Universal description of stochastic supercritical bifurcations: Theory, simulations, and experiments.

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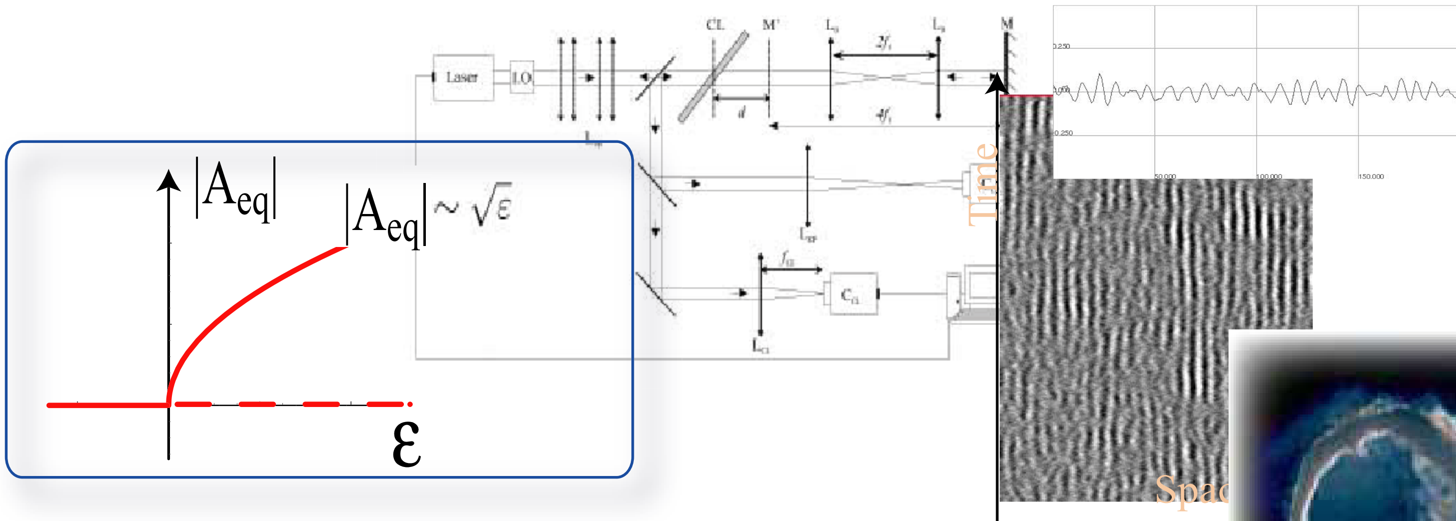
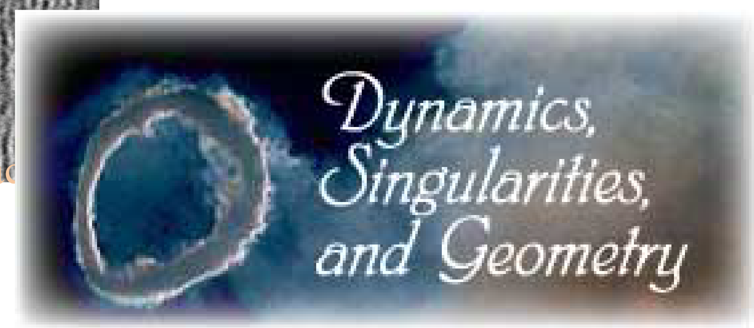




CIMAT

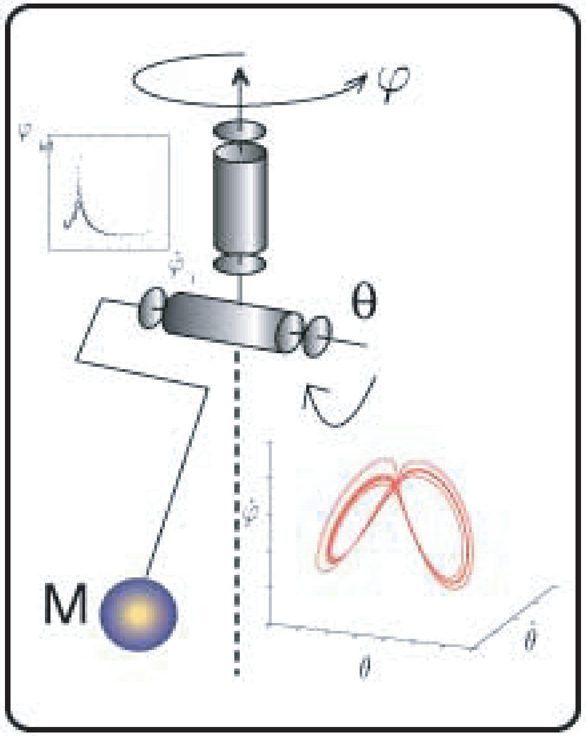
Outline

- Introduction of supercritical bifurcation and supercritical spatial bifurcation
- Universal Behavior close to bifurcation,
- Supercritical bifurcation in presence of noise,
- Experimental observations,
- Problem,
- Unified description: Stationary probability, expectation value and bifurcation criterium,
- Experimental study,
- Conclusions,
- Outlook.

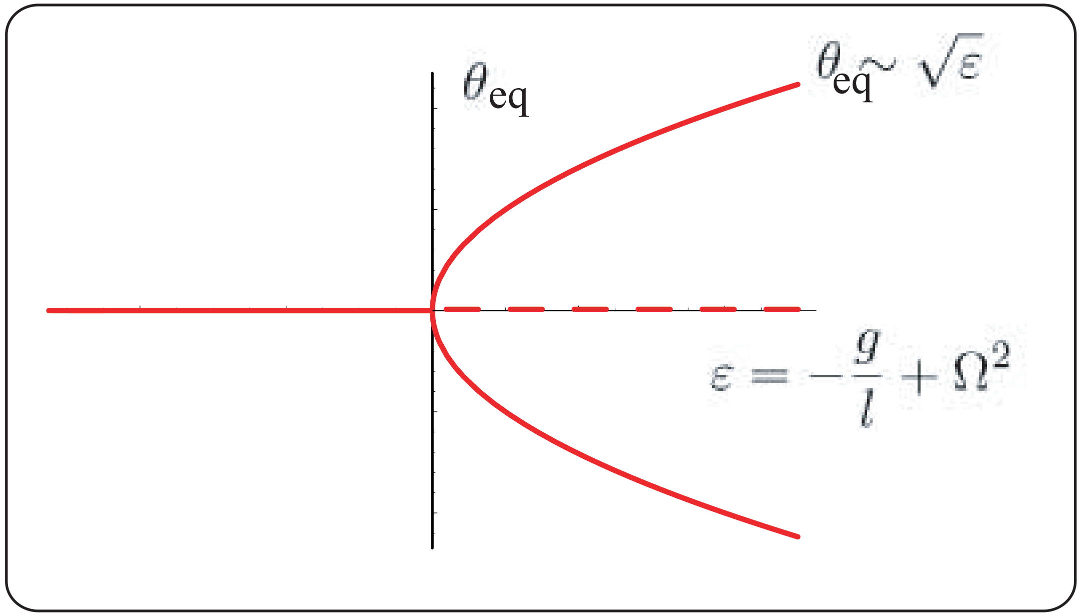
Introduction of supercritical bifurcation and supercritical spatial bifurcation

- Bifurcation: "Qualitative change of the dynamic behavior of the system under study, when one parameter is changed".

- Supercritical instability: "Appearance of fixed points from another one"



Lorenz pendulum,
Phys. Rev. Lett. 83, 3820 (1999).

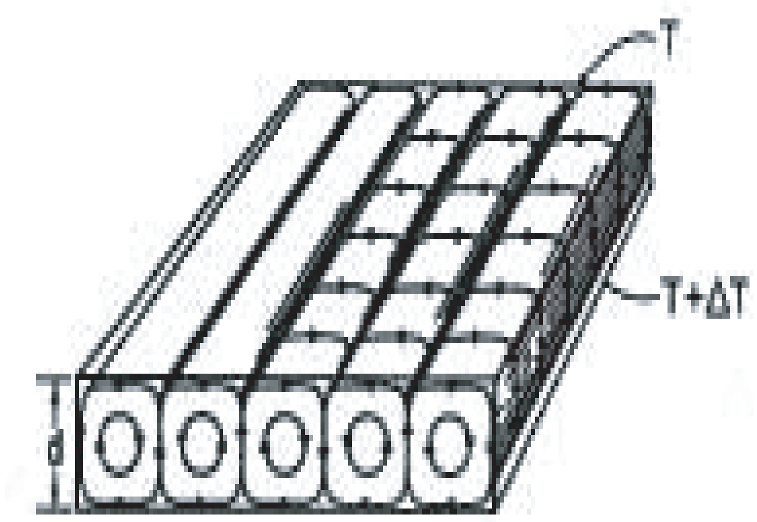


Universal behavior, the fixed point increase as function of bifurcation parameter $\theta_{eq} \sim \sqrt{\varepsilon}$

Supercritical pattern bifurcation

Patterns formation

- Rayleigh-Benard Convection



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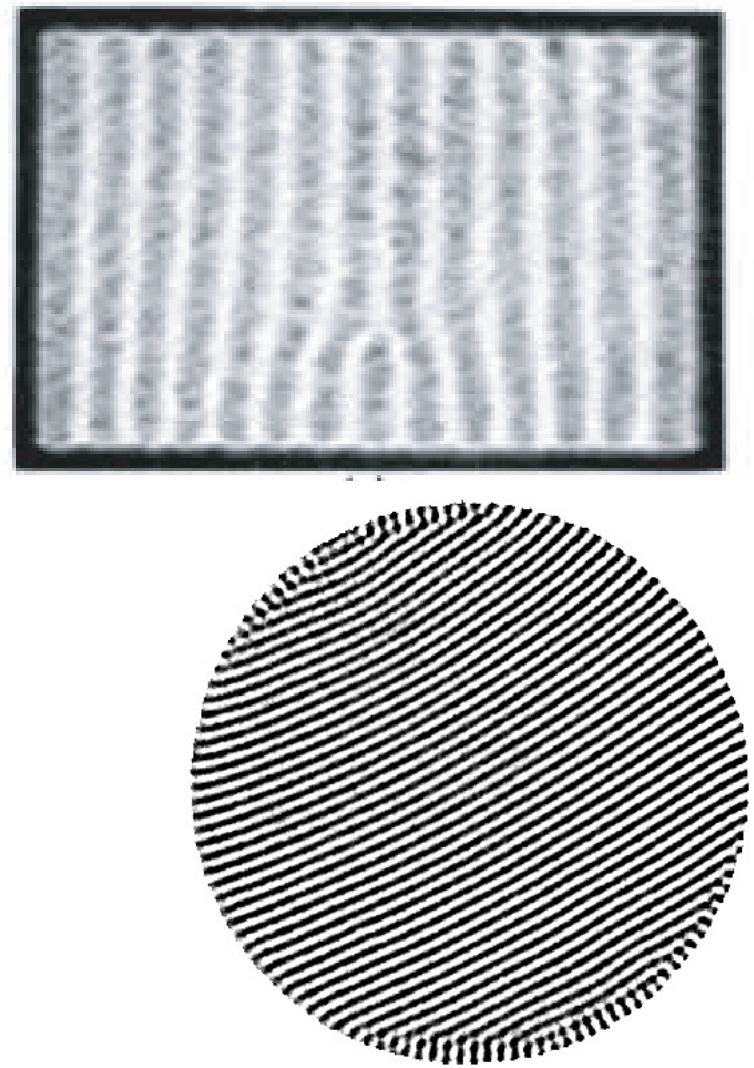
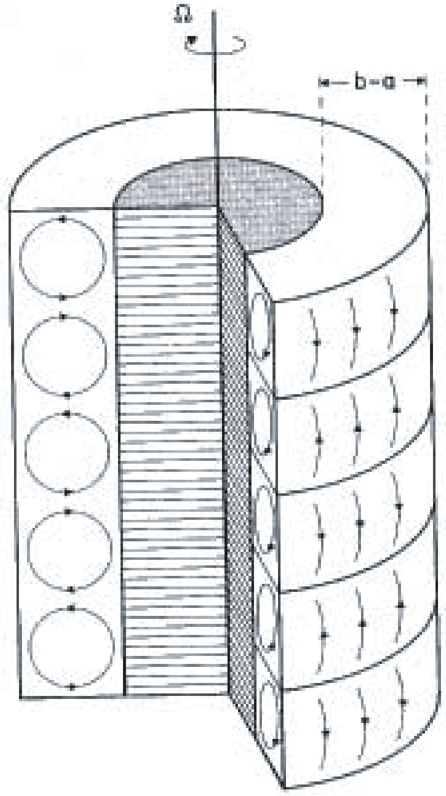


Experiments

Pomeau et. al., Rev. Mod. Phys. 49, 581(1997)

Supercritical pattern bifurcation

- Taylor-Coutte instability



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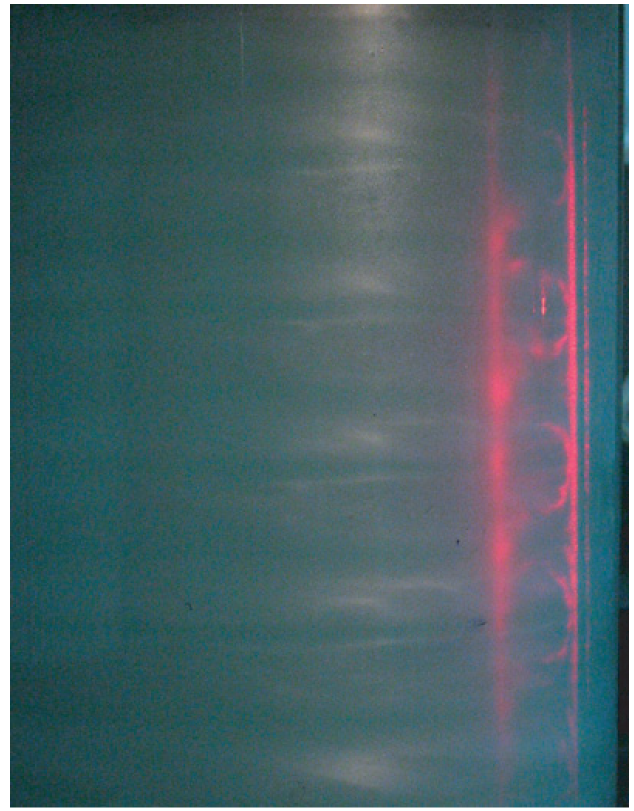
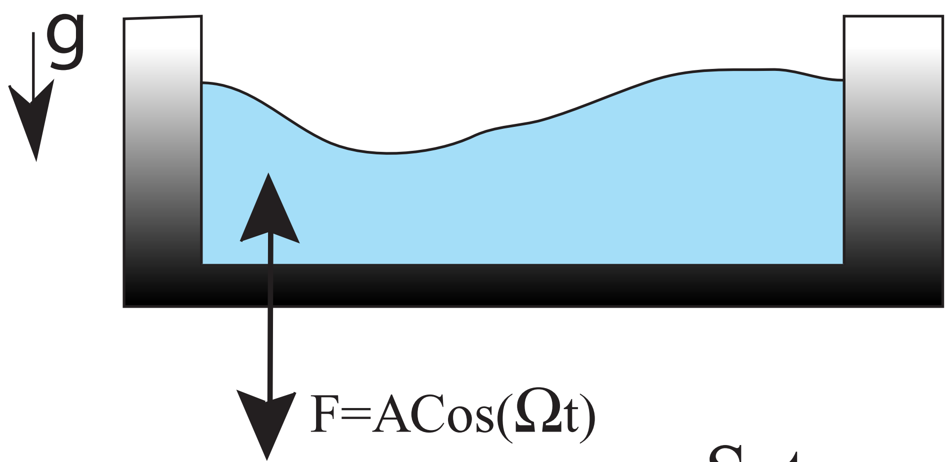


