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The Maxwell-Bloch description of 1/1 resonances

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Abstract

We discuss the generic 1/1 resonance of a reversible system weakly perturbed by dissipative terms. We show that the Maxwell-Bloch equations are the asymptotic normal form of the system when the energy is injected by coupling with a zero frequency mode. © 1999 Elsevier Science B.V. All rights reserved.

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A particularly common instability occurs in reversible systems when, for a critical parameter value, the frequencies of two modes coincide. It is the 1/1 resonance and the instability sets in because the frequencies become complex. The crossing of two eigenvalues, although not generic, is frequently observed when mechanical or electrical oscillators are coupled. As was noted by Rocard [1], an asymmetrical coupling of the two oscillatory modes with an external field is necessary to observe the instability. From a physical point of view the energy flows from an external source into the oscillatory modes through resonant couplings. The properties of the resonance persist when weak dissipation is taken into account. Fluids, elastic membranes and optical cavities among others provide examples of systems which exhibit such ''quasi-reversible'' behaviour. One has in fact two generic instabilities [2,3] in ''quasi-reversible'' systems: the ''Lorenz'' case where the crossing of the eigenvalues is at zero frequency, and the 1/1 resonance where the crossing occurs at a finite frequency. It should be pointed out that in a series of remarkable papers Gibbons and collaborators [4] studied the dispersive instability (crossing at finite frequency) weakly perturbed by dissipation in extended systems. We shall see here that we can arrive at the same conclusions and results with a singularity theory approach which allows us to classify quasi-reversible systems according to their symmetries.

In this paper we focus on the instability at finite frequency called ''confusion of frequencies'' by Rocard and we show that it is described by a set of amplitude equations which are formally equivalent through a simple change of variables to the Maxwell-Bloch (MB) equations describing the interaction of the electromagnetic field and an assembly of two-level atoms. We recall that the MB equations are equivalent to the Lorenz complex equations as shown by Haken [5] and that Abraham and Weiss

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have presented experimental evidence of Lorenz dynamics in lasers [6]. In the context of optics the MB equations are derived from the microscopic equations where from the beginning the electromagnetic field is treated classically (the semi-classical model). One then uses the slowly varying envelope approximation and obtains the MB equations [7]. In fact any reversible system displaying the $1/1$ resonance will be described when it is weakly perturbed by terms violating time reversal and when energy is injected through a zero frequency mode by the MB equations since these equations are equivalent to its normal form.

In the first, mainly pedagogical part we describe the $1/1$ resonance. Then we derive the normal form and show its equivalence to the MB equations. We conclude with some implications of these results.

A reversible system is a dynamical system which is invariant by time reversal. If the dynamical system is $(u \in R^n)$

$$\partial_t u = f(u)\,, \tag{1}$$

reversibility means mathematically [8] that one has a transformation Π of R^n such that $\Pi^2 =$ identity (involution) and

$$\Pi^{-1} f(\Pi u) = -f(u)\,. \tag{2}$$

Then one can immediately check that Eqs. (1) are invariant by $t \to -t, u \to \Pi u$. The most important property of reversibility is restricting the stability spectrum of reversible solutions that is symmetrical with respect to the imaginary and the real axis in the complex plane. A solution is reversible if it is invariant under the reversibility transformation. Let us suppose that Eq. (1) has the simple solution $u=0$. This solution is clearly reversible. The Jacobian operator $\mathscr{L} \equiv \frac{Df}{Du}(0)$ anti-commutes with Π. The trivial solution is stable only if the eigenvalues of the Jacobian matrix lie on the imaginary axis. Let us assume that for some value of a parameter two of these eigenvalues coincide. Generically the corresponding eigenvectors coincide also. For this parameter value the Jordan form of the Jacobian reads

$$\begin{pmatrix} i\Omega & 1 & 0 & 0 & 0 \\ 0 & i\Omega & 0 & 0 & 0 \\ 0 & 0 & -i\Omega & 1 & 0 \\ 0 & 0 & 0 & -i\Omega & 0 \\ 0 & 0 & 0 & 0 & L \end{pmatrix}, \tag{3}$$

where L is a diagonal matrix of dimension $(n-4)$. In the basis of eigenvectors and generalized eigenvectors of $\mathscr{L}$ Eq. (1) linearized around the trivial solution reads

$$\partial_t A = i\Omega A + B\,, \quad \partial_t B = i\Omega B\,, \quad \partial_t v = Lv\,, \tag{4}$$

