Reduced description of the confined quasi-reversible Ginzburg Landau equation

M. Clerc, P. Coullet[*)], E. Tirapegui[†,††].

Institut Non Linéaire de Nice, UMR 6618 CNRS-UNSA,
1361 Route des lucioles, F-065600 Valbonne, France.
[†] *Facultad de Ciencias Físicas y Mat., Depto. Físicas,*
Univ. de Chile, casilla 487-3, Santiago, Chile.
[††] *Centro de Fisica No Lineal y Sistemas Complejos de*
Santiago,Casilla 27122, Correo 27, Santiago, Chile.

We study from the point of view of quasi-reversible instabilities the onset of chaos in the one dimensional quasi-reversible Ginzburg Landau equation with Neuman boundary conditions when the minimum wave number is close to the threshold of the Benjamin-Feir-Newell-Kuramoto instability. The system appears to be described by a Lorenz type model in which chaos arises in two different ways: the usual Lorenz homoclinic bifurcation and the cascade of gluing bifurcations.

The study of chaotic behavior in partial differential equations is a subject of current interest [1)-3)]. A traditional method to study this type of behavior is to put the system in a finite box in which case the partial differential equations reduce to a set of ordinary differential equations. The analysis of the bifurcations of the set of ordinary differential equations allows one to give a description of different behaviors presented by the system, such as instabilities, oscillations, chaos, and so on. There are two generic instabilities in dissipative systems : the saddle node and the Hopf bifurcation [3),4)]. The presence of symmetries changes this picture and a case of special interest arises when the system admits a time reversal symmetry [6)] which is weakly broken (quasi-reversible system) and is forced at zero frequency. We have shown that in this situation one has two generic or codimension one instabilities which are described by Lorenz type equations [8)] and by the Maxwell-Bloch equations [9)]. We shall apply here our general results on quasi-reversible systems to the study of the onset of chaotic behavior in the one dimensional confined complex Ginzburg-Landau equation. This gives a new insight on a previous work of Kuramoto and Koga on the complex Ginzburg-Landau equation far from the reversible limit.

We put the system in a finite box and of length L and we consider the situation in which the minimum wave number which depends on L and the boundary conditions we impose is close to the threshold of the Benjamin Feir-Newell-Kuramoto instability [5)]. For simplicity we shall present the calculations for Neumann boundary conditions and we shall see that in this case the system is described asymptotically by the real Lorenz equations. First we shall consider the nonlinear Schrödinger equation in which we will find a stationary instability characterized by a linear matrix of rank four composed by two Jordan blocks. Then we add the small terms which break the

[*)] Professeur a l'Institut Universitaire de France time reversal symmetry, but respect the phase invariance: the system presents now a stationary instability in the presence of a neutral mode which will lead generically to Lorenz type equations from which we shall give a brief description of the dynamics presented by the system.

We consider the complex one dimensional Ginzburg Landau equation

$$\partial_t A = -i\beta|A|^2 A + i\alpha\partial_x^2 A + \varepsilon_1 A - \varepsilon_2|A|^2 A + \varepsilon_3\partial_x^2 A \qquad (0{\cdot}1)$$

in a closed region, $-\frac{L}{2} \leq x \leq \frac{L}{2}$, and with Neumann boundary conditions

$$\partial_x A|_{x=\left\{-\frac{L}{2},-\frac{L}{2}\right\}} = 0 \qquad (0{\cdot}2)$$

If $\varepsilon_{1,}\varepsilon_2$,and ε_3 vanish equation (1) is invariant under the time reversal transformation $t \to -t, A \to A^\star$. If the previous coefficients, which are associated to irreversible terms (they break the time reversal invariance) are small and can then be considered as unfolding terms, we are in a quasi-reversible situation. The coefficients α and β are of order 0(1) and must have different signs ($\alpha\beta < 0$) to realize the Benjamin-Feir (self-focusing) instability which is the case in which we are interested now, and scaling time and space their modulus can be put equal to one. In the regime of parameters we have described, i.e. when the system is quasi-reversible, the nature of the instability is determined by the reversible part of the equation which is here the nonlinear Schrödinger equation. After the scalings we can put $\alpha = 1$ and $\beta = -1$ and the reversible equation is