Experiments

Supercritical pattern bifurcation

- Faraday instability



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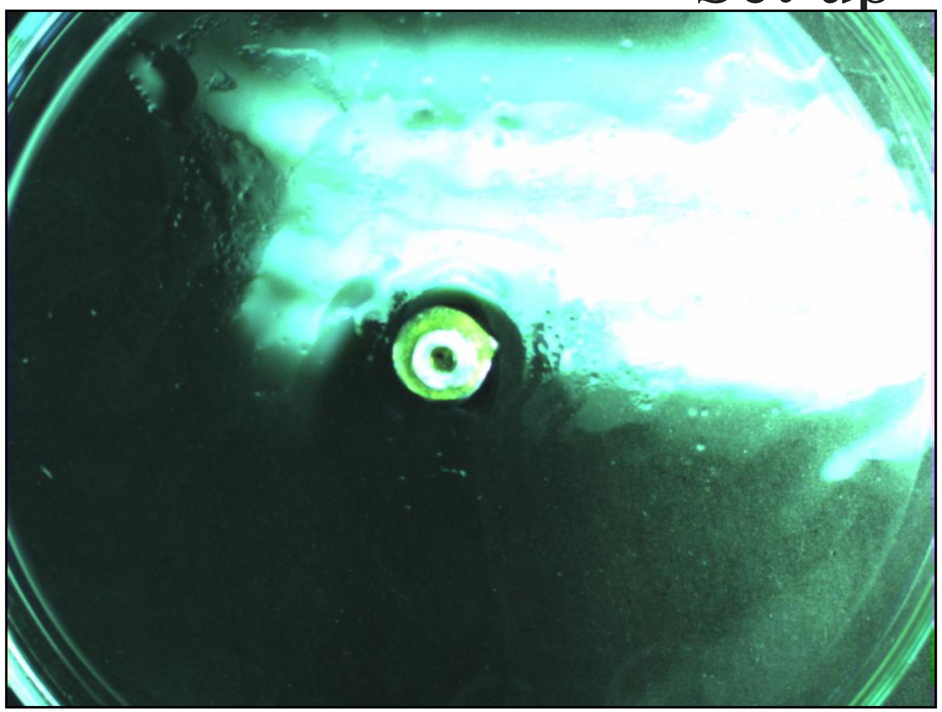
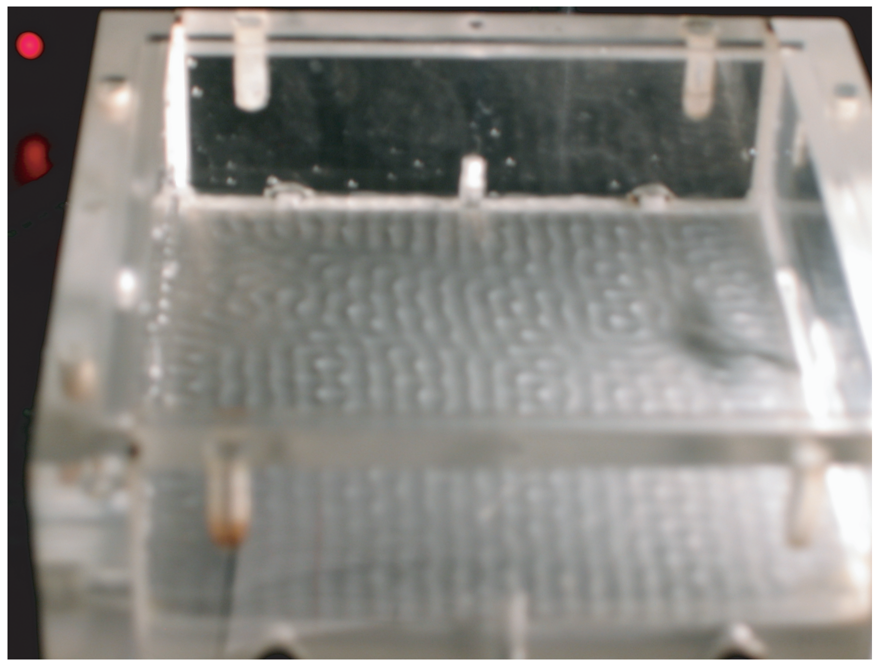
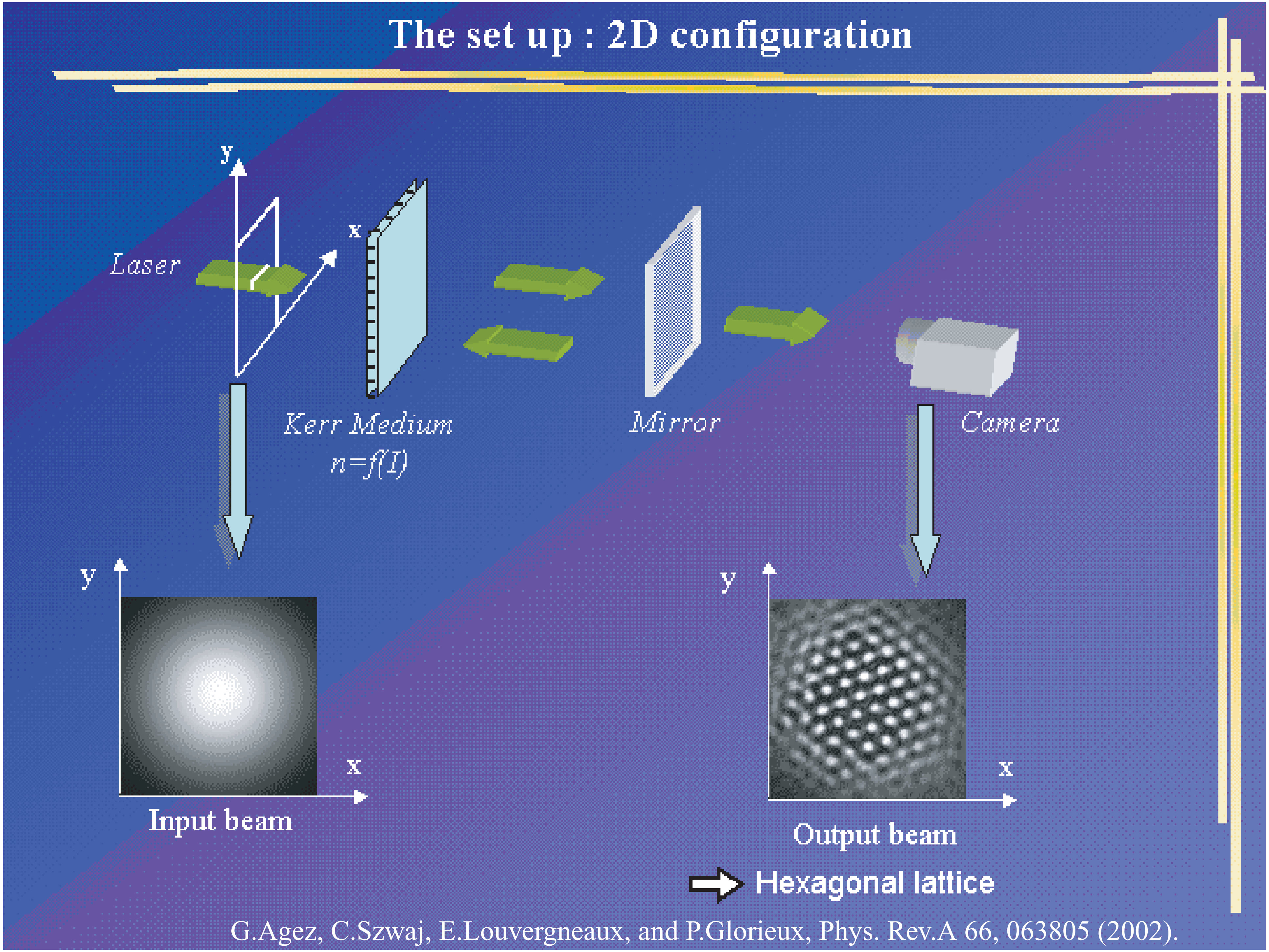




Experiment

Supercritical pattern bifurcation

Liquid crystal with optical feedback



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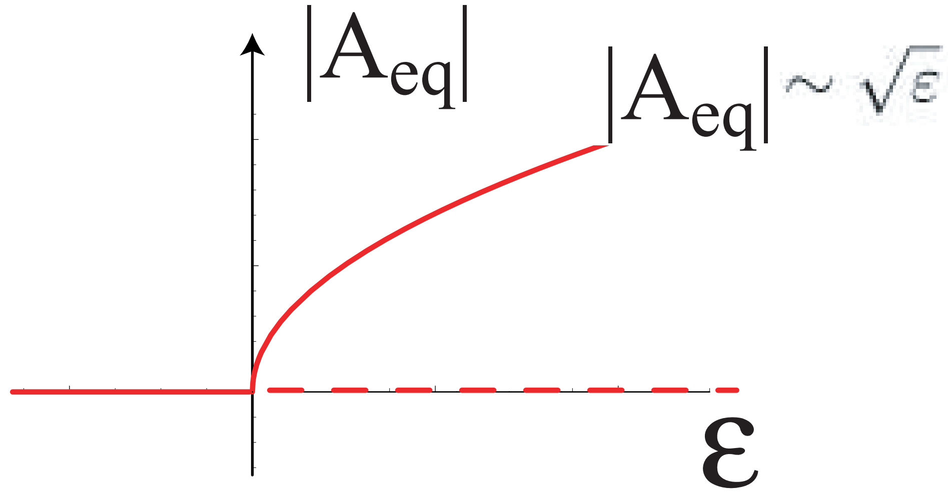
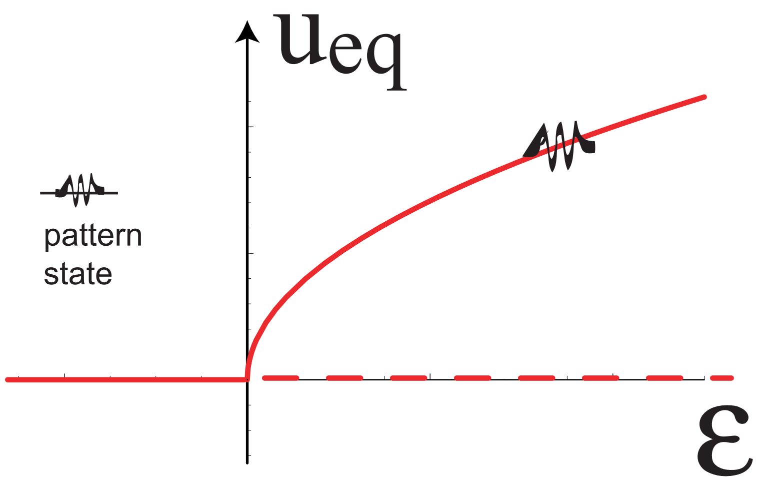
Universal Behavior close to bifurcation

- Whole the previous systems exhibit pattern formation from a uniform state and the amplitude of the pattern close to bifurcation increases with the square root of the bifurcation parameter.



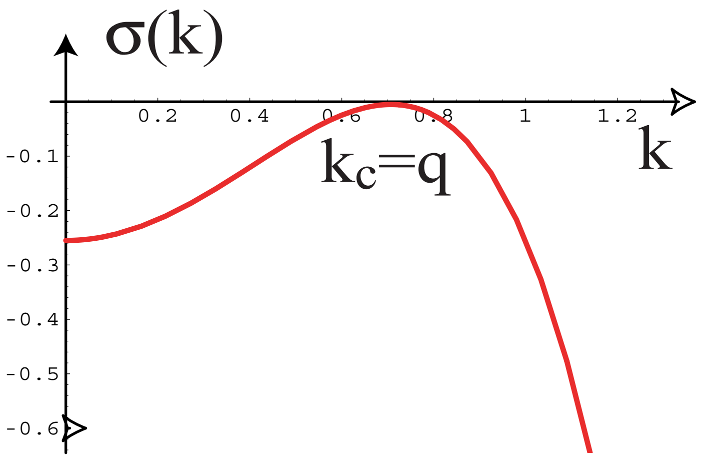


- Prototype model of patterm formation, Swift-Hohemberg equation

$$\partial_t u = \varepsilon u - u^3 - (\partial_{xx} + q^2)^2 u$$

Universal Behavior close to bifurcation

- The spectrum as function of wave number ($\vec{u}(x,t) = \vec{u}_o + e^{\sigma(k)t+ikx}\hat{u}$).



