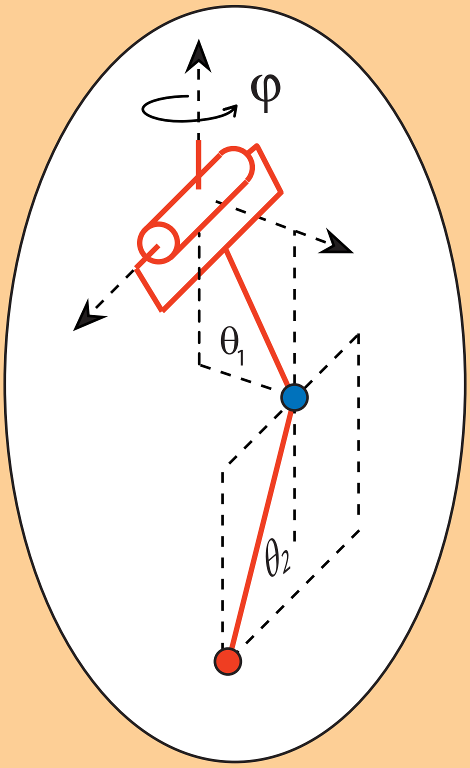

Mechanical analog of a laser close to the 1:1 resonance

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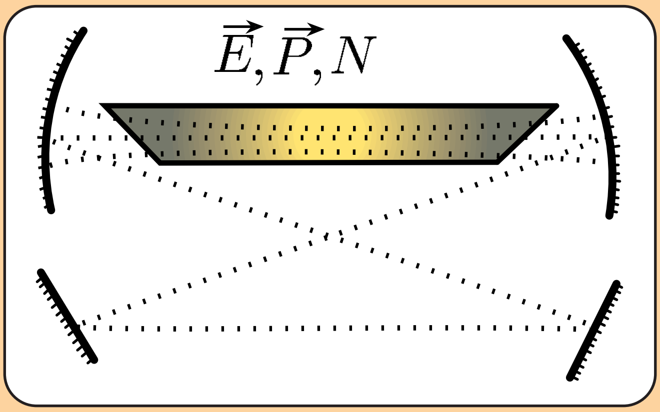
Outline

- Introduction to the laser Physics.
- Laser Instability.
- Hamiltonian bifurcation and quasi-reversible instability.
- Examples of 1:1 Resonances
- Mechanical analog of the laser
- Latent Bifurcation.
- Conclusion.

Semiclassical Laser Model



$$\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial x^2} - \frac{\partial^2 P}{\partial t^2} - \kappa \frac{\partial E}{\partial t},$$

$$\frac{\partial^2 P}{\partial t^2} = -\gamma_\perp \frac{\partial P}{\partial t} - (\gamma_\perp^2 + \Omega^2) P - \mu^2 N E,$$

$$\frac{\partial N}{\partial t} = -\gamma_{||}(N - N_0) + E \left(\frac{\partial P}{\partial t} + \gamma_\perp P \right),$$

where the terms proportional to $\kappa, \gamma_\perp$ and $\gamma_{||}$ are dissipatives.

The nonlasing solution (equilibrium) is

$$E = P = 0 \text{ and } N = D_0$$

Non dissipative semiclassical model

$$\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial x^2} - \left(\frac{\mu}{\Omega}\right)^2 \frac{\partial^2 P}{\partial t^2},$$
$$\frac{\partial P}{\partial t} = G, \quad \frac{\partial G}{\partial t} = -P - NE,$$
$$\frac{\partial N}{\partial t} = EG.$$

where $P \to \mu/\Omega P$, $E \to \Omega/\mu E$, $t \to t/\Omega$, $x \to x/\Omega$

This model has Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\left(D - \left(\frac{\mu}{\Omega}\right)^2 P\right)^2 + \frac{(\partial_x A)^2}{2} + \left(\frac{\mu}{\Omega}\right)^2 N,$$

with $E = \partial_t A = D - (\mu/\Omega)^2 P$.