$$\partial_t A = i|A|^2 A + i\partial_x^2 A. \qquad (0{\cdot}3)$$

This equation has as solutions a one parameter family of limit cycles ($A(t, R_o) = R_o e^{iR_o^2 t}$), which describe uniform oscillations. In order to study the dynamic around this solution we write the ansatz

$$A = R_o e^{i\left(R_o^2 t+\varphi(t)\right)} \left(1 + X\left(t, x\right) + iY\left(t, x\right)\right) \qquad (0{\cdot}4)$$

where $X\left(t, x\right)$ and $Y\left(t, x\right)$ satisfy

$$\begin{aligned} \partial_t X &= -\partial_{xx} Y - R_o^2\left(2XY + \left(X^2 + Y^2\right)Y\right) \\ \partial_t\varphi + \partial_t Y &= \partial_{xx} X + R_o^2\left(2X + 3X^2 + Y^2 + \left(X^2 + Y^2\right)X\right) \end{aligned} \qquad (0{\cdot}5)$$

The linear growth rates (λ_i) associated to spatially modulated perturbations of the form

$$\begin{aligned} X &= x_o e^{\lambda_o t} + \sum_{i=0} x_{2i+1}\sin(({\scriptstyle 2i+1})\frac{\pi}{L}x)e^{\lambda_{2i+1}t} + \sum_{i=1} x_{2i}\cos(2i\frac{\pi}{L}x)e^{\lambda_{2i}t} \\ Y &= \sum_{i=0} y_{2i+1}\sin(({\scriptstyle 2i+1})\frac{\pi}{L}x)e^{\lambda_{2i+1}t} + \sum_{i=1} y_{2i}\cos(2i\frac{\pi}{L}x)e^{\lambda_{2i}t} \end{aligned}$$

satisfy the relations

$$\lambda_i^2 = k_i^2\left(2R_o^2 - k_i^2\right) \qquad (0{\cdot}6)$$

where $k_i = \frac{\pi}{L} i$. The function $Y(t,x)$ is inhomogeneous since its homogeneous part has already been taken into account in the ansatz through the time dependent function $\varphi(t)$ (0·4). For k_i^2 smaller than $2R_o^2$ the perturbations are unstable since λ_i is a real number, in the opposite case, the perturbations oscillate around the original uniform oscillation (λ_i is pure imaginary).

When the minimum wave number decreases until it coincides with $\sqrt{2}R_o$ $\left(k_1 = \sqrt{2}R_o\right)$ the system presents a stationary instability, i.e., two pure imaginary values of λ_1 become real, one positive and one negative. One also has that as a consequence of the phase invariance of the complex Ginzburg-Landau equation and the reversibility of the nonlinear Schrödinger equation there are two new marginal values of λ_0 for zero wavenumber, i.e. $(\lambda_0 = 0, k = 0)$. In order to determine the dynamics of these unstable modes we use the following ansatz

$$\begin{aligned} X &= x_o + x_1 \sin(kx) \\ Y &= y_1 \sin(kx) \end{aligned} \tag{0·7}$$

in the original equations (0·5). We replace the ansatz and then we integrate $\int\limits_{-L/2}^{L/2} dx$ and $\int\limits_{-L/2}^{L/2} \sin(kx)\,dx$. We find the following equations

$$\begin{aligned} \partial_t \varphi &= R_o^2 \left(2x_o + \frac{y_1 x_o}{2} + 3x_o^2 + \frac{y_1^2}{2} + \frac{3}{2}x_1^2 + \frac{3}{2}x_o x_1^2\right) \\ \partial_t y_1 &= R_o^2 \left(2x_1 + 6x_1 x_o + 3x_o^2 x_1 + \frac{3}{4}y_1^2 x_1 + \frac{3}{4}x_1^3\right) - k^2 x_1 \\ \partial_t x_o &= -R_o^2 \left(x_1 y_1 + x_1 y_1 x_o\right) \\ \partial_t x_1 &= -R_o^2 \left(2y_1 x_o + y_1 x_o^2 + \frac{3}{4}y_1^3 + \frac{3}{4}y_1 x_1^2\right) + k^2 y_1 \end{aligned}$$

where $k = k_c + \varepsilon$, and ε is the bifurcation parameter ($\varepsilon \ll 1$). Introducing the new variables