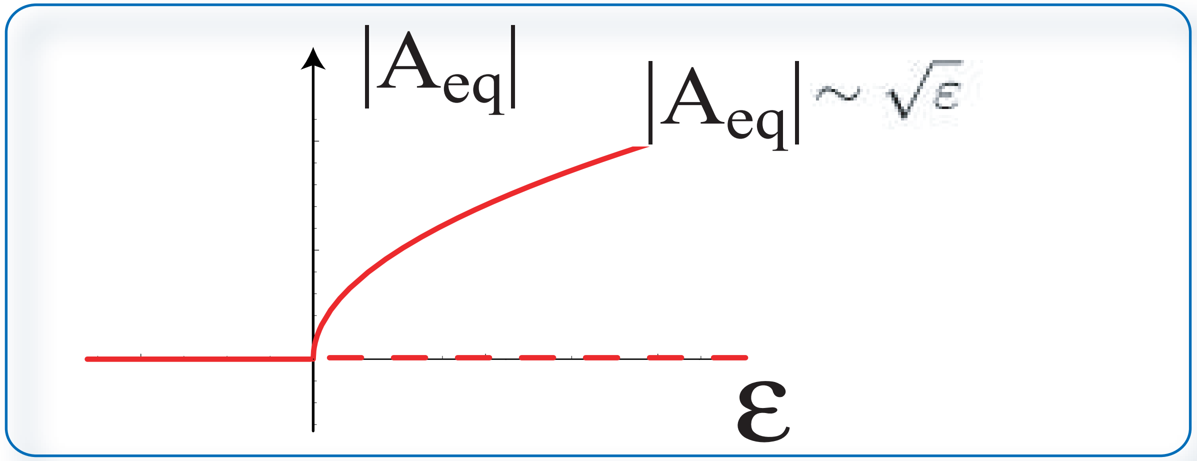
- In order to understand the universal behavior exhibits by the previous systems, we consider the ansatz

$$\vec{u}(x,t) = \vec{u}_o + A\left(T=\varepsilon t, X=\sqrt{\varepsilon}x\right)e^{ik_cx}\hat{u}_k + \bar{A}(T,X)\,e^{-ik_cx}\hat{u}_k + h.o.t.,$$

Imposing the solvability condition one deduces (real GL equation)

$$\boxed{\partial_T A = \varepsilon A - |A|^2 A + \partial_{XX} A}$$

Bifurcation Diagram



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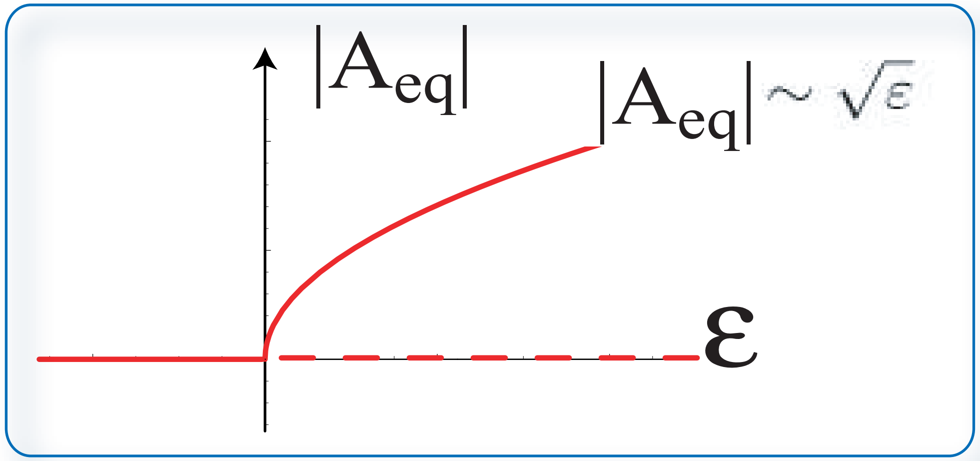
Supercritical bifurcation in presence of noise

The description of macroscopic matter is usually done using a small number of coarse-grained or macroscopic field. This reduction is possible due to a separation of time scales, which allows a description in terms of the slowly varying macroscopic variables, which are in fact fluctuating variables due to the elimination of a large number of fast variables whose effect can be modelized including suitable stochastic terms (noise) in the PDE.



- Stochastic protoype model

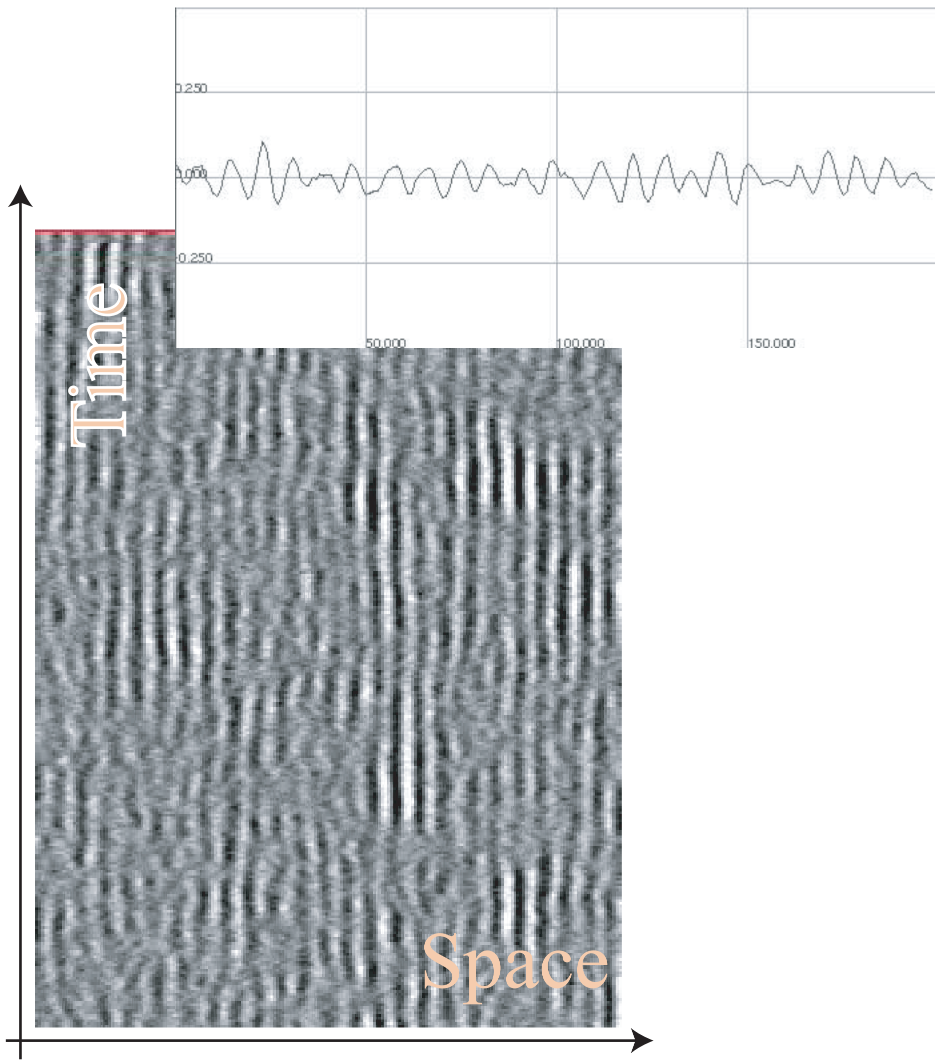
$$\partial_t u = \varepsilon u - u^3 - (\partial_{xx} + q^2)^2 u + \sqrt{\eta}\zeta(x,t)$$

where ζ(x,t) is a guassian white noise with

$$\langle \zeta(x,t) \rangle = 0$$
$$\langle \zeta(x,t), \zeta(x',t') \rangle = \delta(x - x')\,\delta(t - t')$$

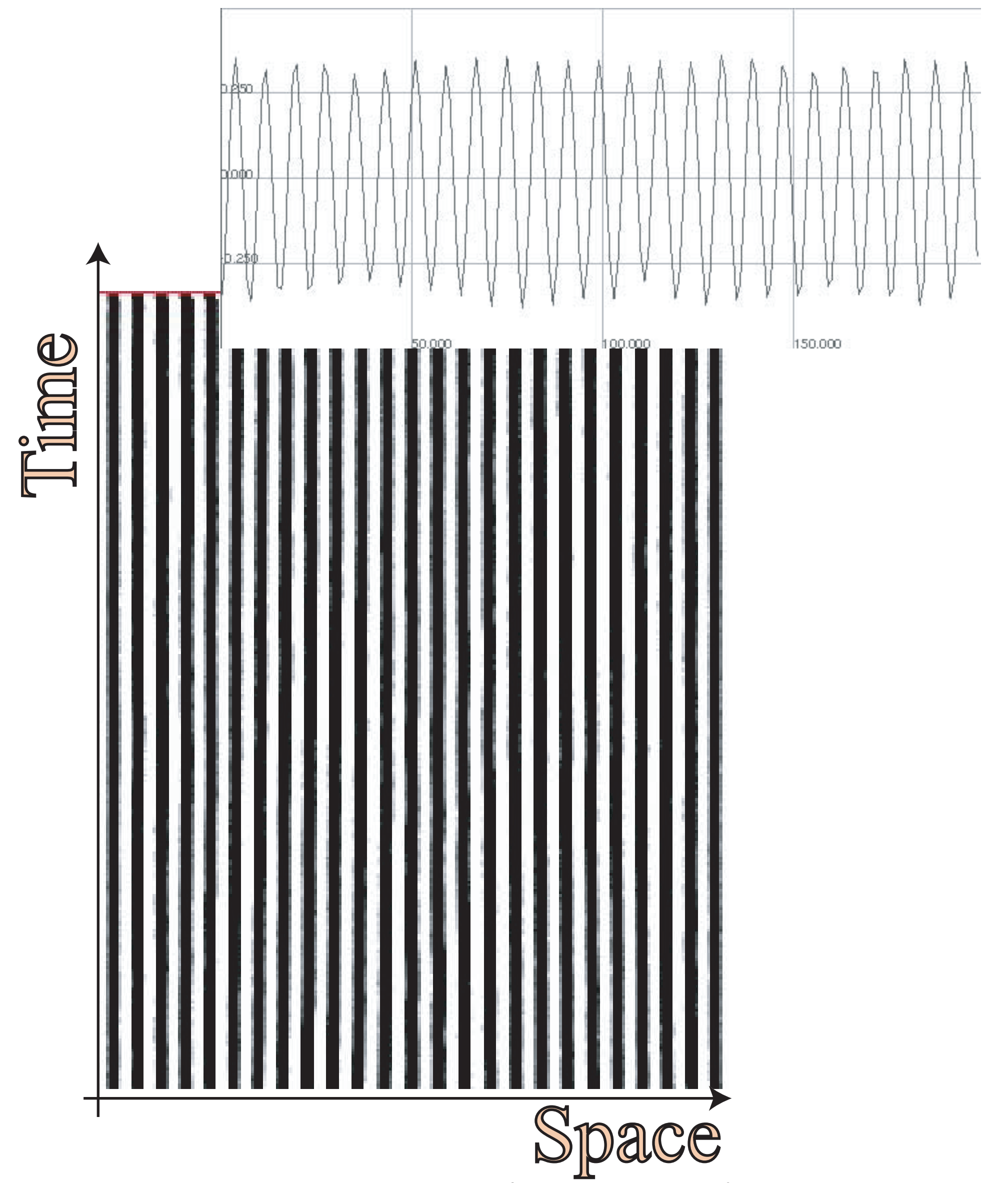
Supercritical bifurcation in presence of noise

- Below bifurcation



The system exhibits the precursor

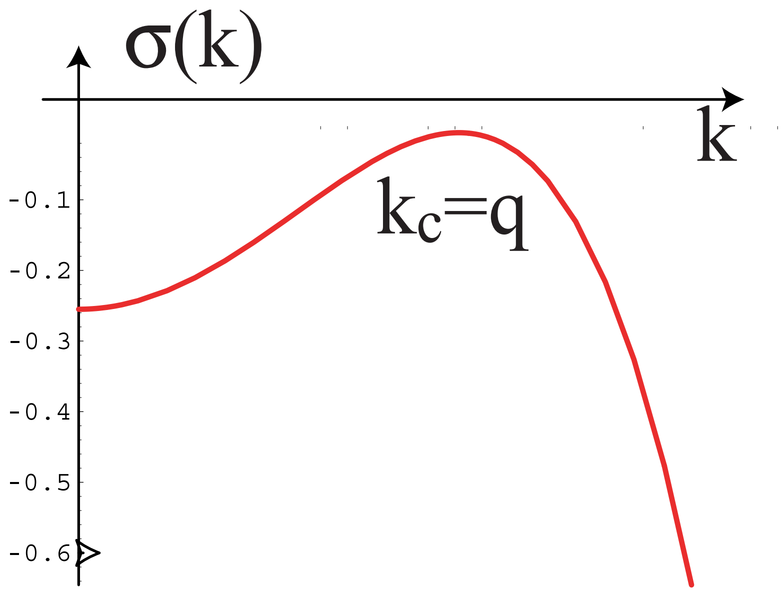
- Above bifurcation



Pattern Fluctuations

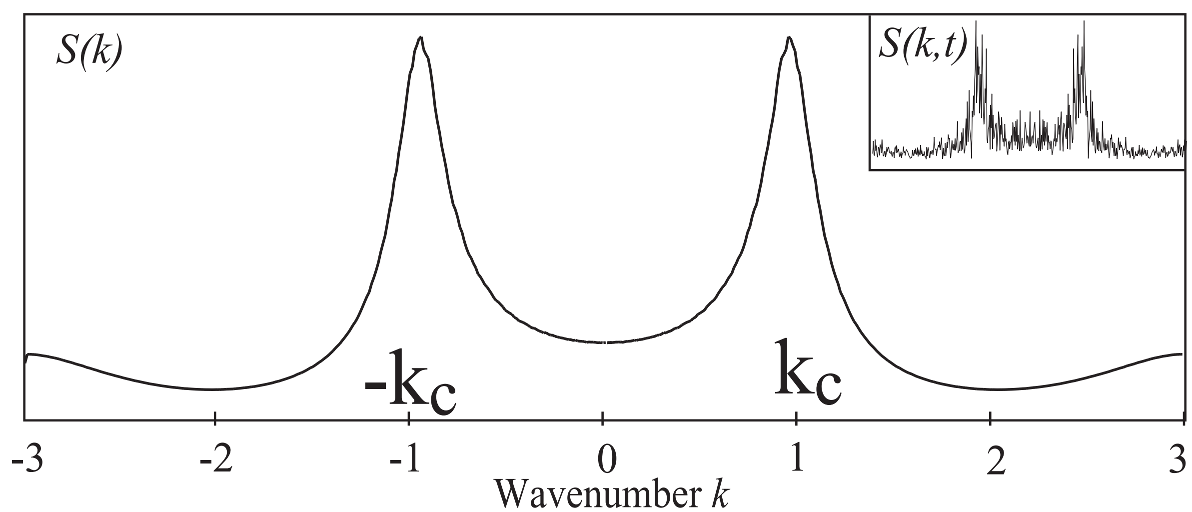
Supercritical bifurcation in presence of noise

- The origin of precursor is noise excites all spatial modes, but the slowest mode leads the dynamics



