys. Rev. Lett. **96**, 214501 (2006).