$$\begin{aligned} x &\equiv x_1 \sqrt{\frac{3R_o^2 k^2}{4}}, \\ Z &= -\left(x_o + \frac{x_1^2}{4}\right) 6R_o^2 k_c^2, \end{aligned} \tag{0·8}$$

and $\varepsilon' = \varepsilon k_c^2$, the preceding equations become asymptotically

$$\begin{aligned} \partial_{tt} x &= \varepsilon' x + zx - x^3, \\ \partial_t z &= 0, \\ \partial_t \varphi &= R_o^2 \left(2x_o + \frac{3}{2}x_1^2\right), \end{aligned}$$

for the asymptotic relations $x \sim \sqrt{\varepsilon'}$, $z \sim \varepsilon'$, $\partial_t \varphi \sim \varepsilon'$,

$\partial_t z \sim (\varepsilon')^{3/2}$ and $\partial_t x \sim \varepsilon'$.

Due to phase invariance the first two equations do not depend on φ and then the last equation is a first integral.. The first equation of the preceding set of equations is associated to the two first unstable spatial modes which become unstable through a reversible pitchfork bifurcation. The second equation is related to the fact that in the reversible system the limit cycles appear as a one parameter family [7)], parametrized by the constant value of z. These equations are invariant under the time reversal transformation

$$t \to -t, x \to x, z \to z, \psi_o \to -\psi_o \tag{0·9}$$

This is a consequence of the time reversal invariance of the nonlinear Schrödinger equation. If we had considered all the spatial modes, we would have found the latter equations coupled to a set of non linear oscillators, but when we switch the small irreversible terms (sources and dissipative terms), the system will be asymptotically described only by the preceding variables since the amplitudes of the oscillators will have asymptotically only dissipative terms.

Using the change of variables (0·8) we can write the complex Ginzburg-Landau equation in the form

$$\begin{aligned}
\partial_t X &= -\partial_{xx} Y - R_o^2\left(2XY + \left(X^2+Y^2\right)Y\right)\\
&\quad -\varepsilon_1\left(2X + 3X^2 + Y^2 + \left(X^2+Y^2\right)X\right) + \varepsilon_3 \partial_{xx} X,\\
\partial_t \varphi + \partial_t Y &= \partial_{xx} X + R_o^2\left(2X + 3X^2 + Y^2 + \left(X^2+Y^2\right)X\right)\\
&\quad -\varepsilon_1\left(2XY + \left(X^2+Y^2\right)Y\right) + \varepsilon_3 \partial_{xx} Y
\end{aligned} \tag{0·10}$$

A consequence of the small irreversible terms which we have added is that only one limit cycle of the family of the reversible system persists and it has the following analytical expression in term of its phase and modulus

$$R_o = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}, \theta = \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 t, \tag{0·11}$$

It is important to note that ε_1 is related to the linear injection of energy and $\{\varepsilon_2, \varepsilon_3\}$ are related to the nonlinear dissipation and the diffusion.

Using the ansatz (0·7), that describes the attractive manifold of the system; we obtain the following equations when we use the same projections of the reversible case

$$\begin{aligned}
\partial_t \varphi &= R_o^2\left(2x_o + \frac{y_1 x_o}{2} + 3x_o^2 + \frac{y_1^2}{2} + \frac{3}{2}x_1^2 + \frac{3}{2}x_o x_1^2\right)\\
&\quad -\varepsilon_1\left(x_1 y_1 + x_1 y_1 x_o\right)\\
\partial_t y_1 &= R_o^2\left(2x_1 + 6x_1 x_o + 3x_o^2 x_1 + \frac{3}{4}y_1^2 x_1 + \frac{3}{4}x_1^3\right) - k^2 x_1\\
&\quad -\varepsilon_1\left(2y_1 x_o + \frac{3}{4}x_1^2 y_1 + \frac{3}{4}y_1^3 + y_1 x_o^2\right) - \varepsilon_3 k^2 y_1
\end{aligned}$$