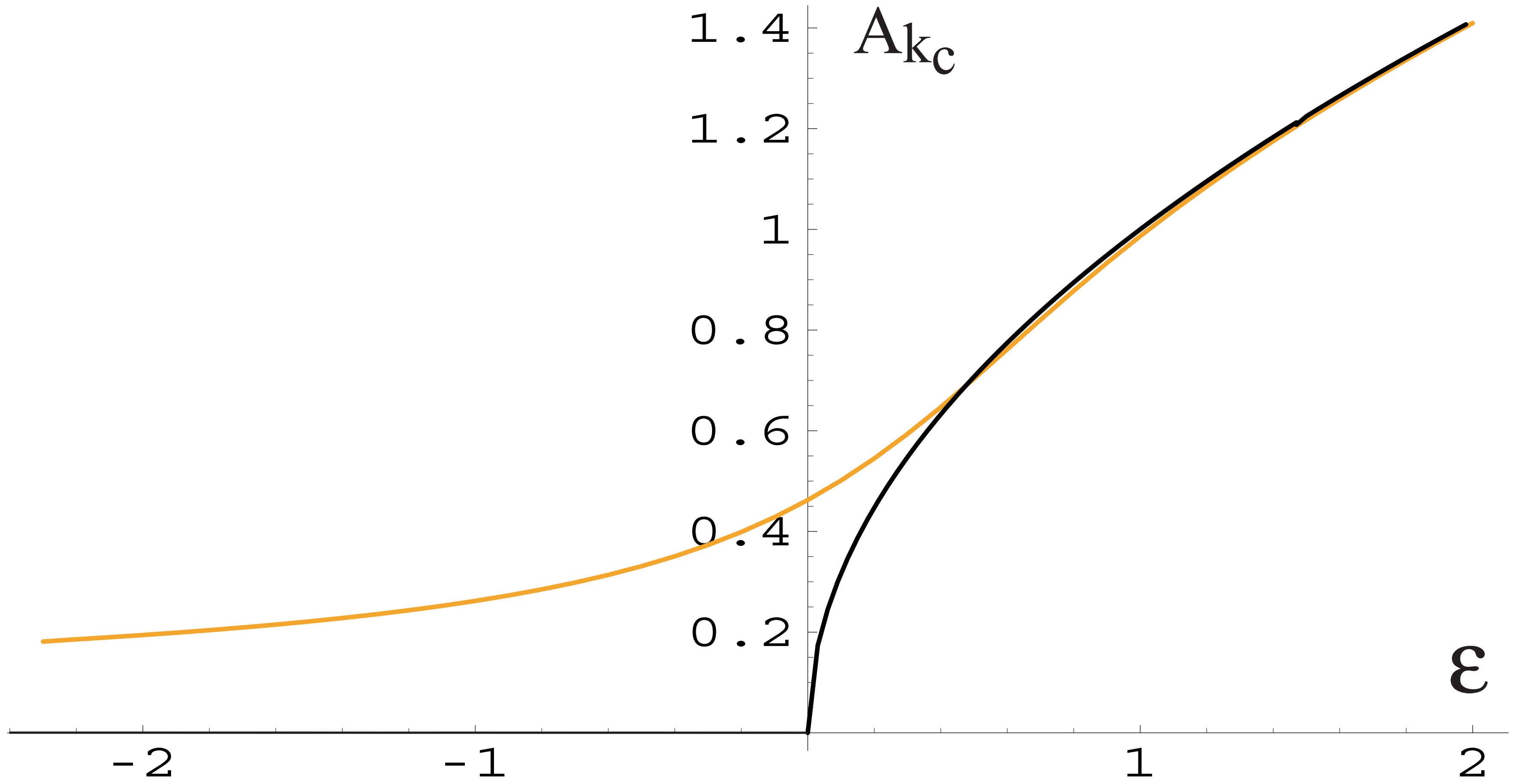
- Dynamic structure factor

$$S(k,t) = \int dx e^{-ikx} |\vec{u}(x,t)|^2$$



Supercritical bifurcation in presence of noise

- From the recognition of critical wave number, one can deduce the bifurcation diagram



Bifurcation diagram

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Experimental observations

- Thermally induced hydrodynamic fluctuations below the onset of electroconvection

Linear theory

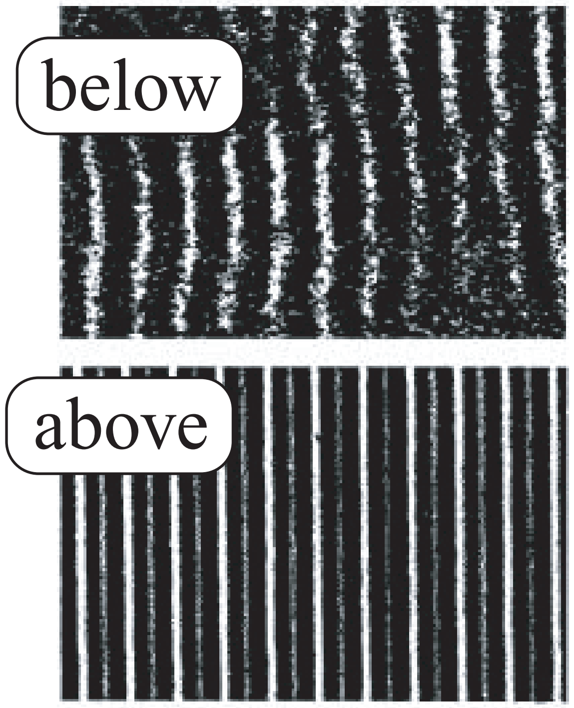


FIG. 2. Spatial structure of the fluctuations below V_c (top), and of fully developed convection (bottom), in a rectangle of $13.4d \times 9.3d$. The contrast in the top part is enhanced by a factor of 5.

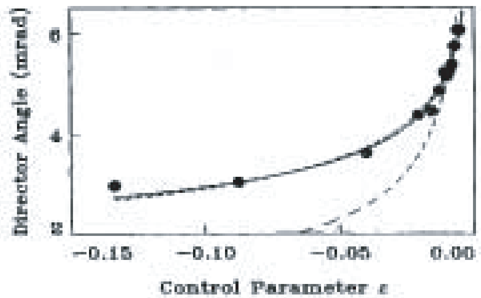
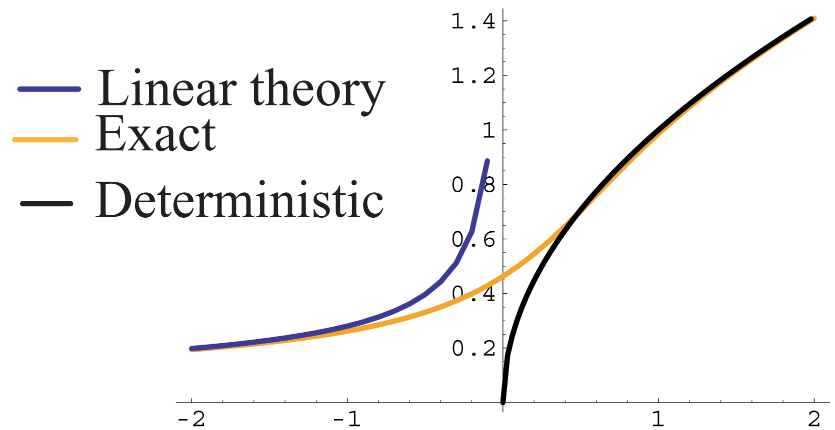


FIG. 6. $\bar{A}$ measured at subcritical values of ε. The lines are fits according to the single-mode (long-dashed), one-dimensional (solid), and two-dimensional (short-dashed) theories.



G. Ahlers et. al. Physical Review Letters 67, 596 (1991)

Experimental observations

- Noise, coherent fluctuations, and the onset of order in an oscillated granular

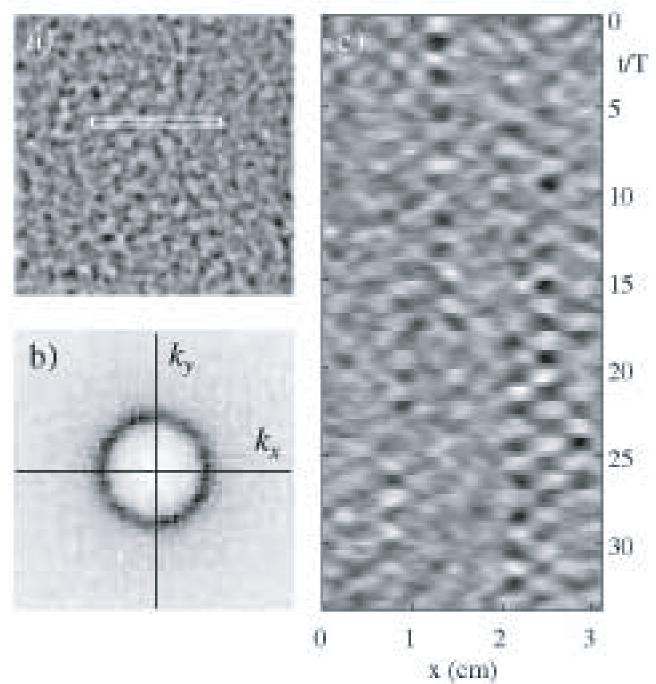


FIG. 1. (a) Snapshot of an area 6.25×6.25 cm^2 in a container oscillating with $\Gamma = 2.6$. (b) The spatial power spectrum of (a) has an intense ring corresponding to randomly oriented spatial structures with a length scale of 0.52 cm (100 spectra were averaged to obtain the spectrum shown). (c) Space-time diagram for the row of pixels in the box in (a); the period of the localized transient oscillations is $2T = 2/f_d$.

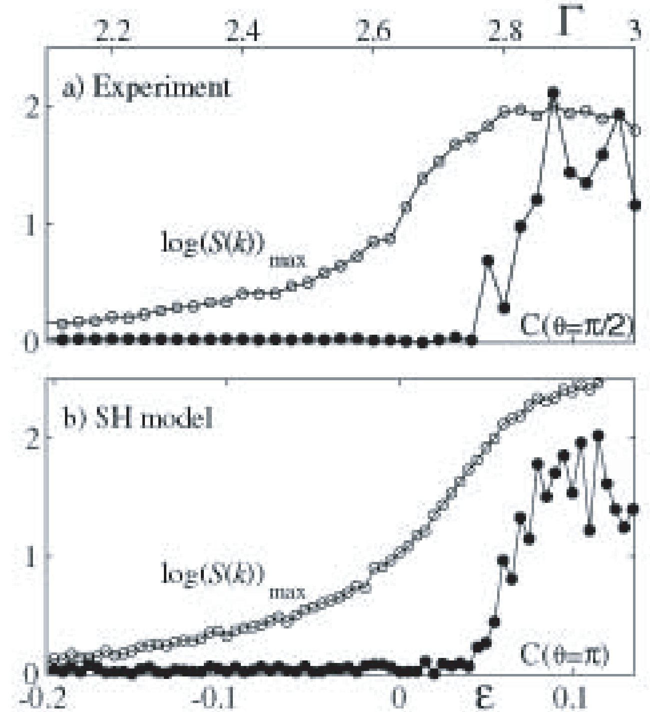


FIG. 3. The growth of the noise power and the onset of long-range order in (a) experiment and (b) the Swift-Hohenberg model. The log of the maximum of $S(k)$ (○) increases through the mean field onset ($\epsilon_c^{MF} = 0$), while the onset of long-range order, indicated by the appearance of angular correlations of the radially averaged structure factor [$C(\theta = \pi/2)$ for the experiment and $C(\theta = \pi)$ for SH equation (●)], is delayed to $\epsilon_c^{LR} \approx 0.04$. The integration of Eq. (1) uses a scheme described in [21]; the solution is obtained on a 128×128 grid with $k_0 = 1$ and integration time step 0.5.

H. Swinney, Physical Review Letters 92, 174302 (2004)

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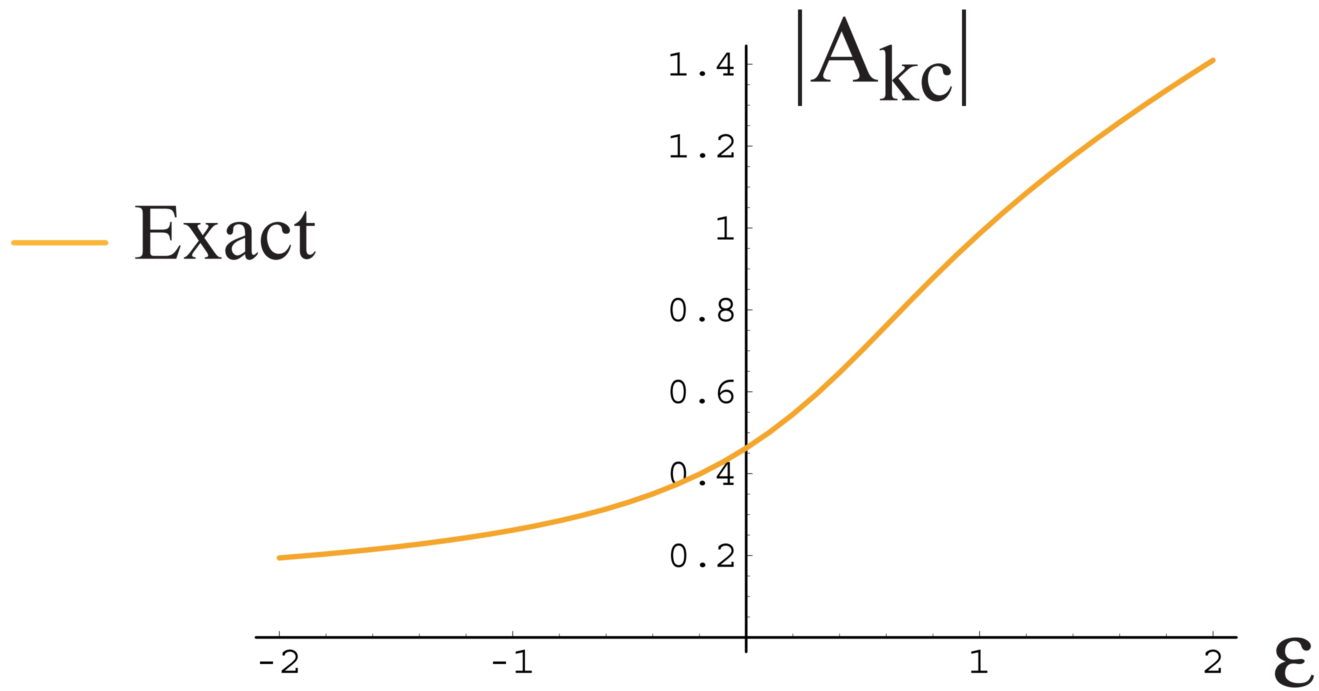
Problem

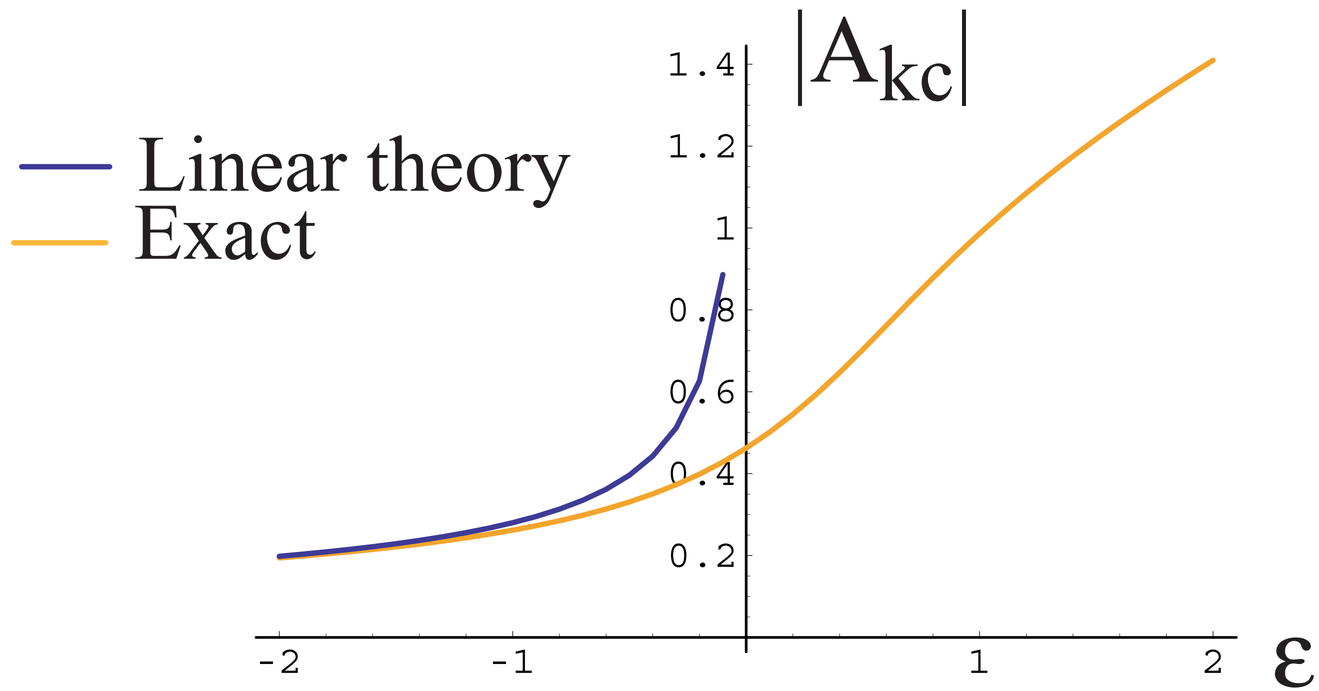
- From different theoretical and experimental studies, the supercritical bifurcation in presence of noise



