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Estimating Detection Statistics within a Bayes-Closed Multi-Object Filter

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***Abstract*—In multi-target tracking, correct models of detection statistics, namely the probability of detection and clutter rate, are required for effective multi-target state estimation. Within a multi-target filter, the detection statistics are usually assumed as known and static. Estimating the detection statistics' parameters before the execution of the filtering algorithms is not always feasible and in some scenarios, these parameters could be time varying, which would invalidate offline estimation. To overcome these issues, this paper presents a Random Finite Set (RFS) based algorithm which is capable of estimating both the probability of detection and the clutter rate, while jointly estimating the multi-target state of the system. The proposed algorithm is based on previous work, the Kronecker Delta Mixture and Poisson (KDMP) filter, which is a Chapman-Kolmogorov and Bayes closed solution to the RFS-based filtering problem. Importantly, the resulting robust filter remains closed under the filtering procedure. Results show that the algorithm converges to the correct detection statistics in simulated environments and, as opposed to other methods, it can even continue to estimate the probability of detection when no targets are present in the environment.**

I. Introduction

The multi-target tracking problem is of interest in many areas of science, from bio-engineering [1] to robotics [2]. A solution to the multi-target tracking problem has to estimate both the number of targets and their respective states. To do so, a model of the detection statistics, namely the probability of detecting a target and a description of the clutter measurements of the sensor and/or detection algorithm, is required. Recently, new RFS-based methods that estimate the full multi-target posterior have been proposed. These filters are closed under the filtering procedure, namely the Chapman Kolmogorov equation and Bayes rule, which results in more accurate estimates than other RFS-based methods. Despite these advances, these filters still require the parameters describing the detection statistics. Solutions to the multi-target tracking problem which are robust to unknown or time varying detection statistics and which are closed under Bayes rule have yet to be explored.

In this work, a solution based on the Kronecker Delta Mixture with Poisson (KDMP) RFS filter, which in [3] was shown to be closed under the Chapman Kolmogorov equation and Bayes rule, is examined. The proposed robust filter jointly estimates the multi-target state, the probability of detection and the clutter rate (or expected number of clutter measurements). To provide a joint robust filtering solution which remains closed under the filtering procedure, the probability of detection and clutter rate are modelled as random variables with Beta and Gamma distributions respectively. Simulations show that the proposed method correctly estimates the detection statistics and even allows for time variability in these parameters.

This article is structured as follows. In Section II related work is presented, followed in Section III by a basic introduction to Random Finite Sets required to develop the proposed robust filter. The proposed model for the joint estimation of RFS state and detection statistics is then presented in Section IV and the implementation details discussed in Section V. Section VI presents results based on simulations with both static and time varying detection statistics. Finally, in Section VII the conclusions to this work are presented.

II. Related Work

Clutter estimation within RFS-based methods, in particular using the Cardinalized Probability Hypothesis Density (CPHD) filter, has been theoretically analysed in [4], [5], while estimating the probability of detection was discussed in [6]. An analytical implementation of such ideas was presented in [7], where a CPHD filter that estimated the detection statistics was presented. This filter used "clutter generators" which are analogous to targets, but only generate clutter measurements. The problem then becomes one of estimating the multi-target state and the multi-target clutter generators.

A similar approach was adopted in [8] where the same idea was implemented using a multi-Bernoulli (MeMBer) filter. New developments of these filters were discussed in [9] and [10], where not only the full clutter RFS is estimated, but also the process of target creation, or birth.

A solution for tracking unknown and time varying detection statistics was presented in [1]. Their solution uses the λ-p_D-CPHD filter, which is a CPHD filter that estimates the clutter rate and probability of detection, and a standard CPHD filter. The proposed solution first uses the λ-p_D-CPHD filter to estimate the detection statistics, which are then fed into the standard CPHD filter, which estimates the tracks.

A different approach is taken in this work, where both, the probability of detection and clutter rate are regarded as continuous random variables (as opposed to an RFS variable) and estimated in a joint manner with the multi-target state, as shown in the next section. In this article it is assumed that

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the probability of detection and clutter rate are properties of the sensor and/or detection algorithm and it is thus intuitive to model them as random variables rather than RFSs. This model has an advantage over previous work proposed in [4], [5], [6], [7], [8], [9], [10]. Since the value for P_D does not depend on the target's state, it is theoretically possible to estimate P_D even when no target is present in the environment. This helps in the (re-)initialisation of targets when there has been a long period of miss detections or only clutter measurements.

III. Background

This section introduces the basic concepts from RFS theory required to derive the equations of the Robust-KDMP (R-KDMP) filter. For a complete description, the reader is referred to [11].

An RFS X is a random finite set, and as such it has two sources of uncertainty, namely randomness in the number of elements, and the values of the set elements themselves. The RFS X can be described by its probability density function $f_X(X)$, or alternatively, by its Probability Generating Functional (PGFl) $G_X[h]$, which is defined as follows:

$$G_X[h] = \int h^X f_X(X)\delta X,$$

where the integration is carried out using the set integral[1], and $h^X = \prod_{\mathbf{x}\in X} h(\mathbf{x})$, for a function $0 \leq h(\mathbf{x}) \leq 1$.

By using RFSs, it is possible to formulate the recursive Bayesian estimation problem in terms of PGFls. If at time t the estimated RFS is X_t, then the predicted RFS $X_{t+1|t}$ has the following PGFl:

$$\begin{aligned} &G_{X_{t+1|t}}[h] = G_B[h] \\ &\times G_{X_t}\left[1 - P_S + P_S \int f_{t+1|t}(\mathbf{x}_{t+1|t}|\mathbf{x}_t) h(\mathbf{x}_{t+1|t}) d\mathbf{x}_{t+1|t}\right], \end{aligned} \tag{1}$$

where $G_B[h]$ is the PGFl of the birth process, which models where new targets are predicted to be born, P_S the probability of surviving between time t and $t+1$ and $f_{t+1|t}(\mathbf{x}_{t+1|t}|\mathbf{x}_t)$ a single target prediction function.