$$\begin{aligned}
\partial_t x_o &= -R_o^2\left(x_1 y_1 + x_1 y_1 x_o\right) - \varepsilon_1\left(2x_o + \frac{y_1^2}{2} + \frac{3}{2}x_1^2 + 3x_o^2\right) \\
&\quad -\varepsilon_1\left(\frac{3}{2}x_1^2 x_o + \frac{1}{2}y_1^2 x_o + x_o^3\right) \\
\partial_t x_1 &= -R_o^2\left(2y_1 x_o + y_1 x_o^2 + \frac{3}{4}y_1^3 + \frac{3}{4}y_1 x_1^2\right) + k^2 y_1 \\
&\quad -\varepsilon_1\left(2x_1 + 6x_o x_1 + \frac{3}{4}x_1^3 + 3x_o^2 x_1 + \frac{3}{4}y_1^2 x_1\right) - \varepsilon_3 k^2 x_1
\end{aligned}$$

If all the coefficients of the irreversible terms are of the order of the square root of the bifurcation parameter ($\varepsilon_1 \sim \varepsilon_2 \sim \varepsilon_3 \sim \sqrt{\varepsilon}$), then we have that after the change of variables (0·8), the system is described asymptotically by the following normal form

$$\begin{aligned}
\partial_{tt} x &= \bar{\varepsilon} x - zx + x^3 - \nu \partial_t x, \\
\partial_t z &= -\mu z + \eta x^2, \\
\partial_t \varphi &= R_o^2\left(2x_o + \frac{3}{2}x_1^2\right),
\end{aligned}$$

where $\bar{\varepsilon} \equiv k_c^2\left(\varepsilon k_c^2 - 2\varepsilon_1\varepsilon_3 - \varepsilon_3^2\right)$, $\nu \equiv 2k_c^2\varepsilon_3 + 2\varepsilon_1$, $\mu = 2\varepsilon_1$, and $\eta = 2\left(4\varepsilon_1 + k_c^2\varepsilon_3\right)$..

Since the irreversible terms respect the phase invariance of the complex Ginzburg Landau equation (0·1), we have that once again, the equation for φ is a first integral, and consequently the attractive manifold is described by three variables $(x_{0,}x_1, y_{1)}$ or equivalently by x, $\dot{x}$, and z).

We note that the coefficients μ and ν are dissipative terms and η depending on its sign is a non linear dissipation or injection of energy. On the other hand the spatial reflection invariance of the Ginzburg-Landau equation implies that we must have the symmetry $(x_0 \to x_0, x_1 \to -x_1, y_1 \to -y_1)$ as it can be seen from equations (5) and consequently the normal form has the symmetry $x \to -x$.

As it has been shown in reference [8)], the two first equations of the normal form written above, i.e. the equations for $x$and z, are the normal form of one of the generic bifurcations of quasi-reversible systems and they are equivalent to the Lorenz equations

$$\begin{aligned}
\dot{x}' &= \sigma\left(y' - x'\right) \\
\dot{y}' &= rx' \mp y' - x'z' \\
\dot{z}' &= -bz' + x'y'
\end{aligned} \tag{0·12}$$

through the change of variables

$$\begin{aligned}
z &= z'\frac{(\eta+\mu)}{\tau_o} - \frac{x'^2}{\tau_o^2}, \\
x &= \frac{x'}{\tau_o}, \\
\dot{x} &= \left(y' - x'\right)\frac{(\eta+\mu)}{\tau_o}
\end{aligned}$$

where $\tau_o = \left|\frac{(\eta+\mu)}{\nu-(\eta+\mu)}\right|$, $\sigma = (\eta+\mu)$, $r = \left(\varepsilon - (\eta+\mu)^2 + \nu(\eta+\mu)\right)$, $b = \frac{\mu}{\tau_o}$, and the sign "$\mp$" is determined by the sign of the expression $-\nu + (\eta+\mu)$. It is important to notice that this sign plays a fundamental role in the different dynamical behaviors presented by the system.

In the case that $-\nu + (\eta+\mu)$ is positive the chaotic behavior is related to the appearance of an unstable homoclinic solution which creates a complex hyperbolic set in the three dimensional phase space (usual Lorenz chaos).