These data are obtained from given theoretical model or experimental system

Problem

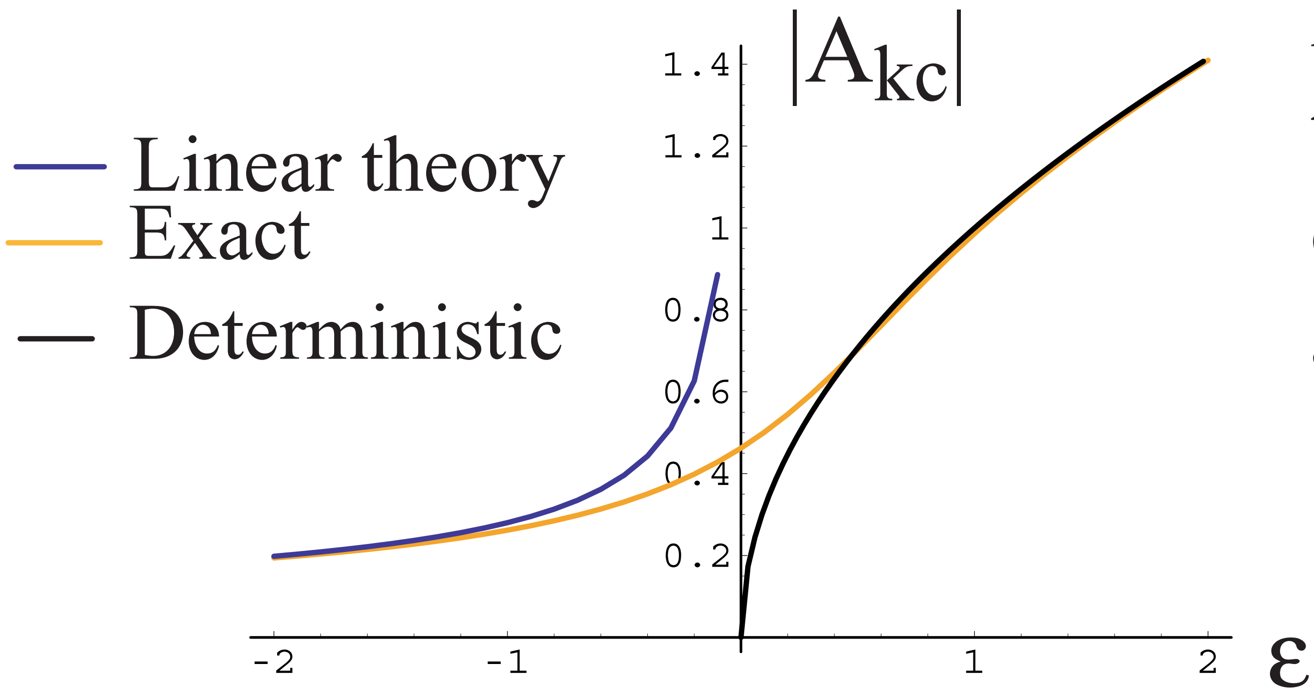
- From different theoretical and experimental studies, the supercritical bifurcation in presence of noise



Below the bifurcation, the linear approach is a good approximation

Problem

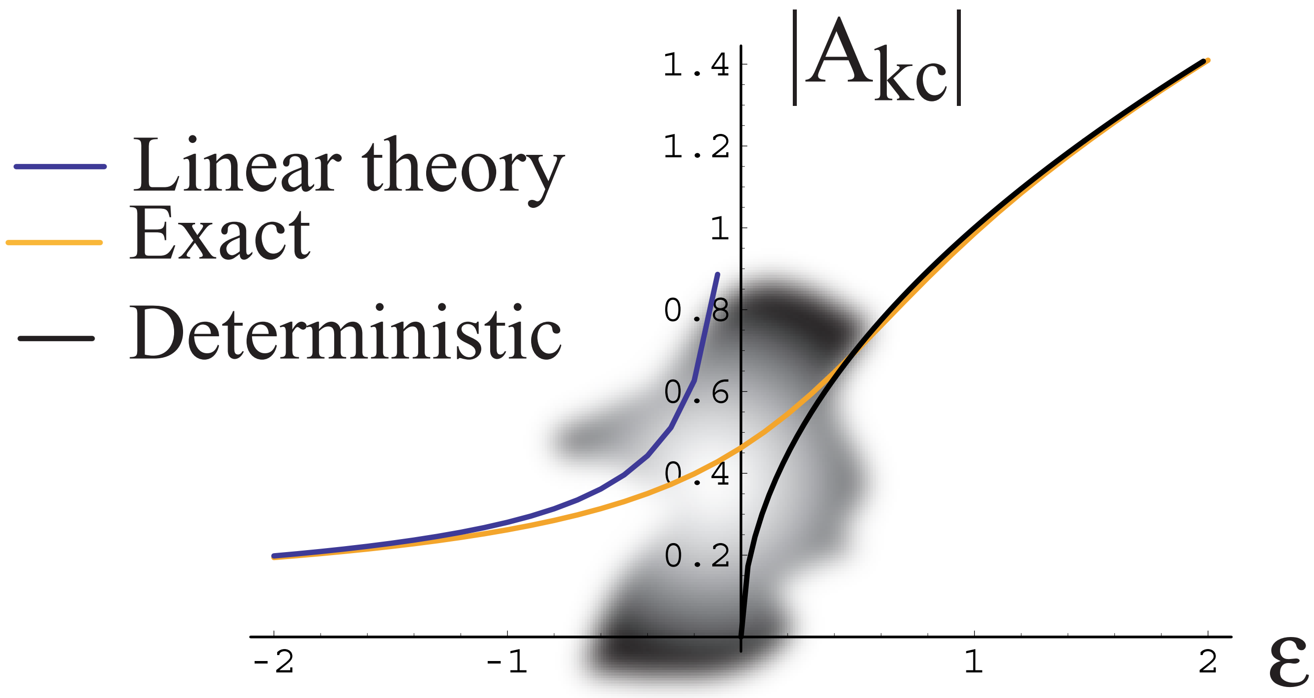
- From different theoretical and experimental studies, the supercritical bifurcation in presence of noise



Above the bifurcation, the deterministic description is a good approximation!

Problem

- From different theoretical and experimental studies, the supercritical bifurcation in presence of noise



There is not a adequate universal description of the supercritical spatial bifurcation close to the instability!

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Unified description

- Let us consider a one-dimensional extended system that exhibits a supercritical spatial bifurcation described by

$$\partial_t \vec{u} = \vec{f}(\vec{u}, \partial_x, \{\lambda\}) + \sqrt{\eta}\vec{\zeta}(x,t),$$

close to the spatial instability, we can introduce the ansatz

$$\begin{aligned} \vec{u}(x,t) &= \vec{u}_o + A\left(T = \varepsilon t, X = \sqrt{\varepsilon} x\right) e^{ik_c x} \hat{u}_k \\ & \quad + \bar{A}(T,X)\, e^{-ik_c x} \hat{u}_k + h.o.t., \end{aligned}$$

the envelope satisfies a Langevin equation

$$\boxed{\partial_T A = \varepsilon A - |A|^2 A + \partial_{XX} A + \sqrt{\eta}\xi(X,T),}$$

where ξ(x,t) is a complex gaussian white noise

$$\begin{aligned} \langle \xi(X,T)\,\xi(X',T') \rangle &= 0, \\ \langle \xi(X,T)\,\bar{\xi}(X',T') \rangle &= \delta(T'-T)\,\delta(X'-X). \end{aligned}$$

Unified description

- Introducing the global amplitude $$a(T)=\frac{1}{L}\int_{-L/2}^{L/2}A(T,X)dX,$$

  which satisfies (Langevin equation)

$$\partial_T a=\varepsilon a-|a|^2 a+\sqrt{\eta}\varsigma(T),$$

where $\varsigma(T)\equiv\int_{-L/2}^{L/2}\xi(T,X)dx/L$ is a complex gaussian white noise with correlation

$$\langle\varsigma(T)\bar{\varsigma}(T')\rangle=\delta(T'-T)$$

From the above langevine equation, one has the corresponding Fokker-Plank equation

$$\begin{aligned}\partial_T P &= \partial_a\left\{-\varepsilon a+|a|^2 a+\frac{\eta}{2}\partial_{\bar{a}}\right\}P\\ &+\partial_{\bar{a}}\left\{-\varepsilon\bar{a}+|a|^2\bar{a}+\frac{\eta}{2}\partial_a\right\}P.\end{aligned}$$

This equation has the stationary probability

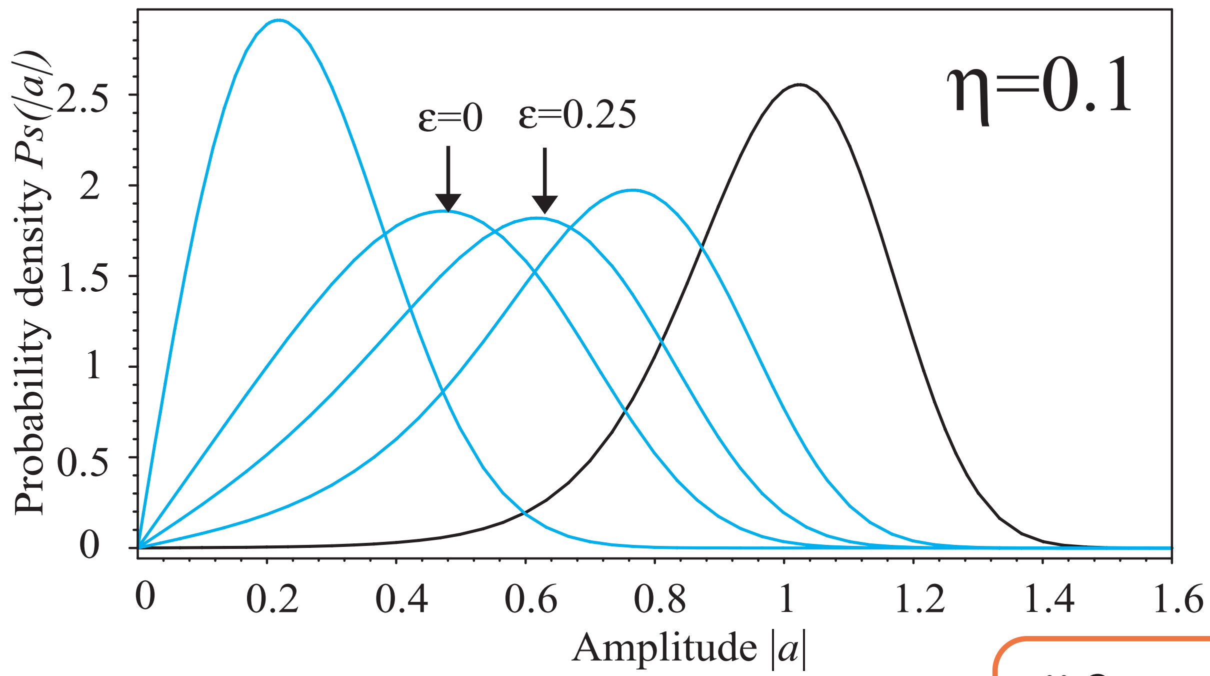
$$\boxed{P_s(a,\bar{a})=Q(\varepsilon,\eta)\,e^{\frac{1}{\eta}(\varepsilon|a|^2-\frac{|a|^4}{2})}}$$

where

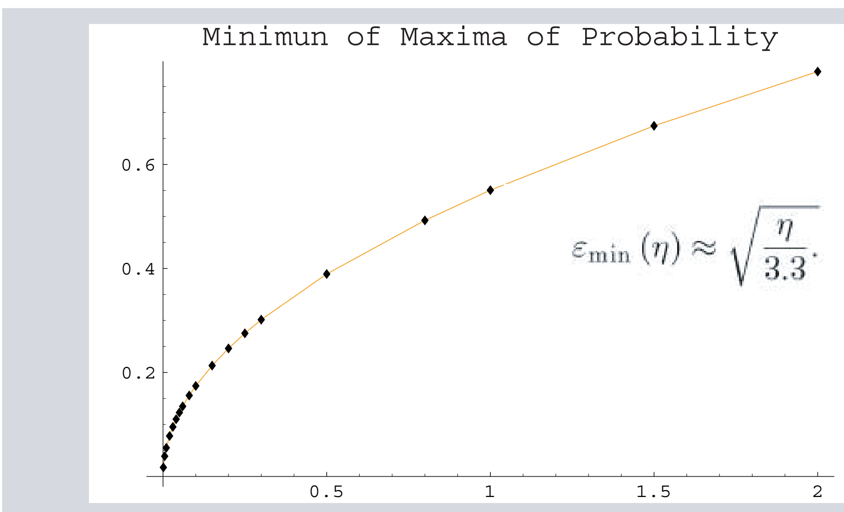
$$Q(\varepsilon,\eta)=\frac{2\sqrt{2}}{\sqrt{\pi\eta}}\frac{e^{-\frac{\varepsilon^2}{2\eta}}}{\mathrm{erf}\,c(-\frac{\varepsilon}{\sqrt{2\eta}})}$$

Unified description

Stationary probability density



$$P_s(a,\bar{a}) = Q(\varepsilon,\eta)\, e^{\frac{1}{\eta}\left(\varepsilon|a|^2 - \frac{|a|^4}{2}\right)}$$



$$\varepsilon_{\min}(\eta) \approx \sqrt{\frac{\eta}{3.3}}.$$

Bifurcation criterium

"for a given noise intensity(η) if one determines the value of the control parameter (λ_{min})for which expectation value has a minimum, **the bifurcation point** is localized at