Similarly, the corrected RFS X_{t+1} has the following PGFl:

$$G_{X_{t+1}}[h] = \frac{\frac{\delta}{\delta Z} F[g,h]\big|_{g=0}}{\frac{\delta}{\delta Z} F[g,h]\big|_{g=0,h=1}}, \tag{2}$$

where $\frac{\delta}{\delta Z}$ is the functional derivative[2] with respect to the observations Z (modelled by function $g(\mathbf{z})$), and

$$\begin{aligned} &F[g,h] = G_\Theta[g] \\ &\quad \times G_{X_{t+1|t}}\left[h\left(1 - P_D + P_D \int f_z(\mathbf{z}|\mathbf{x}) g(\mathbf{z}) d\mathbf{z}\right)\right], \end{aligned} \tag{3}$$

[1]The set integral is defined as follows:

$$\int f_X(X)\delta X = \sum_{n=0}^{\infty} \frac{1}{n!} \int \cdots \int f_X(\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}) d\mathbf{x}_1 \cdots d\mathbf{x}_n.$$

[2]δZ is understood as $\partial\delta_{\mathbf{z}_1} \cdots \partial\delta_{\mathbf{z}_n}$ with $Z = \{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$ and $\delta_{\mathbf{z}_i}$ a Dirac Delta function at $\mathbf{z}_i$.

where $G_\Theta[g]$ is the PGFl of the clutter process Θ, P_D is the probability of detecting a target and $f_z(\mathbf{z}|\mathbf{x})$ is a single target likelihood function.

Depending on the RFS model of $f_X(X)$, different solutions to (1) and (2) can be derived. One such solution, is the KDMP filter [3], where the RFS X is modelled as a mixture of Kronecker Delta RFSs in union with a Poisson RFS. The PGFl of a KDMP distribution is as follows:

$$G_X[h] = e^{\langle D_U, h-1\rangle} \sum_j \omega_j \prod_k \langle f_{j,k}, h\rangle, \tag{4}$$

where $\langle f, g\rangle = \int f(\mathbf{u})g(\mathbf{u})d\mathbf{u}$, $D_U(\mathbf{x})$ is a density function (expected position of the elements of a set) representing unknown, or potential tracks and $f_{j,k}(\mathbf{x})$ are the probability density functions of the different known tracks. The recursive estimation of the parameters of this distribution is achieved by using Algorithm 1.

In the following section, Equation (4) will be extended to account for unknown clutter rate and probability of detection. The resulting PGFl will be used in Equations (1) and (2) to derive the prediction and correction equations for the Robust-KDMP (R-KDMP) filter.

IV. Proposed Model for Joint RFS state and Detection Statistics Estimation

This section describes the model used to jointly estimate the multi-target state X, clutter rate and probability of detection. To account for unknown detection statistics, it will be assumed that both the clutter rate and the probability of detection are variables to be estimated rather than parameters of the filter. To this end, it will be assumed that the probability of detection is a property of the sensor, and as such, independent of the state of any target (e.g. $P_D(\mathbf{x}) = P_D$). This allows P_D to be estimated even if the estimate of the multi target state is the empty set. The full distribution to be estimated is therefore:

$$p(X_t, P_D, \lambda_\Theta | Z_{1:t}),$$

where X_t is the RFS representing the multi-target state, P_D the probability of detection, λ_Θ the clutter rate and $Z_{1:t}$ all the observations received up to and including time t. The clutter rate, or expected number of clutter measurements, will be modelled by a Gamma distribution, and the probability of detection with a Beta distribution. These choices are made, since the number of clutter measurements is assumed Poisson distributed and a Gamma distribution is the conjugate prior for the expected number of clutter measurements. Similarly, the detection or miss-detection of targets can be modelled as a Bernoulli experiment, and thus, a conjugate prior for the detection probability is the Beta distribution. The PGFl used in the R-KDMP filter is therefore of the form:

$$G_{X_t,P_D,\lambda_\Theta}[h] = \underbrace{e^{\sum_i^T (1-P_D)^i \langle D_i, h-1\rangle}}_{\text{Poisson RFS}}$$

Algorithm 1 KDMP filtering algorithm proposed in [3]. The parameters are the maximum number of components of the posterior mixture, N, and a threshold on the smallest weight allowed in the posterior mixture τ. The external function $\textsc{NextBestAssignment}(\mathbf{C}_i)$ is a call to Murty's algorithm that finds the next best assignment and its corresponding cost of the matrix $\mathbf{C}_i$.

Parameters: N, τ
Input: $\texttt{components}_t = \{(\omega_1, \{f_{1,1}, \dots, f_{1,n_1}\}), \dots, (\omega_\ell, \{f_{\ell,1}, \dots, f_{\ell,n_\ell}\})\}, \mathbb{Z}_t = \{\mathbf{z}_1, \dots, \mathbf{z}_m\}, D_{U_t}(\mathbf{x})$
for $i, n \in 1 \dots \ell, 1 \dots n_\ell$ **do** ▷ Update the distribution of each element.
  $f_{i,n} \leftarrow \frac{\langle f_{t+1|t}, p_S f_{i,n} \rangle}{\langle p_S, f_{i,n} \rangle}$
end for
$D_{U_{t+1|t}} = (1 - p_S) \left\langle f_{t+1|t}, D_{U_t}(\mathbf{x}) \right\rangle + D_B(\mathbf{x})$ ▷ Update the distribution of the unknown space.
for $i \in 1 \dots \ell$ **do** ▷ Create the cost matrix for each component of the mixture
  $\mathbf{C}_i = i$ th KDMP Cost matrix (see text).
  $\texttt{assignments}_i, \texttt{cost}_i = \textsc{NextBestAssignment}(\mathbf{C}_i)$ ▷ Get the initial assignment and cost of the $\mathbf{C}_i$ matrix
end for
$\texttt{components}_{t+1} \leftarrow \emptyset, \texttt{totalWeight} = 0$
while $|\texttt{components}_{t+1}| \leq N$ **do** ▷ Generate the new components in decreasing order the posterior weight
  $i^\star = \arg\min \left(\texttt{cost}_i - \log(\omega_i)\right), \qquad \psi = \omega_{i^\star} e^{\texttt{cost}_{i^\star}}$
  $\texttt{totalWeight} \leftarrow \texttt{totalWeight} + \psi, \qquad F \leftarrow \emptyset$
  for all $\texttt{assignment} \in \texttt{assignments}_{i^\star}$ **do**
    if $\texttt{assignment}$ corresponds to the detection sub-matrix $\mathbf{D}^{(i)}$ **then**
      $F \leftarrow F \cup \left\{ \frac{f_z(\mathbf{z}_j|\mathbf{x}_{t+1}) f_{\mathbf{x}_{i^\star,k}}}{\langle f_z(\mathbf{z}_j|\mathbf{x}_{t+1}) f_{\mathbf{x}_{i^\star,k}}, 1 \rangle} \right\}$
    else if $\texttt{assignment}$ corresponds to the miss-detection sub-matrix $\mathbf{M}^{(i)}$ **then**
      $F \leftarrow F \cup \left\{ \frac{f_{\mathbf{x}_{i^\star,k}}}{\langle f_{\mathbf{x}_{i^\star,k}}, 1 \rangle} \right\}$
    else if $\texttt{assignment}$ corresponds to the new components $\mathbf{N}^{(i)}$ **then**
      $F \leftarrow F \cup \left\{ \frac{D_{U_{t+1|t}} f_z(\mathbf{z}_j|\mathbf{x}_{t+1})}{\langle D_{U_{t+1|t}} f_z(\mathbf{z}_j|\mathbf{x}_{t+1}), 1 \rangle} \right\}$
    end if
  end for
  $\texttt{components}_{t+1} \leftarrow \texttt{components}_{t+1} \cup \{(\psi, F)\}$
  if $\psi \leq \tau \cdot \texttt{totalWeight}$ **then** ▷ Stop if the new component does not contribute significantly to the mixture
    break
  end if
  $\texttt{assignments}_{i^\star}, \texttt{cost}_{i^\star} = \textsc{NextBestAssignment}(\mathbf{C}_{i^\star})$ ▷ Next best assignment for the $i^\star$ component
end while
$\textsc{NormaliseWeights}(\texttt{totalWeight}, \{\omega \in \texttt{components}_{t+1}\})$
$D_{U_{t+1}} = (1 - p_D) D_{U_{t+1|t}}(\mathbf{x})$ ▷ Correction for the unknown space
return $\texttt{components}_{t+1}$