and the Poisson-bracket

$$\{F, K\} = \int dx \left\{ \frac{\partial F}{\partial A}\frac{\partial K}{\partial D} - \frac{\partial F}{\partial D}\frac{\partial K}{\partial A} - \left(\frac{\mu}{\Omega}\right)^2 \vec{m} \cdot \left(\vec{\nabla}_m F \times \vec{\nabla}_m K\right) \right\},$$

where $\vec{m} = (N, P, G)$

Energie-Casimir Method

- $\{\Phi(N^2 + P^2 + G^2), F\} = 0$
- The effective energie

$$H_C = \int \left\{ \frac{1}{2}\left(D - \left(\frac{\mu}{\Omega}\right)^2 P\right)^2 + \frac{(\partial_x A)^2}{2} + \left(\frac{\mu}{\Omega}\right)^2 N + \left(\frac{\mu}{\Omega}\right)^2 \frac{(N^2 + P^2 + G^2)}{2D_o} + \alpha^2 (N^2 + P^2 + G^2)^2 \right\},$$

Reversible and Hamiltonian systems

$$\partial_t u = f(u),\ \partial_t Su = -f(Su),\ S^2 = 1$$

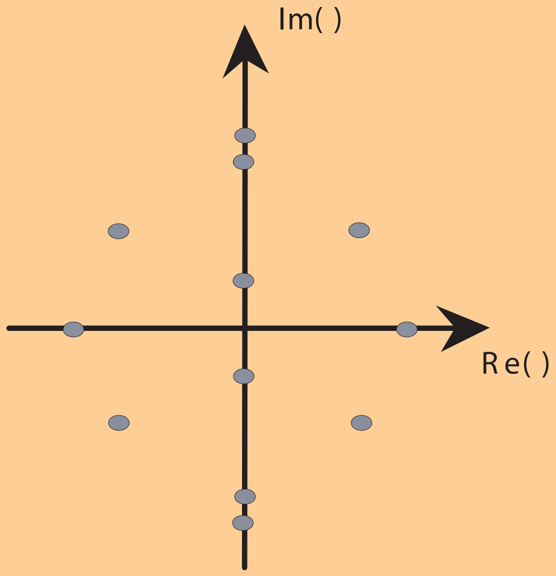
$$t \longrightarrow -t,\ u \longrightarrow Su$$

- Example : Hamiltonian Systems

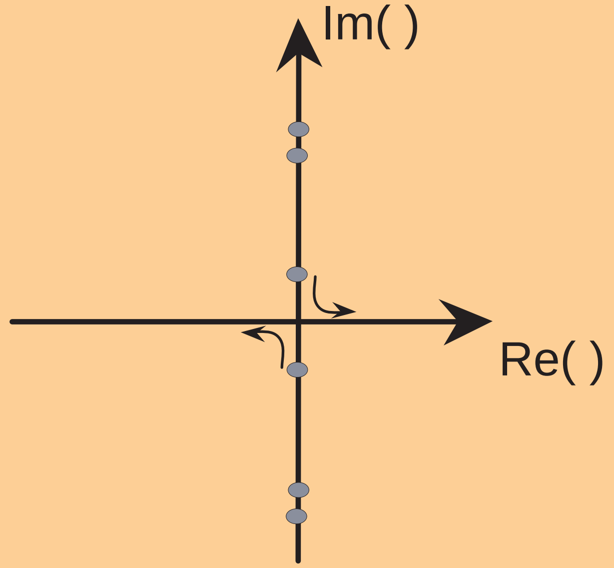
$$H(\vec{p},\vec{q}) = \sum_{i=1}^{n} \frac{p_i p_i}{2} + V(\vec{q})$$

$$t \to -t,\ \vec{q}^i \to \vec{q}^i,\ \vec{p}_i \to -\vec{p}_i, \qquad i = 1, \cdots, n$$

- instabilities ($\lambda \longrightarrow -\lambda$, for reversal solution)