In the case that η and ν are small with respect to μ and ε $(\eta, \nu \ll \varepsilon, \mu)$, the sign in the equations (0·12) is positive, after the pitchfork bifurcation the system presents a supercritical Hopf bifurcation. Contrarily to the previous case, if we increase the bifurcation parameter the two stable limit cycles give rise to a stable double homoclinic solution, and now the system presents chaotic behavior through a cascade of gluings (see [14]). We remark that this last dynamical scenario has been observed for the one dimensional confined Ginzburg Landau equation by Kuramoto and Koga [14), 10)] .For small ν and η $(\eta, \nu \ll \varepsilon, \mu)$, it is possible determine an analytical condition for the homoclinic bifurcation [8] and near this condition we can observe the previous scenarios.

To end let us consider periodic boundary conditions for the Ginzburg Landau equations (0·1). Each spatial mode is now duplicated and we look for the minimum wave number close to the threshold of the Benjamin-Feir instability. We see immediately that the attractive manifold will now be described by five variables associated to even and odd Fourier modes which will satisfy the following asymptotic normal form

$$\begin{aligned}\partial_{tt}A &= \varepsilon A - zA - |A|^2 A - \nu \partial_t A,\\ \partial_t z &= -\mu z + \eta |A|^2.\end{aligned}$$

These equations are equivalent to the complex Lorenz model with real coefficients or to the Maxwell-Bloch equations without detuning [15), 16)]. We remark that the case of periodic boundary conditions has been studied by Malomed et al [11]. They obtain the real Lorenz equations since both the real and imaginary part of the variables (or in fact any fixed linear combination of both) are an invariant manifold and in this manifold one has the real equations.

In conclusion we have studied from the point of view of quasi-reversible instabilities the onset of chaos in the one dimensional quasi-reversible Ginzburg Landau equation with Neuman boundary conditions when the minimum wave number is close to the threshold of the Benjamin-Feir-Newell-Kuramoto instability. In this situation the instability is determined by the reversible part of the equation and the system is described by a Lorenz type model in which chaos appears through two different scenarios: one is the usual Lorenz chaos and the other is the cascade (gluing). Numerical simulations of the quasi-reversible equations show a good agreement with the reduced description of the system.

The authors acknowledge support from the CNRS-CONICYT program, Fondecyt, ECOS, and Cátedra Presidencial. M.C. thanks the ACI Blanche of the French Ministry of Research and P.C. the "Institut Universitaire de France".

References

1) M. Cross, P. Hohenberg, Rev. Mod. Phys. **65** (1993), 851.
2) P. Manneville, Dissipative Structure and weak turbulence (Academic Press, Boston, 1990).
3) Y. Kuramoto, Chemical Oscillations, Waves, and turbulence (Springer-Verlag, Berlin heidelberg, 1984).
4) J. Guckenheimer et P. Holmes, Nonlinear oscillations, dynamical systems and bifurcations of vector fields.(Springer-Verlarg, New York, 1983).
5) T. B. Benjamin and J. E. Feir, J. Fluid Mech. **27** (1967), 417. A. C. Newell, Lect. Appl. Math, **15** (1974), 157, T. Yamada and Y. Kuramoto, Prog. Theor. Phys. **56** (1978), 681.
6) R. Devaney, Trans. Am. Math. Soc. **218** (1976), 89.
7) R. Devaney, Ind. Univ. Math. J. **26** (1977), 247.
8) M. Clerc, P. Coullet, E. Tirapegui, Phys. Rev. Lett. **83** (1999), 3820.
9) M. Clerc, P. Coullet, E. Tirapegui, Optics Communications, **167** (1999), 159.
10) Y. Kuramoto, S. Koga, Phys. Lett. A **92** (1982), 1 .
11) B. Malomed, A. Nepomnyashchy, Phys. Rev. A **42** (1990), 6238.
12) E. Lorenz, J. Atmos. Sci. **20** (1963), 130.
13) C. Sparow, The Lorenz equations: Bifurcation, chaos, and strange attractor (Springer, 1982).
14) Arneodo, P. Coullet, C. Tresser, Phys. Lett.**A81** (1981), 197.
15) H. Haken, Phys. lett. **53A**, 77 (1969).
16) J.D. Gibbon and M. McGuinness, Phys. Lett. A **77** (1980), 295; Proc. R. Soc. Lond. A **377** (1981), 185 ; Physica D **7** (1983), 126; A.C. Fowler, J.D. Gibbon and M. McGuinness, Physica D **4** (1982),139.