$$\times \underbrace{\sum_j^N \omega_j \mathrm{Beta}\left(P_D | a_j, b_j\right) \mathrm{Gamma}\left(\lambda_\Theta | \alpha_j, \beta_j\right) \prod_k^{M_j} \langle f_{j,k}, h \rangle}_{\text{Beta-Gamma Kronecker Delta RFS}}, \tag{5}$$

where (a_j, b_j) and (α_j, β_j) are the parameters of the Beta and Gamma distribution respectively, ω_j and $f_{j,k}$ the parameters of the KDMP multi-object distribution, and $D_i(\mathbf{x})$ are the T density functions (from $i = 1$ to T) modelling the unknown space of the environment. The Poisson component models where targets could be born. In particular, $D_1(\mathbf{x})$ is the density where targets could be born in the current time step, while $D_i(\mathbf{x})$ is the density of targets born i steps in the past, which have not yet been detected. It will be shown (in Equation (7)) that after the prediction stage of the R-KDMP filter, $D_1(\mathbf{x})$ will correspond to the birth density corresponding to $G_B[h]$ in Equation 1.

For clarity, the time sub indices t, $t+1|t$ and $t+1$ will be omitted when they are clear from the context.

A. Prediction

In the prediction step, some care must be taken in the derivation of the equations of the Poisson component. Assuming a Poisson birth process, the predicted Poisson component PGFl is:

$$\begin{aligned} G_{X_{t+1|t}}[h] &= G_B[h] G_{X_t}\left[1 - P_S + P_S \left\langle f_{t+1|t}, h \right\rangle\right] \\ &= e^{\langle D_B, h-1 \rangle + \sum_i^T (1-P_D)^i \left\langle D_{i,t}, -P_S + P_S \left\langle f_{t+1|t}, h \right\rangle \right\rangle} \\ &= e^{\langle D_B, h-1 \rangle + \sum_i^T (1-P_D)^i \left\langle D_{i,t} P_S, \left\langle f_{t+1|t}, h \right\rangle - 1 \right\rangle} \end{aligned}$$

$$= e^{\langle D_B, h-1\rangle + \sum_i^T (1-P_D)^i \left\langle \left\langle D_{i,t} P_S, f_{t+1|t} \right\rangle, h-1 \right\rangle}$$

$$= e^{\sum_i^T (1-P_D)^i \langle D, h-1\rangle}, \quad (6)$$

where:

$$D_{i,t+1}(\mathbf{x}) = \left\langle D_{i,t}(\mathbf{x}_t) P_S(\mathbf{x}_t), f_{t+1|t}(\mathbf{x}|\mathbf{x}_t) \right\rangle + \begin{cases} D_B(\mathbf{x}) & , i=1 \\ 0 & , i>1 \end{cases}. \quad (7)$$

Note that the integration in $\left\langle f_{t+1|t}, h \right\rangle$ is with respect to $\mathbf{x}_{t+1|t}$, while the integration in $\left\langle D_{i,t}(\mathbf{x}_t) P_S(\mathbf{x}_t), f_{t+1|t}(\mathbf{x}|\mathbf{x}_t) \right\rangle$ is with respect to $\mathbf{x}_t$. From Equation (7) it can be seen that the birth intensity at the current time step only adds to the first component of the Poisson part of the R-KDMP.

In the current formulation, there is no prediction model for the Beta and Gamma components and thus their parameters do not change under prediction.

B. Correction

To derive the correction equations, first the joint functional $F[g,h]$ is derived:

$$\begin{aligned} F[g,h] &= G_\Theta[g] G_X \left[h(1-P_D+P_D \langle f_z, g\rangle) \right| P_D, \Theta] \\ &= e^{\langle D_\Theta, g-1\rangle} e^{\sum_i U_i[h,g]} \sum_j \omega_j \mathrm{B}_j \mathrm{G}_j \prod_k T_{j,k}[h,g], \end{aligned}$$

where

$$\begin{aligned} U_i[h,g] &= (1-P_D)^i \langle D_i, h(1-P_D+P_D\langle f_z, g\rangle) - 1\rangle \\ T_{j,k}[h,g] &= \langle f_{j,k}, h(1-P_D+P_D\langle f_z, g\rangle)\rangle \\ \mathrm{B}_j &= \mathrm{Beta}(P_D | a_j, b_j), \qquad \mathrm{G}_j = \mathrm{Gamma}(\lambda_\Theta | \alpha_j, \beta_j), \end{aligned}$$

and the integration in $\langle f_z, g\rangle$ is with respect to $\mathbf{x}$. To obtain an expression for the corrected PGFl, it is required to derive the functional $F[g,h]$ with respect to Z^3:

$$\begin{aligned} G_X[h|P_D, \lambda_\Theta] &\propto \left. \frac{\delta}{\delta Z} F[g,h] \right|_{g=0} \\ &= \sum_{Z_C \uplus Z_N \uplus Z_D = Z} \frac{\delta}{\delta Z_C} \left\{ e^{\langle D_\Theta, g-1\rangle} \right\} \quad (8) \\ &\times \frac{\delta}{\delta Z_N} \left\{ e^{\sum_i U_i[h,g]} \right\} \frac{\delta}{\delta Z_D} \left\{ \sum_j \mathrm{B}_j \mathrm{G}_j \prod_k T_{j,k}[h,g] \right\}. \end{aligned}$$