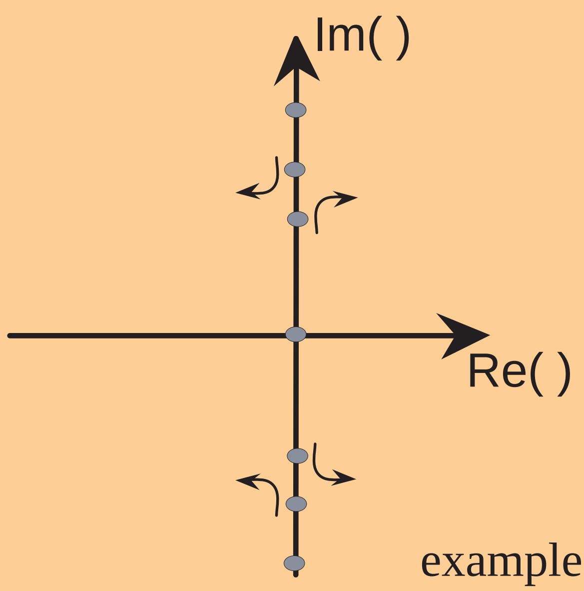


i)Stationary instability



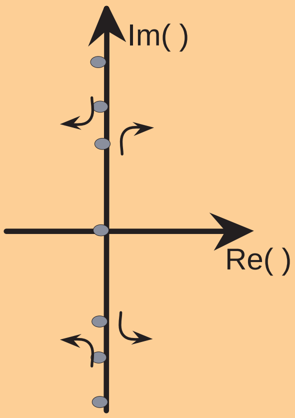
example : rotating pendulum

ii)Confusion of frequencies (Rocard ('43))



example : wing's aircraft,
Laser

- In the quasi-reversible confusion of frequencies, the asymptotic normal form



$$A_{tt} = \epsilon A - (\nu + i\Delta)A_t - |A|^2 A$$
$$- \;\; zA$$
$$z_t = -\mu \; z + \eta \; |A|^2$$

- J. Gibbon et al ('80): the dispersive instability with small dissipation.

Introducing the change of variables

$$P = \kappa E + \partial_t E, \quad E = e^{-i\frac{\Delta\kappa}{\gamma+\kappa}t}\frac{A}{\sqrt{g}}, \qquad N = \frac{z - D_o}{g} + |E|^2,$$

the equations read (Maxwell-Bloch or complex Lorenz eqs.)

$$\partial_t E = -\kappa E + P$$

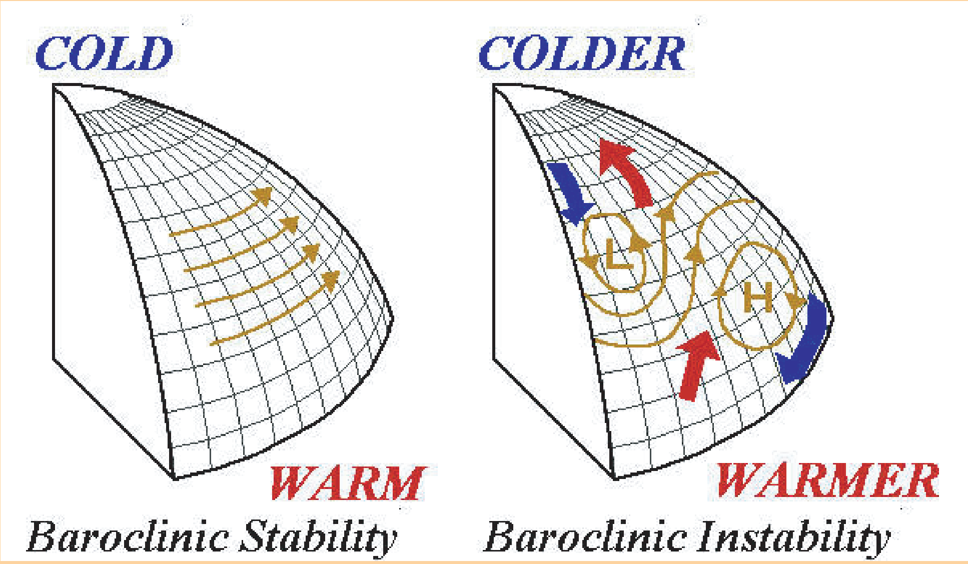
$$\partial_t P = -(\gamma_\perp + i\Delta)P - gNE$$

$$\partial_t N = -\gamma_{||}(N - N_0) + (E\bar{P} + \bar{E}P)$$

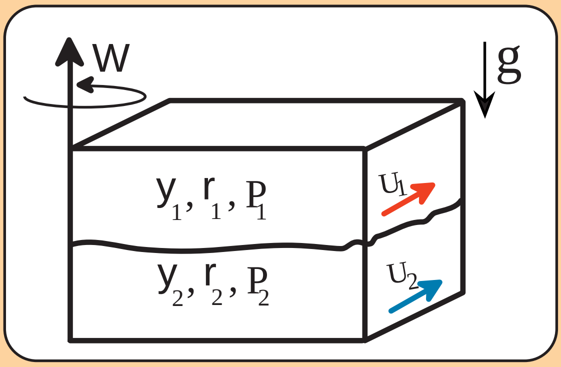
Physical examples

Baroclinic instability

This instability gives rise waves motion due to vertical shear of the basic current in the presence of Coriolis and buoyancy forces