with

$$u = A\Phi + B\Psi + \text{c.c.} + \sum_{l=1}^{l=n-4} v_l\,\chi_l\,. \tag{5}$$

Here the χ_l are the eigenvectors corresponding to the eigenvalues $i\omega_l$, Φ the eigenvector corresponding to the eigenvalue $i\Omega$, and Ψ the generalized eigenvector $(\mathscr{L}\Psi = i\Omega\Psi + \Phi)$. Linearly the modes v decouple and the time reversal transformation for the first variable in Eq. (4) is $t \to -t$, $A \to \bar{A}$, $B \to -\bar{B}$ ($\bar{A}$ and $\bar{B}$ are the complex conjugates of A and B). The change of variables

$$A = \tilde{A}\exp(i\Omega t)\,, \quad B = \tilde{B}\exp(i\Omega t) \tag{6}$$

renders the linear equation in the form (omitting the tildes)

$$A_{tt} = 0\,. \tag{7}$$

Then the solution restricted to the generalized $i\Omega$ eigenspace reads

$$u = (A\Phi + A_t\Psi)\exp(i\Omega t) + \text{c.c.} \tag{8}$$

Close to the confusion of frequencies the reversible linear unfolding depends on two small parameters δ and ϵ and one has

$$A_{tt} = i\delta A_t + \epsilon A\,, \tag{9}$$

whose interpretation is the following:

- The parameter δ is the ‘‘detuning’’. When $\epsilon = 0$, the amplitude equation (9) reads

  $$A_{tt} = i\delta A_t\,. \tag{10}$$

  Its general solution is

  $$A = A_0 + A_1\exp(i\delta t)\,,$$

  where A_0 and A_1 are arbitrary complex numbers. The solution is then

  $$u = A_0\exp(i\Omega t)\Phi + A_1\exp(i(\Omega t + \delta)) \times [\Phi + i\delta\Psi]\,,$$

  which explicitly shows that δ is the mismatch frequency between the frequencies of the two modes.

- The parameter ϵ is the control parameter. The change of variables $A = A\exp(i\frac{\delta}{2}t)$ transforms Eq. (9) in

$$A_{tt} = \left(\epsilon - \frac{\delta^2}{4}\right) A . \tag{11}$$

Its general solution reads

$$A = A_{-}\exp\left(-i\sqrt{\frac{\delta^2}{4} - \epsilon\, t}\right) + A_{+}\exp\left(i\sqrt{\frac{\delta^2}{4} - \epsilon\, t}\right),$$

where $A_{+,-}$ represent two arbitrary complex numbers. The effect of negative ϵ is to split away the frequencies. The minimum distance between the two eigenvalues is reached when $\delta = 0$ and it is given by $\Delta\omega = 2\sqrt{\epsilon}$. Positive values of ϵ lead to instability. More precisely, the instability occurs when $-2\sqrt{\epsilon} < \delta < 2\sqrt{\epsilon}$. In this parameter regime the amplitude grows exponentially and its frequency is locked on $\Omega + \frac{\delta}{2}$.

The simplest example of a $1/1$ resonance is an aircraft wing. Following Rocard [1] we can model the behaviour of a wing with a flexion and a torsion mode. From elasticity theory it follows that these two modes are coupled symmetrically when the plane is at rest [1]. However, if the aircraft is in motion with constant velocity, the modes acquire an asymmetric coupling through a term proportional to the velocity squared. Let θ_1 and θ_2 represent the torsion and the flexion modes, ω_1 and ω_2 the equilibrium frequencies of these two modes, and V the velocity of the aircraft. The equations of motion are

$$\frac{d^2\theta_1}{dt^2} + I\frac{d^2\theta_2}{dt^2} = -\omega_1^2\theta_1 - \rho V^2\theta_2 ,$$
$$\frac{d^2\theta_2}{dt^2} + I\frac{d^2\theta_1}{dt^2} = -\omega_2^2\theta_2 , \tag{12}$$

where I is the moment of mutual inertia and ρ a parameter.

We can easily understand that the asymmetric term is responsible for the instability, writing the equations in the form (we take $\omega_1 = \omega_2 = \Omega$ for simplicity)

$$(1 - I^2)\frac{d^2\theta_1}{dt^2} = -\Omega^2\theta_1 + \left(I\Omega^2 - \rho V^2\right)\theta_2 ,$$
$$(1 - I^2)\frac{d^2\theta_2}{dt^2} = \left(I\rho V^2 - \Omega^2\right)\theta_2 + I\Omega^2\theta_1 , \tag{13}$$

where we can see directly that for some values of $I\rho V^2$ the aircraft wing becomes unstable. When the parameters satisfy $I\rho V^2 = 2\Omega^2(1 - \sqrt{1 - I^2})$ which fixes $V = V_c$, the system presents an instability with finite frequency $\pm\Omega(1 - I^2)^{1/4}$. A linear change of variables from $(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2)$ to $(A, \bar{A}, A_t, \bar{A}_t)$ gives for the amplitude A the equation

$$A_{tt} = \frac{2I\rho V_c \Delta V}{4} A - i\frac{2I\rho V_c \Delta V}{4\Omega(1 - I^2)^{1/4}} A_t , \tag{14}$$

where $\Delta V = V - V_c$, and we see that the system is unstable for positive ΔV.