This derivative divides the set of observations Z into three disjoint sets, namely the set of clutter measurements Z_C, the set of measurements from existing targets Z_D and the set of measurements from new targets Z_N. The derivative with respect to the clutter Z_C, results in the clutter component of the KDMP filter:

$$\left. \frac{\delta}{\delta Z_C} \left\{ e^{\langle D_\Theta, g-1\rangle} \right\} \right|_{g=0} = e^{-\langle D_\Theta, 1\rangle} \prod_{\mathbf{z}\in Z_C} D_\Theta(\mathbf{z}),$$

whereas the derivative with respect to the detected measurements Z_D, results in the detected components of the KDMP filter:

$$\left. \frac{\delta}{\delta Z_D} \left\{ \sum_j \mathrm{B}_j \mathrm{G}_j \prod_k T_{j,k}[h,g] \right\} \right|_{g=0} = \sum_{\substack{j \\ 1\le k_1 \neq \cdots \neq k_{|Z_D|} \le |Z_D|}} \mathrm{B}_j \mathrm{G}_j \prod_{\mathbf{z}_i \in Z_D} P_D \langle f_z(\mathbf{z}_i|\cdot) f_{j,k_i}, h\rangle.$$

The derivative with respect to the new components Z_N is different from that of the standard KDMP filter and is given by:

$$\frac{\delta}{\delta Z_N} \left\{ e^{\sum_i U_i[h,g]} \right\} = e^{\sum_i U_i[h,g]} \times \prod_{\mathbf{z}\in Z_n} \sum_i^T (1-P_D)^i P_D \left\langle D_{i,t+1|t} f_z(\mathbf{z}|\cdot), h \right\rangle. \quad (9)$$

Evaluating at $g=0$ and expanding the summation results in:

$$\left. \frac{\delta}{\delta Z_N} \left\{ e^{\sum_i U_i[h,g]} \right\} \right|_{g=0} = e^{\sum_i (1-P_D)^i \left\langle D_{i,t+1|t}, (1-P_D)h - 1\right\rangle} \times \sum_{1\le i_1 \le \cdots \le i_{|X|} \le T} \prod_{\mathbf{z}_k \in Z_N} (1-P_D)^{i_k} P_D \left\langle D_{i_k, t+1|t} f_z(\mathbf{z}_k|\cdot), h \right\rangle. \quad (10)$$

Since the exponential in (10) does not depend of any value of Z, it can be taken outside of the sum of (8), and when normalised (which is achieved by dividing by the exponential term with $h = 1$), the updated values for the Poisson component of the R-KDMP filter are:

$$\frac{e^{\sum_i (1-P_D)^i \left\langle D_{i,t+1|t}, (1-P_D)h-1\right\rangle}}{e^{\sum_i (1-P_D)^i \left\langle D_{i,t+1|t}, (1-P_D)-1\right\rangle}} = e^{\sum_i (1-P_D)^{i+1} \left\langle D_{i,t+1|t}, h-1\right\rangle}$$

$$\Rightarrow D_{i,t+1}(\mathbf{x}) = D_{i-1,t+1|t}(\mathbf{x}). \quad (11)$$

Note that (11) implies that $D_{1,t+1}(\mathbf{x}) = 0$.

Without loss of generality, the clutter density can be written as $D_\Theta(\mathbf{x}) = \lambda_\Theta f_\Theta(\mathbf{x})$, where λ_Θ is the expected number of clutter measurements and $f_\Theta(\mathbf{x})$ is the probability density function of clutter occurrences. By normalising the probability densities, the PGFl (8) can be further expanded as:

$$\begin{aligned} G_X[h|P_D, \lambda_\Theta] &\propto e^{-\langle f_\Theta, 1\rangle + \sum_i (1-P_D)^i \langle D_i, h-1\rangle} \\ &\times \sum \omega_j \left(\prod_{\mathbf{z}\in Z_C} f_\Theta(\mathbf{z}) \right) \left(\prod_{\mathbf{z}_k \in Z_N} \langle D_{i_k} f_z(\mathbf{z}|\cdot), 1\rangle \right) \\ &\times \left(\prod_{\mathbf{z}_i \in Z_D} \langle f_z(\mathbf{z}_i|\cdot) f_{j,k_i}, 1\rangle \right) \left(\prod_{\mathbf{x}_{k'} \in X_M} \cancel{\langle f_{j,k'}, 1\rangle} \right) 1 \end{aligned}$$

[3]This derivative can be calculated using the product rule for set derivatives [11]:

$$\frac{\delta}{\delta X} \prod_i^n F_i = \sum_{X_1 \uplus \cdots \uplus X_n = X} \prod_i^n \frac{\delta}{\delta X_i} F_i,$$

where $\uplus$ is the disjoint union.

$$\times P_D^{|Z_D|+|Z_N|}(1-P_D)^{|X_M|+\sum_k (i_k-1)}\mathrm{Beta}(P_D|a_{t-1},b_{t-1})$$
$$\times e^{-\lambda_\Theta}\lambda_\Theta^{|Z_C|}\mathrm{Gamma}(\lambda_\Theta|\alpha_{t-1},\beta_{t-1})$$
$$\times\left(\prod_{\mathbf{z}_i\in Z_D}\left\langle h,\frac{f_z(\mathbf{z}_i|\cdot)f_{j,k_i}}{\langle f_z(\mathbf{z}_i|\cdot)f_{j,k_i},1\rangle}\right\rangle\right)\left(\prod_{\mathbf{x}_{k'}\in X_M}\left\langle h,\frac{f_{j,k'}}{\cancelto{1}{\langle f_{j,k'},1\rangle}}\right\rangle\right)$$
$$\times\left(\prod_{\mathbf{z}_k\in Z_N}\left\langle h,\frac{D_{i_k}f_z(\mathbf{z}|\cdot)}{\langle D_{i_k}f_z(\mathbf{z}|\cdot),1\rangle}\right\rangle\right). \quad (12)$$

From this point, the new parameters for the distributions of λ_Θ and P_D, and the new weights, can be calculated. The new parameters and the weight associated with the clutter rate λ_Θ can be determined from:

$$e^{-\lambda_\Theta}\lambda_\Theta^{|Z_C|}\mathrm{Gamma}(\lambda_\Theta|\alpha_{j,t-1},\beta_{j,t-1}) =$$
$$\frac{\Gamma(\alpha_{j,t+1})}{(\beta_{j,t+1})^{\alpha_{j,t+1}}}\frac{\beta_{j,t-1}^{\alpha_{j,t-1}}}{\Gamma(\alpha_{j,t-1})}\mathrm{Gamma}(\lambda_\Theta|\alpha_{j,t+1},\beta_{j,t+1}), \quad (13)$$

with $\Gamma(z)$ being the Gamma function. The new Gamma distribution parameters are:

$$\alpha_{j,t+1}=\alpha_{j,t}+|Z_C|, \qquad \beta_{j,t+1}=\beta_{j,t}+1 \quad (14)$$