Quasi-Geostrophic Two-layer Model



two layers of immiscible, incompressible, homogeneous fluid of slightly different densities ($r_2 > r_1$). The dimensionless quasi-geostrophic vorticity equations are

$$[\partial_t + \psi_{1,x}\partial_y - \psi_{1,y}\partial_x]\left[\vec{\nabla}^2\psi_1 + F\left(\psi_2 - \psi_1\right) + \beta y\right] = -r\vec{\nabla}^2\psi_1,$$

$$[\partial_t + \psi_{2,x}\partial_y - \psi_{2,y}\partial_x]\left[\vec{\nabla}^2\psi_2 + F\left(\psi_1 - \psi_2\right) + \beta y\right] = -r\vec{\nabla}^2\psi_2,$$

- This system is Lagrangean

$$\mathcal{L} = \int \frac{1}{2}\left(\vec{\nabla}\psi_1\right)^2 + \frac{1}{2}\left(\vec{\nabla}\psi_1\right)^2 + \frac{F}{2}\left(\psi_1 - \psi_2\right)^2 - \beta y\left(\psi_1 + \psi_2\right)\ dtdxdy,$$

constrained to Euler-Poincaré variations

$$\delta\psi_1 = \frac{\partial}{\partial t}\delta\phi_1 + \hat{z}\cdot\left(\vec{\nabla}\psi_1 \times \vec{\nabla}\delta\phi_1\right),$$

$$\delta\psi_2 = \frac{\partial}{\partial t}\delta\phi_2 + \hat{z}\cdot\left(\vec{\nabla}\psi_2 \times \vec{\nabla}\delta\phi_2\right).$$

Where the stream functions depent of the horizontal coordinate and time. The respective Hamiltonian is

$$\mathcal{H} = \int \frac{1}{2}\left(\vec{\nabla}\psi_1\right)^2 + \frac{1}{2}\left(\vec{\nabla}\psi_1\right)^2 + \frac{F}{2}\left(\psi_1^2 + \psi_2^2\right) dxdy,$$

If one considers perturbations of the geostrophic basic flow of the form

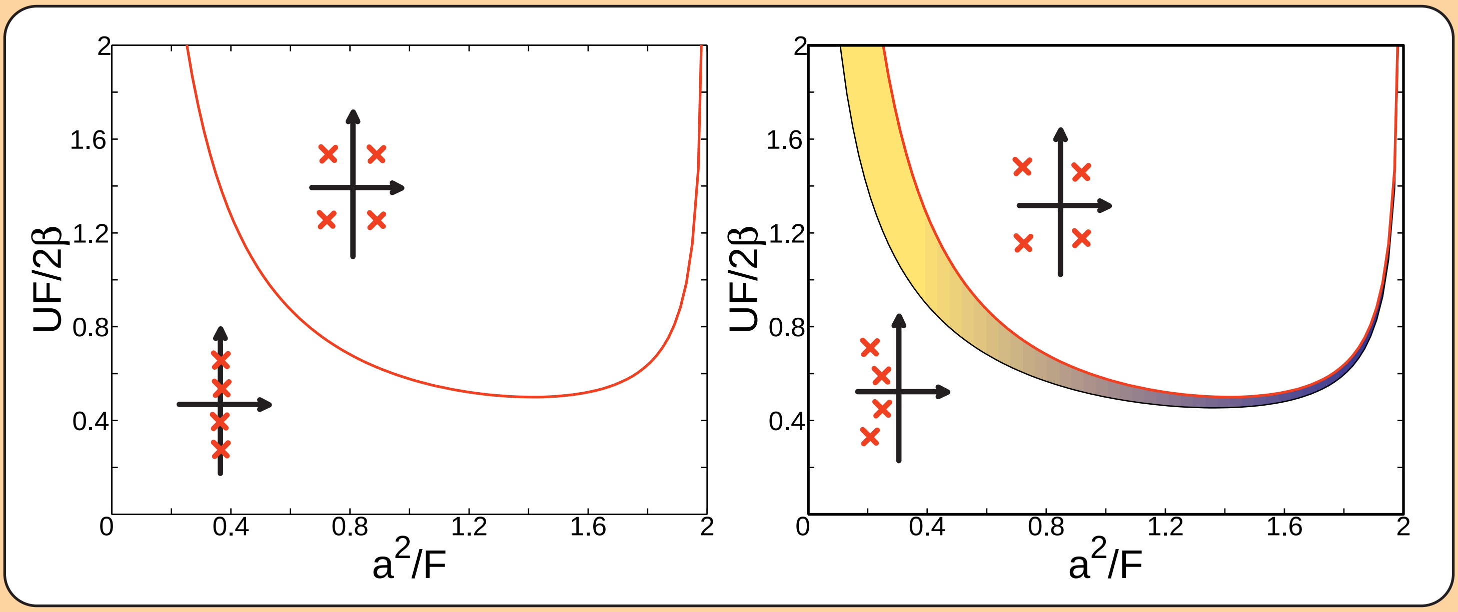
$$\psi_1 = -U_1 y + \operatorname{Re} A e^{i\alpha(x-ct)} \sin(m\pi y),$$
$$\psi_2 = -U_2 y + \operatorname{Re} \gamma A e^{i\alpha(x-ct)} \sin(m\pi y),$$