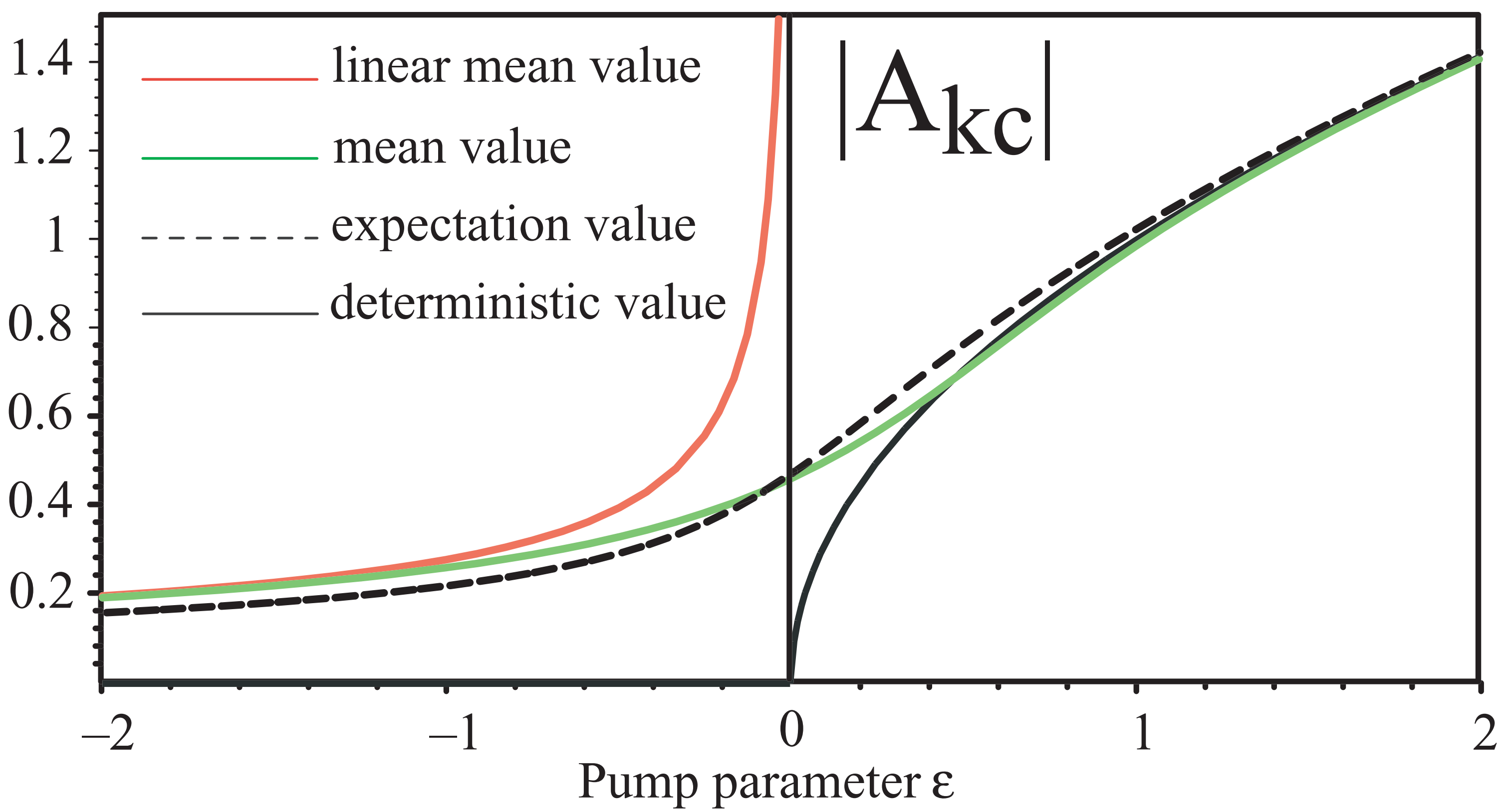
$$\lambda_c = \lambda_{min} - \sqrt{\frac{\eta}{3.3}}.$$

"

$$P_s(a,\bar{a}) = Q(\varepsilon,\eta)\, e^{\frac{1}{\eta}\left(\varepsilon|a|^2 - \frac{|a|^4}{2}\right)}$$

Unified description

- From the stationary probability density, we can compute the average of the critical amplitude ($|A_{kc}|$), the expectation value (a_{max}) and neglecting the higher order term we deduce the liner behavior of the average of the critical amplitude.



- There is not a simple expression for $|A_{kc}|$

- The expectation value

$$|a_{\max}| = \sqrt{\frac{-\varepsilon + \sqrt{\varepsilon^2 + 2\eta}}{2}}.$$

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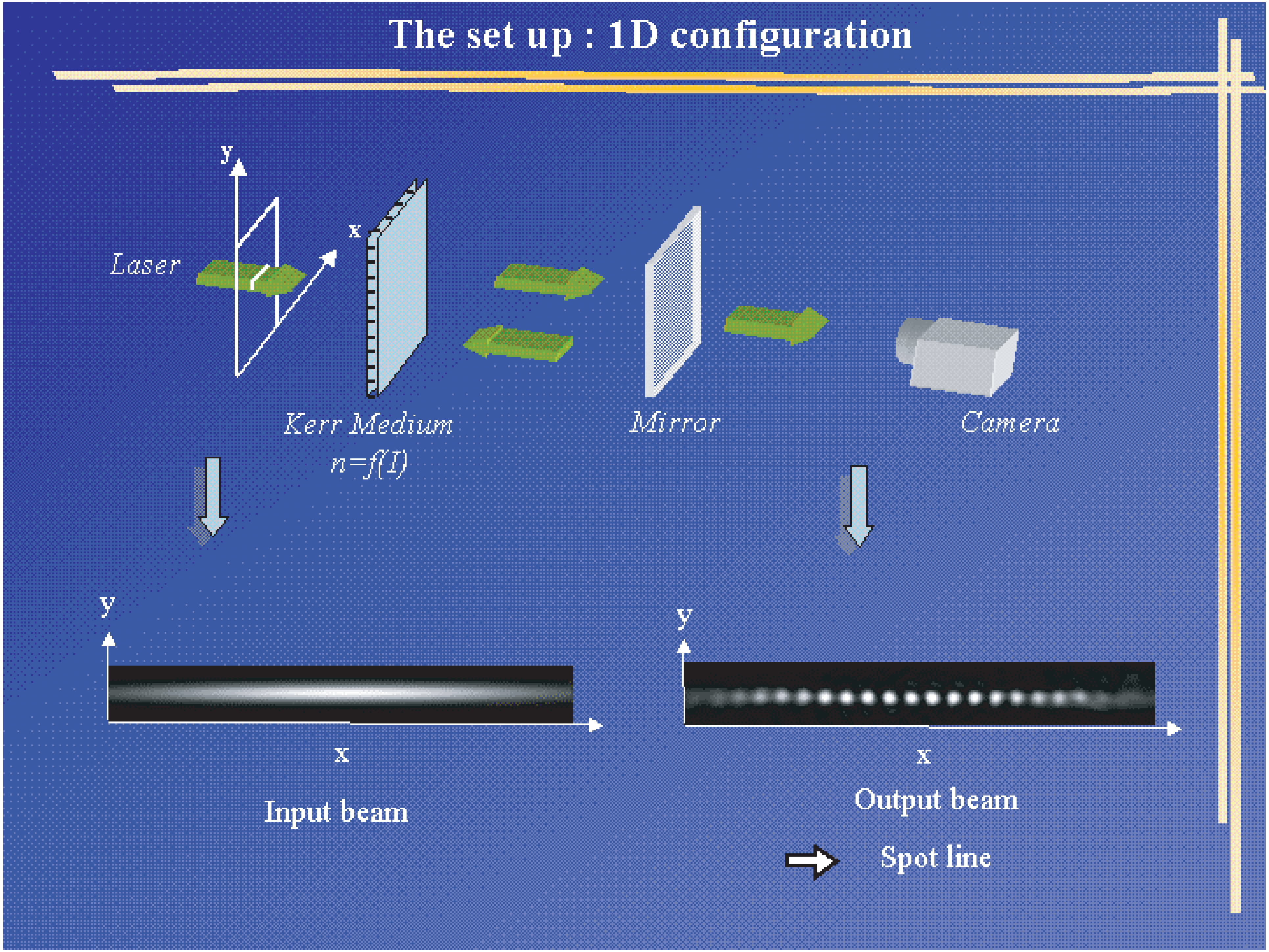
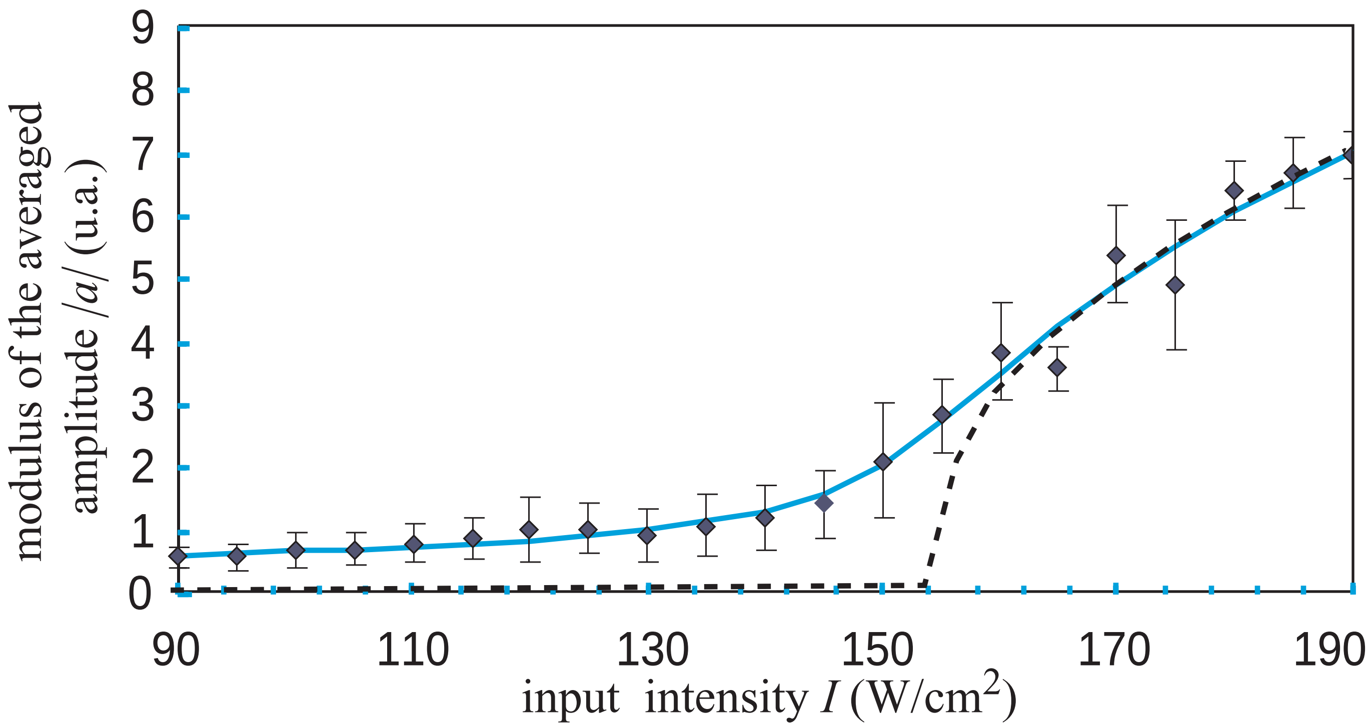
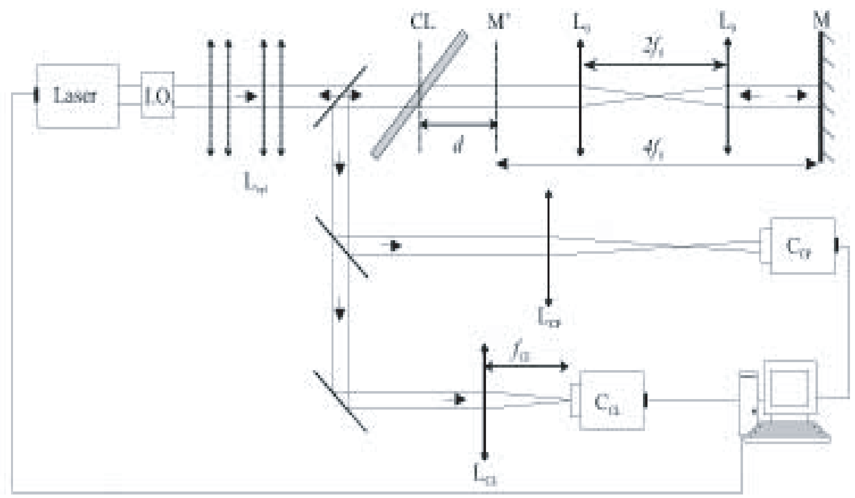
Experimental study



Experimental study

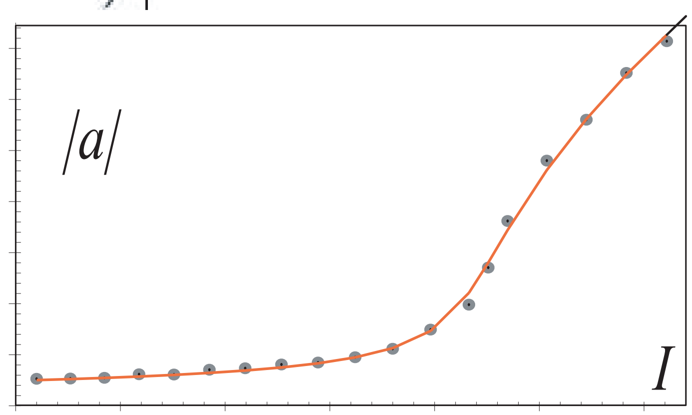


- Experiment



- Theory

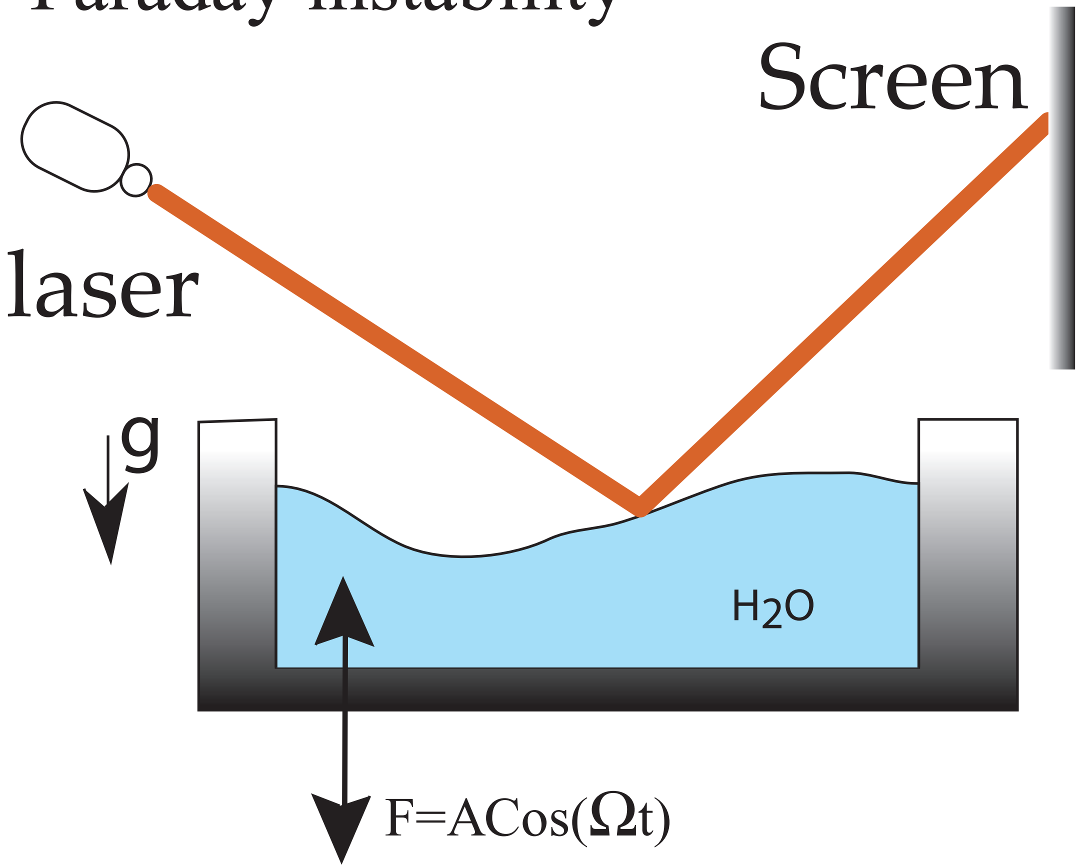
$$\frac{\partial n(\mathbf{r},t)}{\partial t} - \nabla_\perp^2 n(\mathbf{r},t) + n(\mathbf{r},t) = |F_0|^2 + \left|\sqrt{R}e^{i\sigma\nabla_\perp^2}\left(e^{i\chi n(\mathbf{r},t)}F_0 g(\mathbf{r})\right)\right|^2 + \sqrt{\varepsilon}\xi(\mathbf{r},\mathbf{t})$$