The parameters and the weight associated with the probability of detection P_D are determined from:

$$P_D^{|Z_D|+|Z_N|}(1-P_D)^{|X_M|+\sum_k (i_k-1)}\mathrm{Beta}(P_D|a_{j,t},b_t) =$$
$$\frac{B(a_{j,t+1},b_{j,t+1})}{B(a_{j,t-1},b_{j,t-1})}\mathrm{Beta}(P_D|a_{j,t+1},b_{j,t+1}), \quad (15)$$

where the beta function is defined as $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and the new Beta distribution parameters are:

$$a_{j,t+1}=a_{j,t}+|Z_D|+|Z_N| \quad (16)$$
$$b_{j,t+1}=b_{j,t}+|X_M|+\sum_k (i_k-1). \quad (17)$$

Summarising, the weight of a posterior component is:

$$\omega_k\propto\omega_j\left(\prod_{\mathbf{z}\in Z_C}f_\Theta(\mathbf{z})\right)\left(\prod_{\mathbf{z}_k\in Z_N}\langle D_{i_k}f_z(\mathbf{z}|\cdot),1\rangle\right)$$
$$\left(\prod_{\mathbf{z}_i\in Z_D}\langle f_z(\mathbf{z}_i|\cdot)f_{j,k_i},1\rangle\right)$$
$$\times\frac{\Gamma(\alpha_{j,t+1})}{(\beta_{j,t+1})^{\alpha_{j,t+1}}}\frac{\beta_{j,t-1}^{\alpha_{j,t}}}{\Gamma(\alpha_{j,t})}\times\frac{B(a_{j,t+1},b_{j,t+1})}{B(a_{j,t},b_{j,t})}, \quad (18)$$

the new Kronecker Delta components are:

$$\left(\prod_{\mathbf{z}_i\in Z_D}\left\langle h,\frac{f_z(\mathbf{z}_i|\cdot)f_{j,k_i}}{\langle f_z(\mathbf{z}_i|\cdot)f_{j,k_i},1\rangle}\right\rangle\right)\left(\prod_{\mathbf{x}_{k'}\in X_M}\langle h,f_{j,k'}\rangle\right)$$
$$\times\left(\prod_{\mathbf{z}_k\in Z_N}\left\langle h,\frac{D_{i_k}f_z(\mathbf{z}|\cdot)}{\langle D_{i_k}f_z(\mathbf{z}|\cdot),1\rangle}\right\rangle\right), \quad (19)$$

and the parameters of the corrected Gamma and Beta distributions are shown in Equations 14, 16 and 17 respectively.

V. Implementation

In this section, details of the implementation of the proposed filter are presented.

A. Robust Prediction

In the derivation of the R-KDMP filter, the predicted parameters for the distributions of P_D and λ_Θ do not change in time. Inspecting Equations 14, 16 and 17, the corrected parameters are increased by a positive amount at each time step. This results in a decrease in the variance of P_D and λ_Θ at each time step, which would not adapt to varying values of the probability of detection or the clutter rate. To avoid the problem of the distributions of P_D and λ_Θ reducing to Dirac delta functions, and to allow for time variability in the prediction step, the parameters of each Beta and Gamma component distribution are updated in such a way that their expected values are maintained, but their variances do not fall below predefined thresholds σ_B^2 and σ_G^2 respectively[4]. This is achieved by scaling the parameters of each distribution by constant factors γ_j and η_j. These values are derived such that the variances of the estimated values of λ_Θ and P_D never fall below the thresholds σ_G^2 and σ_B^2 respectively, while maintaining their expected values. This allows for a small degree of time-variability of these parameters.

For the Gamma distributions, both parameters are scaled by a factor γ_j[5]:

$$\gamma_j=\begin{cases}1 & \text{if } \mathrm{Var}(\mathrm{G}_j)>\sigma_G^2\\ \frac{\alpha_j}{\sigma_G^2\beta_j^2} & \text{otherwise.}\end{cases} \quad (20)$$

Similarly, both parameters of each Beta distribution are scaled by a factor η_j calculated as follows[6]:

$$\eta_j=\begin{cases}1 & \text{if } \mathrm{Var}(\mathrm{B}_j)>\sigma_B^2\\ \frac{a_jb_j-\sigma_B^2(a_j+b_j)^2}{\sigma_B^2(a_j+b_j)^3} & \text{otherwise.}\end{cases} \quad (21)$$

B. Poisson RFS Maintenance

In the KDMP filter, the measurement set is divided into three groups, clutter measurements, measurements from existing targets and measurements from new targets. By analysing Equation 12, it can be seen that the R-KDMP filter has to further subdivide the set of measurements of new targets

[4]It should be noted that, although this is a necessary implementation detail, strictly speaking the theoretical model assumes no change.

[5]Deriving a value for γ_j can be carried out as follows:

$$\mathrm{Var}(\mathrm{G}_j)=\frac{\gamma_j\alpha_j}{\gamma_j^2\beta_j^2}\geq\sigma_G^2\Rightarrow\frac{\alpha_j}{\sigma_G^2\beta_j^2}\geq\gamma_j.$$

[6]Deriving a value for η_j can be carried out as follows:

$$\mathrm{Var}(B_j)=\frac{\eta_j^2a_jb_j}{\eta_j^2(a_j+b_j)^2(\eta_ja_j+\eta_jb_j+1)}\geq\sigma_B^2$$
$$\frac{a_jb_j}{(a_j+b_j)^2\sigma_B^2}\geq(\eta_ja_j+\eta_jb_j+1)$$
$$\frac{a_jb_j-(a_j+b_j)^2\sigma_B^2}{(a_j+b_j)^3\sigma_B^2}\geq\eta_j.$$

into subsets corresponding to how long ago the target was born, or equivalently from which density $D_i(\mathbf{x})$ is the target initialised. This subdivision makes the number of components of the posterior mixture further increase combinatorially (as it requires the identification of all partitions of the set of measurements of new targets). By inspecting Equation 7, it can be seen that the influence of the i th component decreases exponentially with a factor of P_S (the expected number of elements in the ith component is multiplied by a factor less than one, P_S). This suggests that new targets arising from the density $D_i(\mathbf{x})$ for a large index i should be rare, as the expected number of elements of this density is very low. Following this idea, the number of components D_i is fixed, and for indices greater than a threshold ℓ are regarded as zero, $D_{i\geq\ell}(\mathbf{x}) = 0$. In this work, as a proof of concept and to maintain a low computational cost, the threshold ℓ is set to one component only.