then dispertion relation is

$$c = \frac{U_1+U_2}{2} - \frac{a^2+F}{a^2+2F}\left[\frac{\beta+i\alpha r}{\alpha^2}\right]$$
$$\pm \frac{\left[(\Delta U)^2 a^4 \left(a^4-4F^2\right) + 4F^2\left(\beta + ira^2\alpha^{-1}\right)\right]^{1/2}}{2a^2\left(a^2+2F\right)}$$

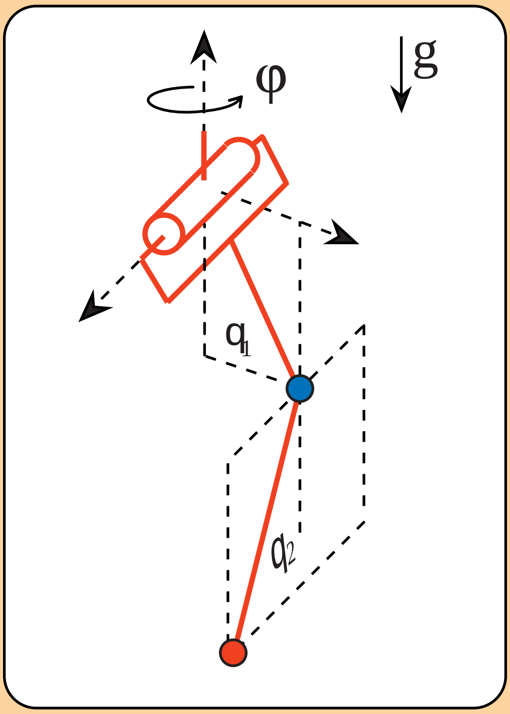
where $a^2 = \alpha^2 + m^2\pi^2$ and $\Delta U \equiv U_1 - U_2$.

- Bifurcation diagram for small viscosity

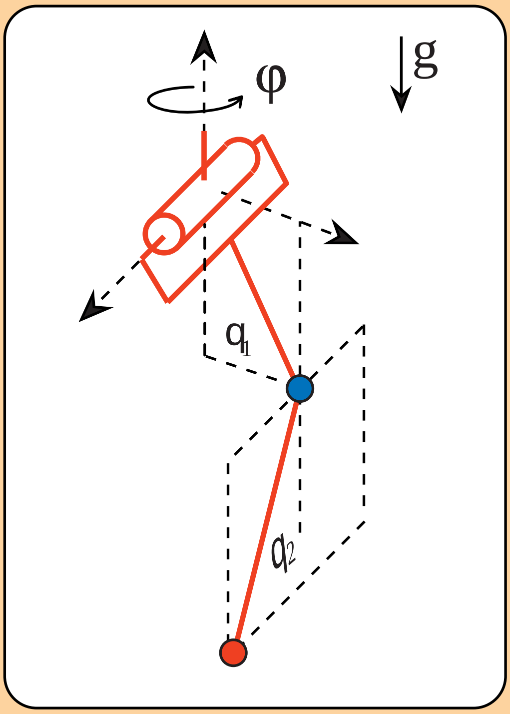


- references(E.O. Holopainen 1961, Romea 1977).