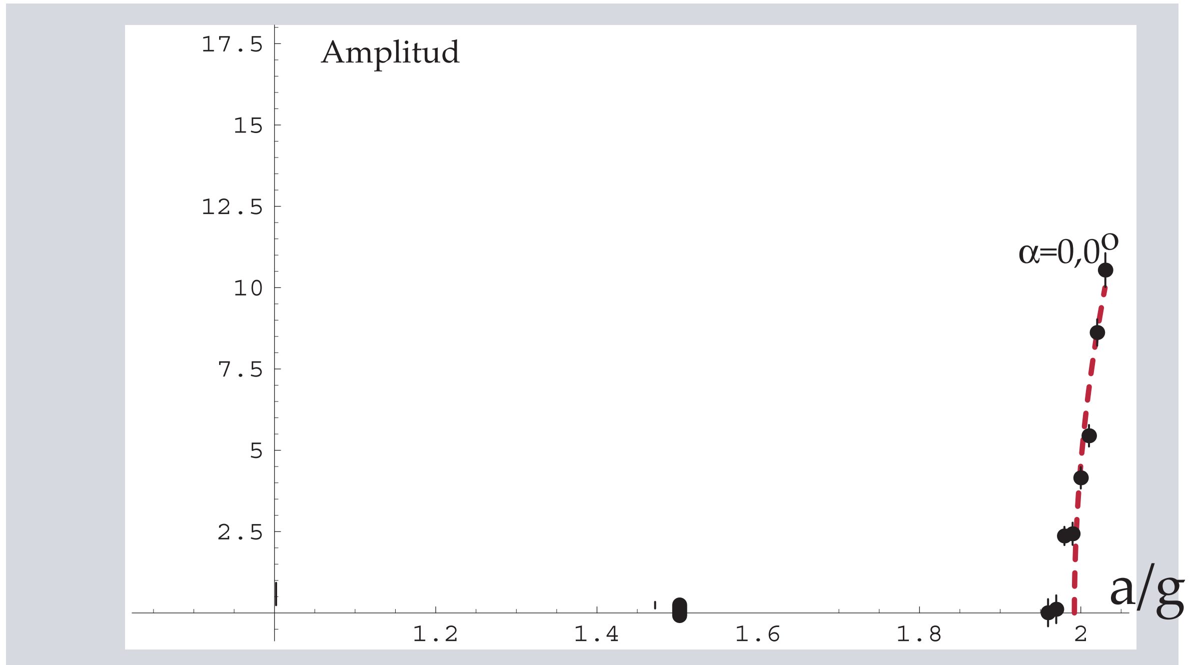


W.J. Firth, J. Mod. Opt. 37, 151 (1990).
G. D'Alessandro and W.J. Firth, Phys. Rev. Lett. 66, 2597 (1991)

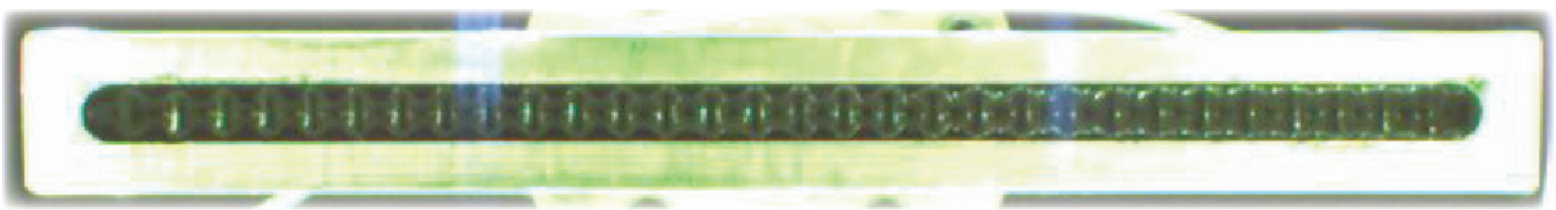
Parametrically driven Newtonian fluid in 1D

- Faraday instability





- The system has multiplicative noise



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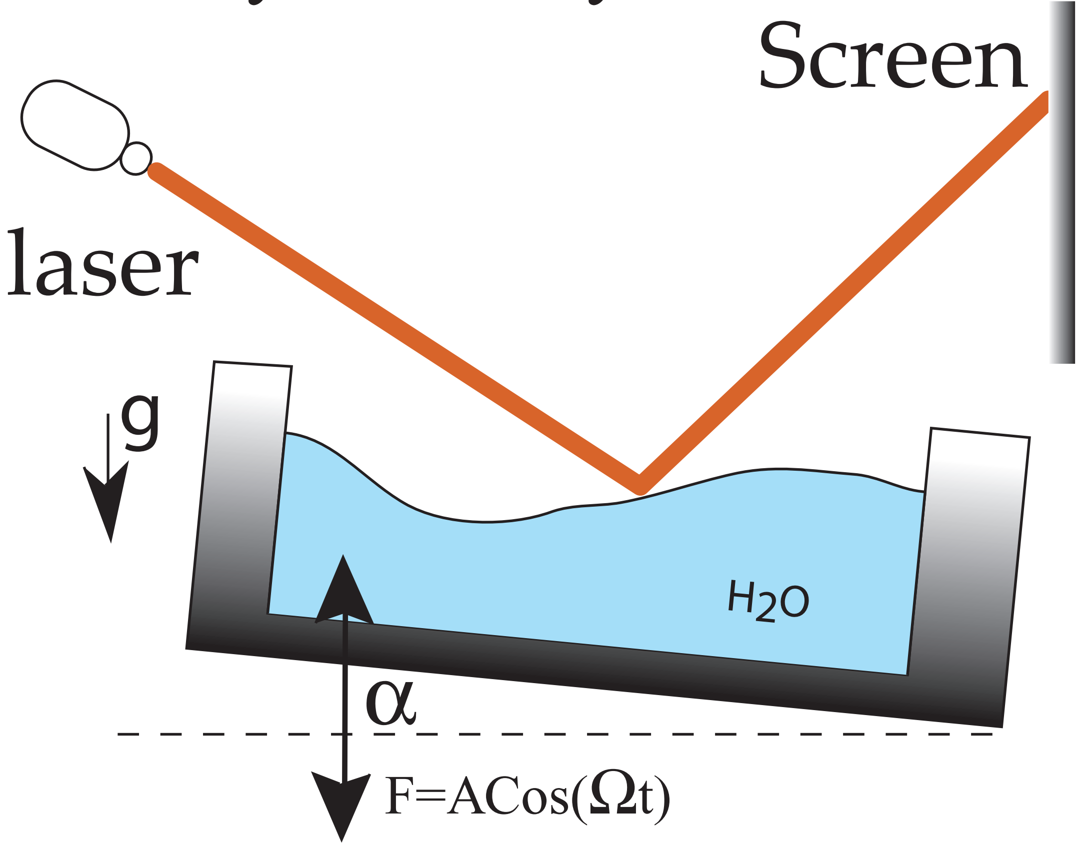
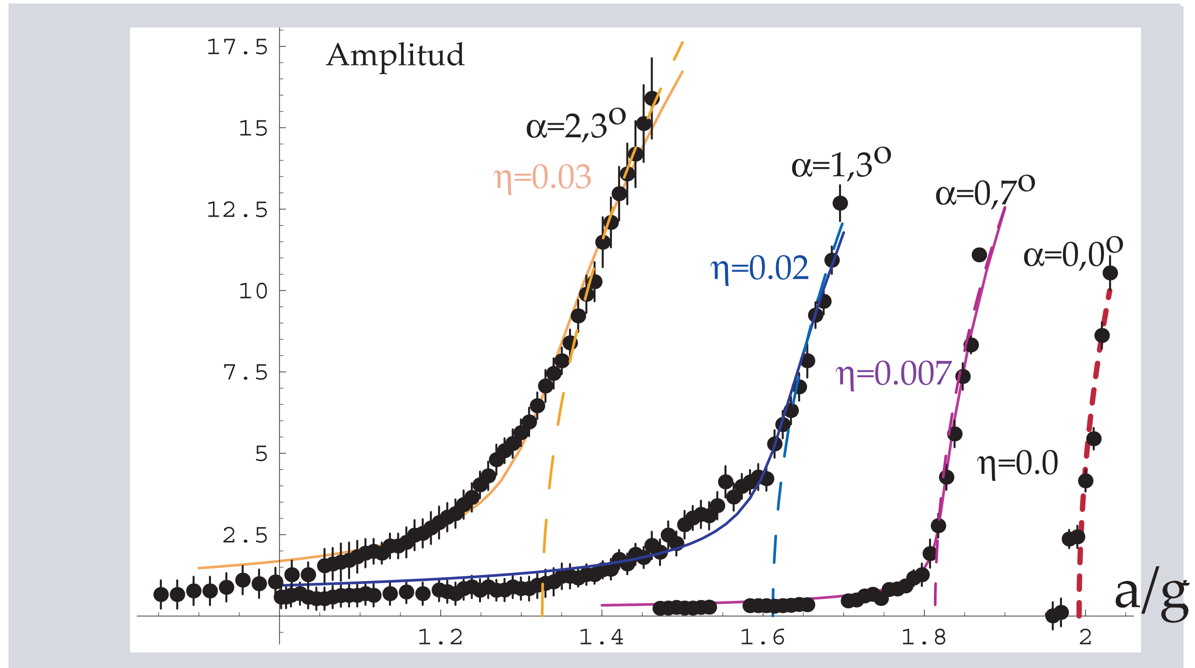
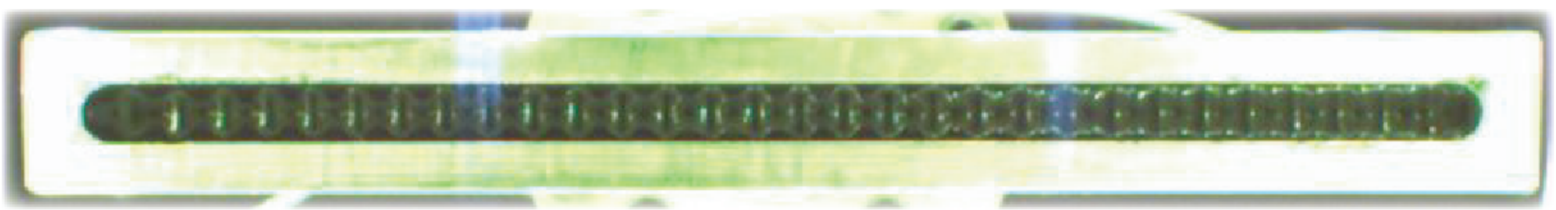
Parametrically driven Newtonian fluid in 1D