C. R-KDMP Filtering Algorithm

In a manner similar to the KDMP algorithm [3], the R-KDMP algorithm determines the components of the R-KDMP mixture in decreasing order of weights now including their corresponding Beta and Gamma distribution parameters. To this end, it has to be noted that by taking the logarithm of the posterior weights, given in Equation (18), then determining the components with the highest weights, is a problem of finding the k-best optimal assignments [12]. The cost matrices from which the R-KDMP mixture components are extracted is of the form:

$$\mathbf{C}_i = \left(\begin{array}{c|c|c|c|c|c} \mathbf{Cl}^{(i)} & \mathbf{D}^{(i)} & \mathbf{N}_1^{(i)} & \cdots & \mathbf{N}_T^{(i)} & \infty \\ \hline \mathbf{0}_F & \mathbf{1}_D & \infty & \cdots & \infty & \mathbf{0}_D \\ \hline \mathbf{0}_D & \infty & \mathbf{0}_D & \cdots & \infty & \infty \\ \hline \vdots & \vdots & \cdots & \ddots & \vdots & \infty \\ \hline \mathbf{0}_D & \infty & \infty & \cdots & \mathbf{0}_D & \infty \\ \hline \infty & \neg\mathbf{S}^{(i)} & \infty & \cdots & \infty & \mathbf{S}^{(i)} \end{array}\right), \quad (22)$$

where i is the index from the prior component used to create this matrix, $\mathbf{0}_D$ is a matrix with zeros in all diagonal entries and ∞ otherwise, $\mathbf{0}_F$ is a matrix filled with zeros and $\mathbf{1}_D$ is a matrix with ones in the diagonal and ∞ otherwise. Intuitively, the sub-matrix $\mathbf{Cl}^{(i)}$ represents the costs of measurements being clutter, the sub-matrix $\mathbf{D}^{(i)}$ the costs of assigning measurements to existing targets, the sub-matrices $\neg\mathbf{S}^{(i)}$ and $\mathbf{S}^{(i)}$ the costs of targets not surviving and surviving respectively, and finally the sub matrices $\mathbf{N}_k^{(i)}$ the costs of initialising new targets using the unknown density $D_k(\mathbf{x})$. The individual entries (row j, column k) of the sub-matrices are calculated as follows:

$$\left[\mathbf{Cl}^{(i)}\right]_{j,k} = \begin{cases} -\log\left(f_\Theta(\mathbf{z}_j)\right) & j=k \\ \infty & \text{otherwise} \end{cases} \quad (23)$$

$$\left[\mathbf{D}^{(i)}\right]_{j,k} = -\log\left(\langle f_z(\mathbf{z}_j|\mathbf{x}_{t+1}) f_{k,i}, 1\rangle\right) \quad (24)$$

$$\left[\mathbf{N}_m^{(i)}\right]_{j,k} = \begin{cases} -\log\left(\langle D_m(\mathbf{x}_{t+1}) f_z(\mathbf{z}_j|\mathbf{x}_{t+1}), 1\rangle\right) & j=k \\ \infty & \text{otherwise} \end{cases} \quad (25)$$

$$\left[\mathbf{S}^{(i)}\right]_{j,k} = \begin{cases} -\log\left(P_S\right) & j=k \\ \infty & \text{otherwise} \end{cases} \quad (26)$$

$$\left[\neg\mathbf{S}^{(i)}\right]_{j,k} = \begin{cases} -\log\left(1-P_S\right) & j=k \\ \infty & \text{otherwise.} \end{cases} \quad (27)$$

Four differences can be observed between the R-KDMP and the standard KDMP cost matrices. The main difference is that there is no P_D or λ_Θ in the R-KDMP cost matrix. This is expected as these values are now part of the estimation problem. Given that there are no values for the detection statistics in the cost matrix, the miss-detection sub-matrix, referred to as $\mathbf{1}_D$ in (22), simplifies to a diagonal matrix with ones in the diagonal and ∞ otherwise. Similarly, instead of using the expected density of targets $D_\Theta(\mathbf{z})$, the clutter sub-matrix uses the spatial density $f_\Theta(\mathbf{z})$ to weight clutter measurements. In contrast to the single new target sub-matrix present in the KDMP filter, multiple sub-matrices corresponding to the different $D_i(\mathbf{x})$ densities from which targets could be born, now exist.

The R-KDMP algorithm also has to determine which sub-matrices are used to create a posterior component. This is to calculate the values of $|Z_C|$, $|Z_D|$, $|X_M|$, $|Z_N|$ and $\sum_k(i_k-1)$ required to update the parameters of the Beta and Gamma distributions. The number of entries in an optimal assignment using the clutter sub-matrix $\mathbf{Cl}^{(i)}$ corresponds to the value $|Z_C|$, while the number of entries of an optimal assignment using the detection sub-matrix $\mathbf{D}^{(i)}$ corresponds to the value $|Z_D|$. The number of miss-detected targets $|X_M|$ corresponds to the number of assignments using the $\mathbf{1}_D$ sub-matrix. Finally, the total number of assignments using any of the sub-matrices $N_m^{(i)}$, corresponds to the total number of new targets $|Z_N|$, while if measurement k is assigned to a new target using the sub matrix $N_{m'}^{(i)}$, then $i_k = m'$.

As mentioned earlier, in this work only one component of the Poisson RFS is used, namely $D_1(\mathbf{x})$. This keeps the dimension of the cost matrices low and maintains a low computational cost of the R-KDMP algorithm.

Finally, to model the state distribution of each target (functions $f_{j,k}(\mathbf{x})$ in Equation (5)), a Normal distribution is used. This results in the single-target update equations being equivalent to those in the Kalman filter, or Extended/Unscented Kalman filter if non-linear transition or observation models are used.