Mechanical Laser

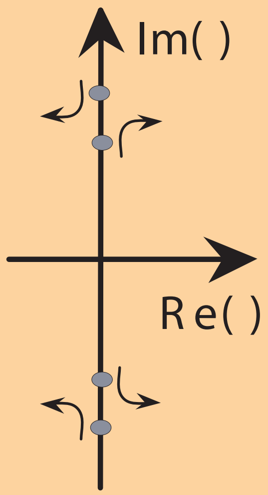


Two coupled spherical pendula in a gravitational field, with a support, which can rotate around the vertical axis. The lower pendulum is constrained to move in a plane that is orthogonal to the plane of the upper pendulum.



$$\ddot\theta_1 = -\sigma^2 \sin\theta_1 \sin\theta_2 \ddot\theta_2 - \sigma^2 \sin\theta_1 \cos\theta_2 \dot\theta_2^2$$
$$-2\sigma^2 \cos\theta_1 \cos\theta_2 \dot\varphi \dot\theta_2 + \sin\theta_1 \cos\theta_1 \dot\varphi^2$$
$$-\sigma^2 \cos\theta_1 \sin\theta_2 \ddot\varphi - \frac{g}{l}\sin\theta_1 - \nu_1 \dot\theta_1,$$
$$\ddot\theta_2 = -\sin\theta_1 \sin\theta_2 \ddot\theta_1 - \cos\theta_1 \sin\theta_2 \dot\theta_1^2$$
$$+2\cos\theta_1 \cos\theta_2 \dot\varphi \dot\theta_1 + \sin\theta_2 \cos\theta_2 \dot\varphi^2$$
$$+\sin\theta_1 \cos\theta_2 \ddot\varphi - \frac{g}{l}\sin\theta_2 - \nu_2 \dot\theta_2,$$
$$\frac{d}{dt}\left\{ \begin{array}{c} \left(\sin^2\theta_1 + \sigma^2 \sin^2\theta_2\right)\dot\varphi + \sigma^2 \cos\theta_1 \sin\theta_2 \dot\theta_1 \\ -\sigma^2 \sin\theta_1 \cos\theta_2 \dot\theta_2 + I\dot\varphi \end{array} \right\} =$$
$$-\nu_\varphi\left(\dot\varphi - \Omega\right) - \mu_1 \sin^2\theta_1 \dot\varphi - \mu_2\left(\sin^2\theta_1 + \sin^2\theta_2\right)\dot\varphi.$$

where $l_1 = l_2 = l$ and $\sigma = \sqrt{m_2/\left(m_1 + m_2\right)}$



The vertical solution

$$\theta_1 = \theta_2 = 0,\ \varphi_t = \Omega_0$$

exhibits a 1:1 resonance when

$$\Omega_0 = \Omega_c = \sqrt{\frac{g}{l}\frac{(m_1+m_2)}{m_1}}$$

with $\omega_c = \pm\sqrt{gm_2/lm_1}$ frequencies. The system is described by

$$A_{tt} = \frac{2g\,(\Omega-\Omega_c)}{l\Omega}A + i\,(2\sigma\,(\Omega-\Omega_c))\,A_t$$

- Quasi-reversible instability

$$\begin{aligned}\partial_{tt}A &= (\varepsilon - Z)\,A - (\mu - i\delta)\,\partial_t A - \alpha|A|^2 A\\ \partial_t Z &= \nu Z + \eta|A|^2.\end{aligned}$$

where

$$\begin{aligned}\varepsilon &= 2\frac{g}{l}\frac{(\Omega-\Omega_c)}{\Omega_c},\ \alpha = \frac{g}{4l}\left(\frac{\sigma^4 - 2\sigma^3 - 2\sigma^2 + 3}{1-\sigma^2}\right),\\ \delta &= 2\sigma\,(\Omega-\Omega_c)\,,\ \nu = \frac{\nu_\varphi}{I},\quad \mu = \frac{1}{2l^2}\,(\nu_1+\nu_2)\,,\\ \eta &= \frac{1}{I}\left[\mu_1 + \left(1+\frac{1}{\sigma^2}\right)\mu_2 - 2\frac{\nu_\varphi}{\sigma^2 I}\left(\frac{\Omega}{\Omega_c}-1\right)\right];\end{aligned}$$

FOUCAULT PENDULUM

- As consequence of Earth rotation, the vertical solution exhibits a latent bifurcation!.

$$\ddot{x} = -\left(\frac{g}{l} - \Omega^2\right) x + 2\Omega\dot{y}$$
$$\ddot{y} = -\left(\frac{g}{l} - \Omega^2\right) y - 2\Omega\dot{x}$$

- Latent bifurcation.
- Dissipation induced instability.
- Applications : simple mechanical system, Laser, restricted three-body problem, Foucault pendulum, Chemical systems, nuclear physics and Baroclinic instability.