The limitation of the amplitude A in Eq. (9) arises from the nonlinear unfolding of the instability whose asymptotic normal form reads at leading order

$$A_{tt} = i\delta A_t + \epsilon A - \alpha|A|^2 A , \tag{15}$$

where we have assumed that the eigenmodes associated with the frequencies ω_l uncouple from the dynamics. This is the generic situation since it corresponds to the case in which these frequencies are not rationally related to Ω. The truncation of the amplitude equations is obtained assuming the following scaling relations between the parameters, the time and the amplitude: $\partial_t \sim \delta \sim \sqrt{\epsilon} \sim A$. The sign of the nonlinear term in Eq. (15) controls the saturation of the instability: when α is positive we have saturation, and if α is negative the instability is not saturated. The mechanism of saturation is, as usual, an amplitude frequency effect. When the amplitude grows the frequency changes, reaching the border of the resonance when $\alpha|A|_s^2 = \epsilon - \frac{\delta^2}{4}$. Besides the stationary solution Eq. (15) has periodic and quasiperiodic solutions. In order to calculate the normal form up to cubic order we must calculate the solution u (see Eq. (9)) up to quadratic order:

$$u = (A\Phi + \dot{A}\Psi) + \text{c.c.} + A^2\Phi_{20}\exp(2i\Omega t) + |A|^2\Phi_{11} + |\dot{A}|^2\Psi_{11} + \dot{A}^2\Psi_{20} + \text{c.c.} , \tag{16}$$

and one can show that in this case the normal form to any polynomial order reads [9]

$$A_{tt} = iA_t f\left(|A|^2, i\left(A\bar{A}_t - \bar{A}A_t\right)\right) + Ag\left(|A|^2, i\left(A\bar{A}_t - \bar{A}A_t\right)\right), \quad (17)$$

where f and g are real-valued functions of their arguments. This system is an integrable dynamical system [10]. One can easily check that the asymptotic form of the general reversible normal form (17) the above mentioned scalings is Eq. (15). We remark that in the case of non reversible, i.e., generic dynamical systems, f and g are complex-valued functions [9].

Weak dissipation breaks the reversibility transformation $t \to -t$, $A \to \bar{A}$, and if we add to Eq. (15) the terms which break the time reversal invariance, two new parameters $\mu \sim \sqrt{\epsilon}$ and $\nu \sim \epsilon$ will appear. The asymptotic amplitude equation then reads

$$A_{tt} = i\delta A_t - \mu A_t + \epsilon A + i\nu A - \alpha A|A|^2. \quad (18)$$

Positive values of μ will correspond to the standard dissipation, and linear stability analysis of Eq. (18) leads to the stability condition

$$\mu^2 - 4\epsilon \geq \sqrt{\left(\mu^2 + 4\epsilon\right)^2 + 16\nu^2}. \quad (19)$$

The term $i\nu A$ in Eq. (18) can be eliminated by putting $A = \tilde{A}\exp\left(i\frac{\nu}{\mu}t\right)$, and from now on we ignore it.

We now turn to the particularly important question of the energy. In the amplitude equation the energy injection depends on the parameter ϵ. If $\epsilon = 0$ and α is positive, all solutions converge to zero, while for negative α the initial conditions above a certain amplitude diverge in finite time. The resonant couplings have a finite frequency: for example if one considers the effect of a parametric forcing at frequency $2(\Omega - \Delta)$, we end up with an amplitude equation which reads

$$A_{tt} = i\delta A_t - \mu A_t - \alpha A|A|^2 + \gamma\bar{A}\exp(-2i\Delta t), \quad (20)$$

where γ measures the intensity of the forcing. A more interesting question concerns the possibility of coupling the system of oscillators with a time-independent source. In order to realize such a coupling we have to assume the existence of a zero frequency mode which can directly couple with the external field. Additional neutral modes can generically exist in the case of reversible dynamical systems and are usually associated with fundamental symmetries. Let us assume that at the bifurcation the Jordan block of the Jacobian operator is

$$\begin{pmatrix} i\Omega & 1 & 0 \\ 0 & i\Omega & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (21)$$

For the time reversal transformation $t \to -t$, $A \to \bar{A}$, $z \to z$, where z is the amplitude of the neutral mode, the reversible amplitude equation in leading order will be

$$A_{tt} = i\delta A_t + zA - \alpha A|A|^2, \quad z_t = 0, \quad (22)$$

and asymptotically $z \sim |A|^2$. The interpretation is now very clear: the additional neutral mode is actually a conserved quantity z, and for values $z_0 > 0$ of this quantity the instability sets in. We recall that in the case of aircraft wings the conserved quantity is related to the kinetic energy of the plane.