D. State Extraction

To extract the estimated multi-target state, the first moment, or density of targets will be used. Since the variable P_D affects the density of the Poisson components of the R-KDMP, it is not directly clear if the density of targets (modelled by the

KDMP component) is affected. The density of targets can be obtained using the following equation [11]:

$$D_X(\mathbf{x}) = \frac{\partial}{\partial \delta_{\mathbf{x}}} G_X[h]\bigg|_{h=1}. \tag{28}$$

To obtain a value for $G_X[h]$, the full PGFl $G_X[h|P_D, \lambda_\Theta]$ has to be integrated with respect to the variables P_D and λ_Θ:

$$\begin{aligned} G_X[h] &= \int\int G_X[h|P_D,\lambda_\Theta] d\lambda_\Theta dP_D & (29)\\ &= \sum_j^N \omega_j \left[\int e^{\sum_i^T (1-P_D)^i \langle D_i, h-1\rangle} \mathrm{Beta}(P_D|a_j,b_j) dP_D\right] \\ &\quad\times \left[\int \mathrm{Gamma}(\lambda_\Theta|\alpha_j,\beta_j) d\lambda_\Theta\right] \prod_k^{M_j} \langle f_{j,k}, h\rangle & (30)\\ &= \sum_j^N \omega_j \underbrace{\left[\int e^{\sum_i^T (1-P_D)^i \langle D_i, h-1\rangle} \mathrm{Beta}(P_D|a_j,b_j) dP_D\right]}_{s[h|a_j,b_j,D_0,\dots,D_T]} \\ &\quad\times \prod_k^{M_j} \langle f_{j,k}, h\rangle. & (31) \end{aligned}$$

The integral involving the probability of detection P_D in (31), does not have a closed-form solution. Using:

$$q(\mathbf{x}, a_j, b_j, D_0, \dots, D_T) = \frac{\partial}{\partial \delta_{\mathbf{x}}} s[h|a_j, b_j, D_0, \dots, D_T]\bigg|_{h=1},$$

and observing that $s[1|a_j, b_j, D_0, \dots, D_T] = 1$, the density of targets can be computed as follows:

$$\begin{aligned} D(\mathbf{x}) &= \frac{\partial}{\partial \delta_{\mathbf{x}}} G_X[h]\bigg|_{h=1} & (32)\\ &= \sum_j \omega_j \Bigg(q(\mathbf{x}, D_0, \dots, D_T) & (33)\\ &\quad + s[1|a_j, b_j D_0, \dots, D_T] \sum_k f_{j,k}(\mathbf{x}) \Bigg) & (34)\\ &= \underbrace{q(\mathbf{x}, D_0, \dots, D_T)}_{\text{Density of the Poisson component}} + \underbrace{\sum_{j,k} \omega_j f_{j,k}(\mathbf{x})}_{\text{Density of the KDMP component}}, & (35) \end{aligned}$$

with $q(\mathbf{x}, D_0, \dots, D_T) = \sum_j \omega_j q(\mathbf{x}, a_j, b_j, D_0, \dots, D_T)$[7]. It can be appreciated that the introduction of the Beta distribution does not affect the density of the KDMP component. In a similar manner to the standard KDMP filter, the density related to the unknown space, $q(\mathbf{x}, D_0, \dots, D_T)$, is discarded and the density of the Kronecker Delta, which models the known tracks is used to estimate the multi-target state. Since the density functions $f_{j,k}(\mathbf{x})$ are modelled as Normal distributions, the resulting density is a mixture of Gaussian distributions. To extract the multi-target state, Gaussian components with a weight greater than 0.5 are identified and reported to be part of the state.

[7]Furthermore, it is possible to show that:

$$q(\mathbf{x}, a_j, b_j, D_0, \dots, D_t) = \sum_i^T \frac{B(a_j, b_j + i)}{B(a_j, b_j)} D_i(\mathbf{x}),$$

with $B(a, b)$ being the beta function.

VI. Experimental Results

This section shows the performance of the proposed filter in estimating the state and detection statistics of a multi-target tracking problem. To do so, a simulation study is carried out. Figure 1 shows the x and y coordinates as a function of time of different tracks using different colours and the simulated measurements as black crosses.

The state of the targets is composed of their position and velocities in 2-dimensional space, $\mathbf{x}_t = [x_t, y_t, \dot{x}_t, \dot{y}_t]$. For proof of concept purposes, linear Gaussian motion and observation models are used. These models are described by the following motion (F) and observation (H) matrices:

$$F = \begin{pmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{36}$$

where Δt is the sampling time, and the process covariance Q and measurement covariance R matrices of the noise components are as follows:

$$Q = 0.2^2 I_{4\times4} + q \cdot q^\intercal \qquad R = 0.1^2 I_{2\times2},$$
$$q^\intercal = \begin{pmatrix} 0.5\Delta t^2 & 0 & \Delta t & 0 \\ 0 & 0.5\Delta t^2 & 0 & \Delta t \end{pmatrix},$$

with I being the identity matrix.

The expected number of false positive measurements per time step is 12 over the detection area of $20\text{m} \times 20\text{m}$, resulting in a clutter density of $3 \times 10^{-2}\text{m}^{-2}$. The probability of detection is set to 90%, and the probability of survival to 99%. The ground truth clutter rate and probability of detection values from which the data was created, are not available to the filter. The time step used to simulate the evolution of each target's state was $\Delta t = 0.1$s. To evaluate the performance of the R-KDMP filter, 100 Monte Carlo simulations were executed.

Figure 2 shows the estimated and ground truth clutter rate and probability of detection as functions of time. It can be appreciated that, for the clutter rate, the proposed filter converges to the true value, while for the probability of detection, a small bias, after approximately time step 150, exists. This effect seems to be due the filter not correctly differentiating between tracks disappearing and tracks not being detected. A deeper analysis of this behaviour will be the subject of further research.

Figure 3 shows the OSPA [13] errors incurred by the KDMP and the R-KDMP algorithms for the scenario with static detection statistics. Despite the fact that the parameters of the detection statistics must be estimated by the R-KDMP filter,

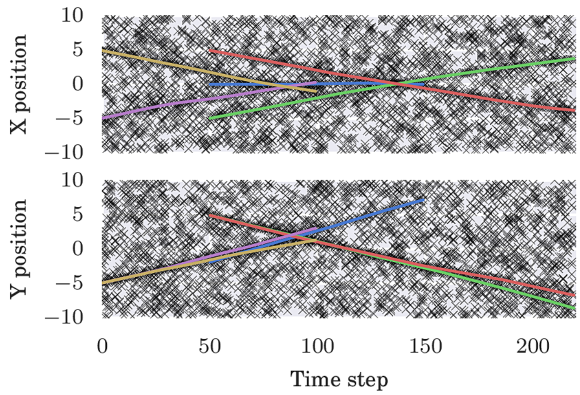

Fig. 1. Simulated environment. The black crosses represent the simulated measurements of the sensor, while the coloured lines represent the different ground truth tracks.