- Faraday instability with small tilt





- The tilt control the effective additive noise



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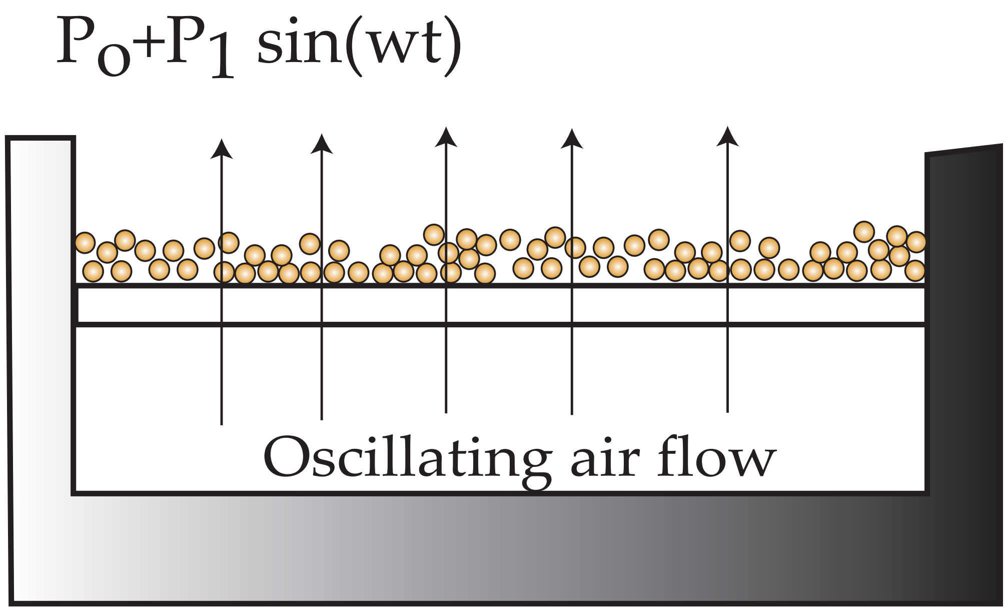
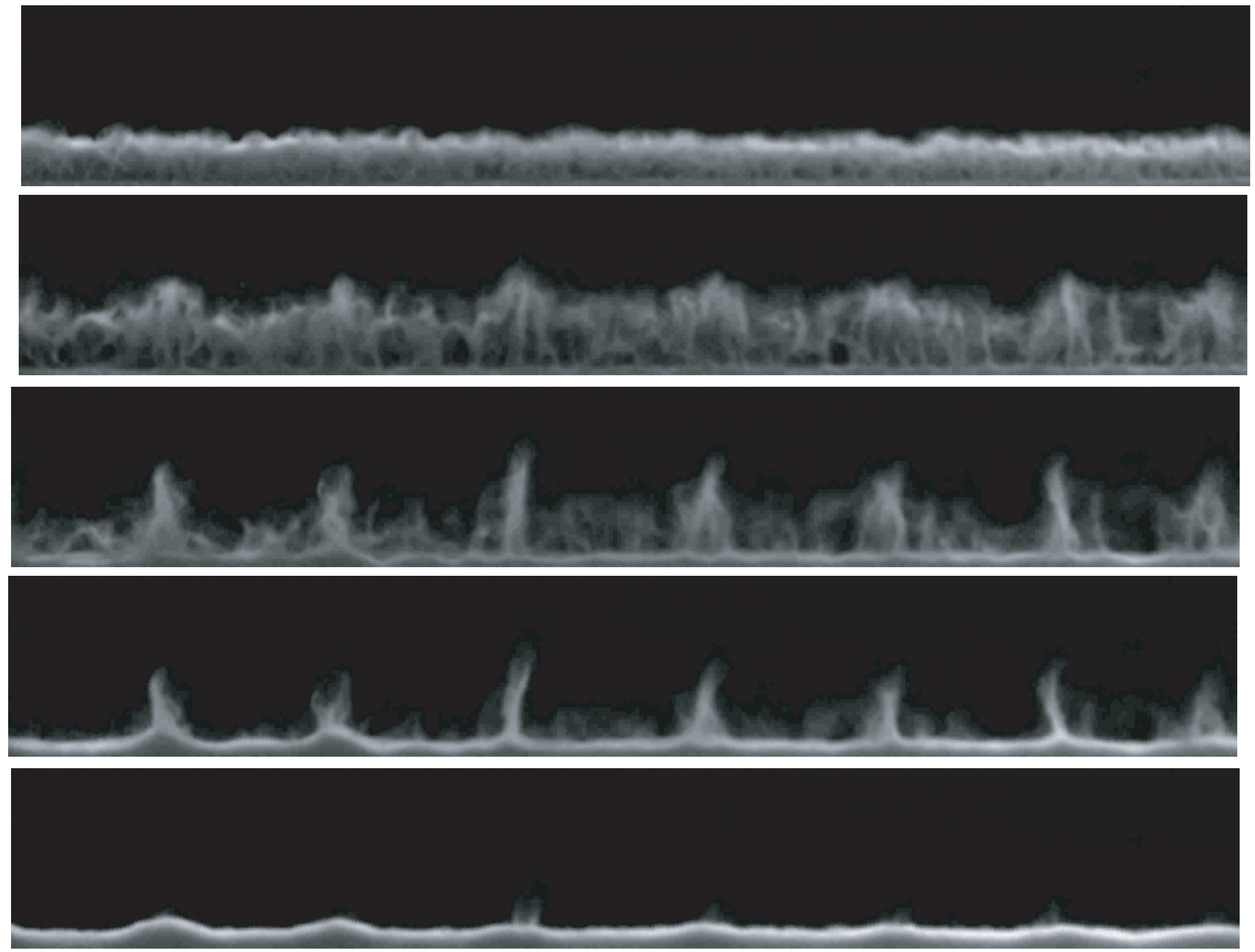
fluidized granular system





Carlos Orellana, Nicolas Mujica

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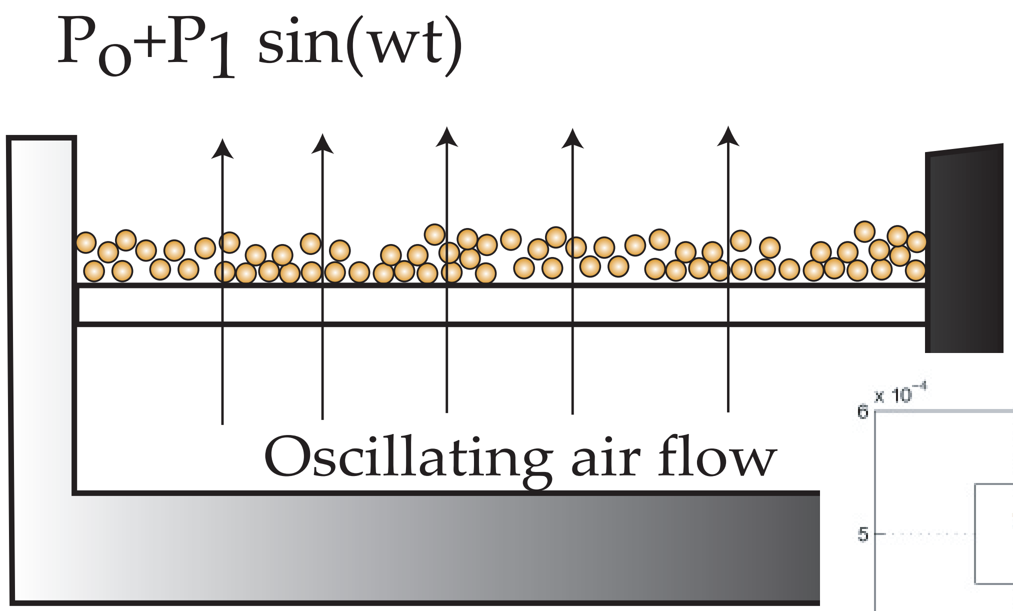
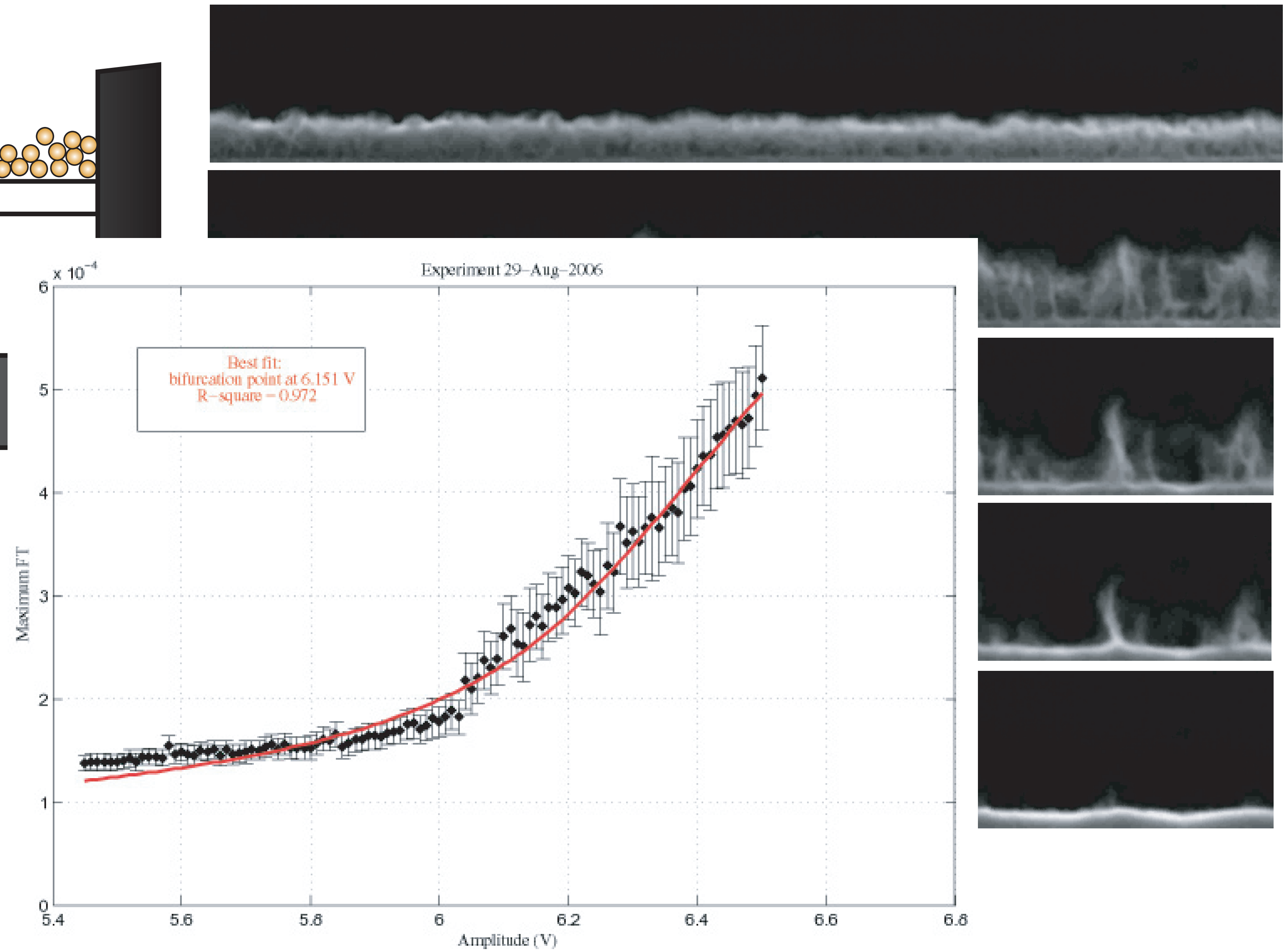




Outline

- Introduction of supercritical bifurcation and supercritical spatial bifurcation
- Universal Behavior close to bifurcation,
- Supercritical bifurcation in presence of noise,
- Experimental observations,
- Problem,
- Unified description: Stationary probability, expectation value and bifurcation criterium,
- Experimental study,
- **Conclusions,**
- Outlook.

Conclusions

- Auniversal description on the effect of additive noise on supercritical spatial bifurcation in one dimension is theoretically and experimentally studied.

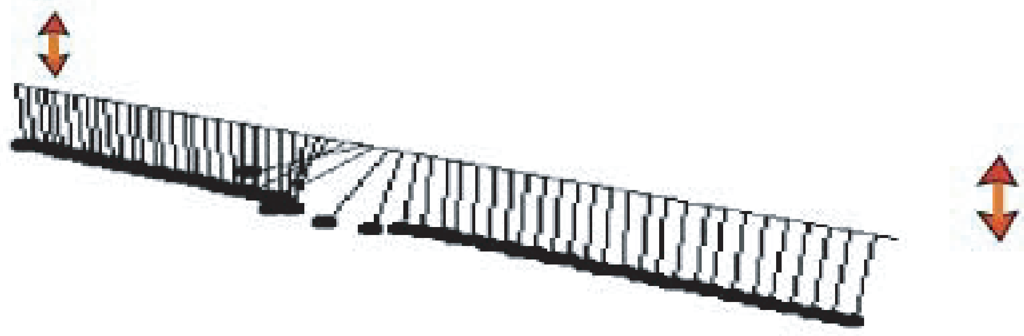
- A stationary distribution for spatial mode is derived (Ps). From this distribution, we characterized the shape of the noisy bifurcation for the mean value and the most probable value.

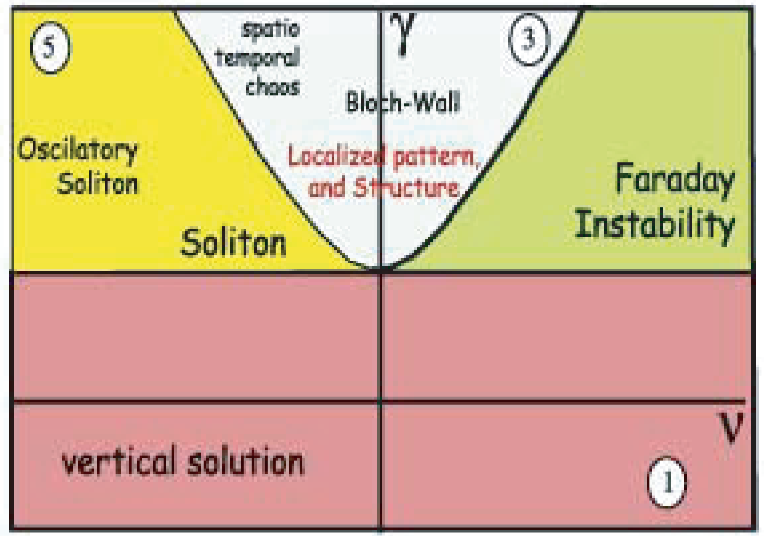
- We propose a criterium for the determination of bifurcation point.

- Comparison with numerical simulation and experimental results obtain in a Kerr-like slice subjected to optical feedback are quite good agreement.

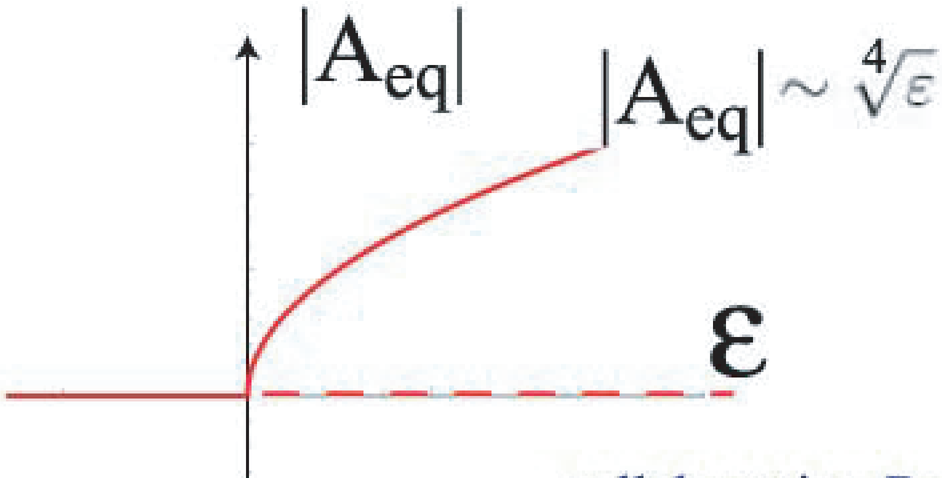
Outlook

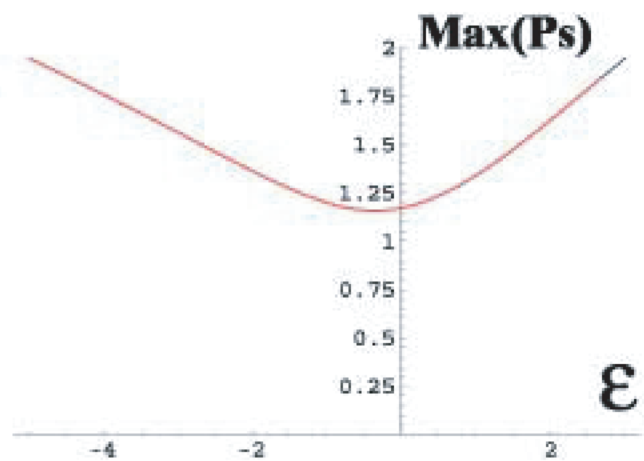
- Degenerated case





$$\ddot{\theta}(x,t) = -\left(\omega_o^2 + \gamma\sin(\omega t)\right)\sin\left(\theta\right) - \mu\dot{\theta} + k\partial_{xx}\theta,$$





collaboration Rene Rojas