We now turn our attention to the more realistic situation where weak dissipation is taken into account. We introduce in Eqs. (22) terms breaking time reversal invariance and we obtain

$$A_{tt} = (-\mu + i\delta)A_t + zA - \alpha A|A|^2,$$

$$z_t = -\nu(z - z_0) + \eta|A|^2. \quad (23)$$

We have here dissipation terms allowed by the asymptotic normal form and also a forcing term z_0 which measures the distance to the instability threshold. These equations are the asymptotic normal form, i.e., the leading order truncation, of the instability associated with the Jordan block given by (21). Two new dissipative parameters appear in the equations: $\nu \sim A$ associated with the relaxation of the conserved quantity, and $\eta \sim A$ associated with the feedback of the oscillation on the zero frequency mode.

We recall now that physically the resonance between an electromagnetic cavity mode and the atomic oscillators is at the origin of the laser phenomenon. 0ne of the simplest descriptions of lasers consists in the study of the interaction of an assembly of two-level atoms, described by the Bloch equations, and the electromagnetic modes described classically by the Maxwell equations. In the frame of the semi-classical approximation and using the slowly varying envelope approximation, the equations describing the laser instability take the simple form of five amplitude equations known as the Maxwell-Bloch equations [7]:

$$\partial_t E = -\kappa E + P\,, \quad \partial_t P = -(\gamma_\perp + i\Delta) P - gNE\,,$$

$$\partial_t N = -\gamma_\parallel (N - N_0) + (E\bar{P} + \bar{E}P)\,, \tag{24}$$

where E,P and N represent, respectively, the electric field, the polarization and the population inversion, κ and $\gamma_\perp$, $\gamma_\parallel$ are the damping constants, Δ is the detuning, and N_0 is the pump parameter. If we remark now that dissipative effects are weak in lasers, we have a quasi-reversible system whose linear part at threshold is the matrix (21) since energy is injected here through the population inversion which is a zero frequency mode. We should expect then Eqs. (24) to be equivalent to the normal form (23) and this is easily proved by the following change of variables:

$$P = \kappa E + \partial_t E\,, \quad E = \exp\left(-i\frac{\Delta\kappa}{\gamma+\kappa}t\right)\frac{A}{\sqrt{g}}\,,$$

$$N = \frac{z - D_o}{g} + |E|^2\,, \tag{25}$$

where

$$D_o \equiv \gamma\kappa + \frac{\Delta^2\kappa}{\gamma+\kappa} - \frac{(\Delta\kappa)^2}{\gamma+\kappa}\,,$$

which transform Eqs. (23) in the MB equations (24). This shows then that the complex Lorenz equations are together with the MB equations alternative ways of writing the asymptotic normal form of the quasi-reversible $1/1$ resonance instability with a zero frequency mode. We also remark that Eqs. (23) with real coefficients ($\delta = 0$, i.e., no detuning) are the normal form when the eigenvalues coincide at zero frequency in the presence of the $O(2)$ symmetry [2]. Physical examples of this last situation are the weakly dissipative baroclinic instability [11], and the self focusing in the quasi-reversible Ginzburg-Landau equation studied by Malomed et al. [12].

In conclusion, we have shown that the classical $1/1$ resonance in quasi-reversible systems is described by amplitude equations which are formally equivalent to the Maxwell-Bloch equations describing the interaction between two-level atoms and the electromagnetic field. An immediate consequence of this result is the possibility of constructing a mechanical or electrical analog of lasers. On the other hand, since the Maxwell-Bloch equations appear as an asymptotic normal form, it is possible to generalize easily these equations by adding next order terms or spatial effects. As a simple example, the term $i\alpha NA$ in the equation for the complex amplitude of the oscillation is known as the α effect and it is used in the modelling of semi-conductor lasers [13]. It describes an N-dependent detuning effect and in our language it appears as a higher order dissipative resonant term in the normal form. Another example is the additional term used in Ref. [14] to generalize the MB equations for a Raman laser.

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