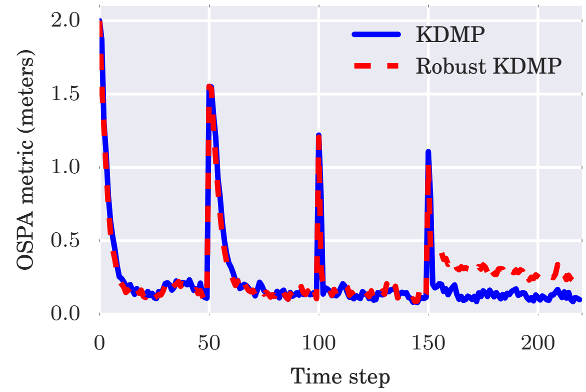

Fig. 3. OSPA error between the standard KDMP and the Robust KDMP algorithm for an environment with static detection statistics.

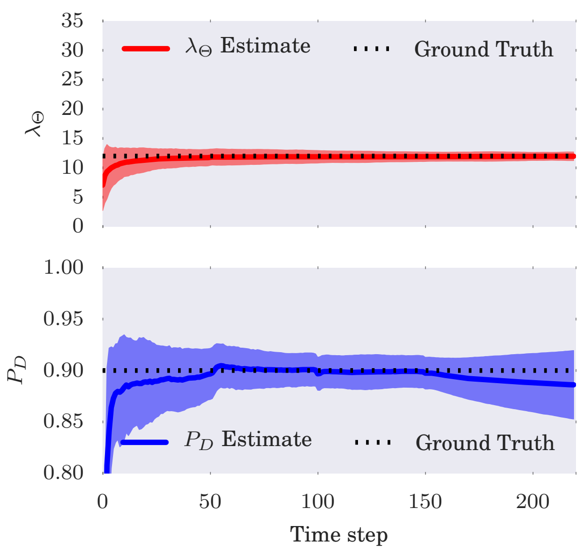

Fig. 2. Results for an environment with static detection statistics. The shaded area represents the standard deviation of the estimate. The mean and standard deviation were obtained using 100 Monte Carlo simulations.

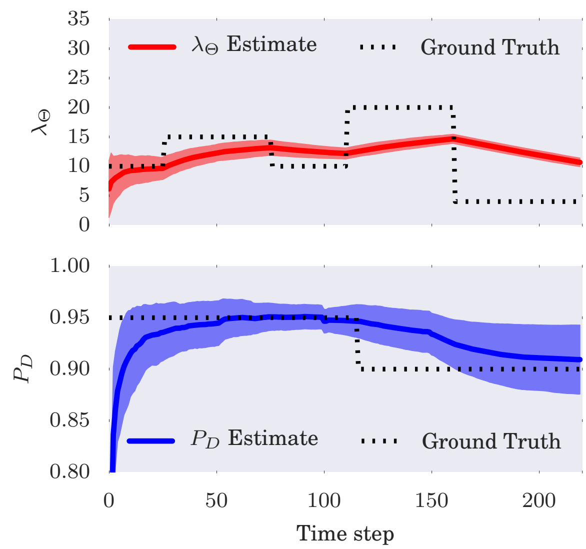

Fig. 4. Results for an environment with varying detection statistics. The shaded area represents the standard deviation of the estimate. The mean and standard deviation were obtained using 100 Monte Carlo simulations.

its OSPA errors for time steps less than 150 are comparable to the KDMP filter. For time steps greater that 150, the OSPA error is worse due to the bias shown in Figure 2.

To analyse how the filter adapts when the detection statistics vary with time, a scenario similar to the previous one is simulated except that the detection statistics now vary with time. The results are shown in Figure 4.

Despite the lack of a prediction model for the evolution of the detection statistics, it can be appreciated that the R-KDMP filter manages to adapt to changing detection statistics. Figure 4 shows that the clutter rate tends towards its ground truth values and a similar behaviour is observed for the estimated probability of detection. The main problem seen in these figures is that the lack of prediction models results in delays for the estimates to converge. An improved study of the evolution of the detection statistics and the incorporation of this knowledge into the corresponding prediction models should greatly improve the performance of the R-KDMP filter.

The OSPA error for the environment with varying detection statistics is shown in Figure 5. As the KDMP algorithm does not allow for changing detection statistics, the initial parameters of the detection statistics were used as parameters of the KDMP algorithm. As expected, the proposed R-KDMP filter outperforms the standard KDMP filter, as it adapts to the changing detection statistics, while the KDMP filter uses the static parameters provided at initialisation.

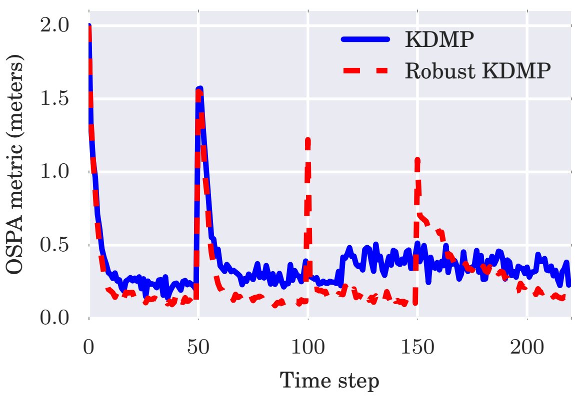

Fig. 5. OSPA error between the standard KDMP and the Robust KDMP algorithm for an environment with varying detection statistics.

VII. Conclusions

This paper extended the KDMP filter to estimate both the probability of detection and the clutter rate of a sensor and/or detection algorithm for multi-object tracking. To achieve this, the probability of detection is modelled with a Beta distribution while the clutter rate is modelled as a Gamma distribution. Both these distributions are the conjugate prior to the Bernoulli process, which models the detection or miss-detection of targets, and to the Poisson distribution, modelling the number of clutter measurements respectively. By using these models, the filter remains closed under both, the prediction and correction steps of the filtering process. Since, contrary to previous methods, the probability of detection is not included in the state to be estimated, the resulting filter manages to provide an estimate of the probability of detection even when no targets are present in the environment. Results show that the R-KDMP filter manages to correctly estimate both detection statistics, namely the clutter rate and the probability of detection, in scenarios where these values are unknown and even time varying.

As future work, the full cost matrix using multiple initialisation cost matrices $\mathbf{N}_i$ will be implemented. This will require a more detailed analysis into the performance of the algorithm, since using the complete cost matrix will increase the computational requirements. It will also be explored how to model the evolution of the detection statistics' parameters in time and to use these models in the prediction step of the filtering procedure. This work estimates both the probability of detection and clutter rate assuming their independence of the state and measurement space respectively. Two possible extensions are the estimation of the spatial clutter density, and a space-dependent probability of detection.

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