An algebraic description of supersymmetric neutral spin-(1/2) systems(*)

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Summary. — The linear-spectrum–generating algebra technique that has been used for confluent Natanzon potentials is extended to deal with supersymmetric neutral spin-(1/2) particles. The generators are exhibited.

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1. – Introduction

For some time the relation between the generators of the $SO(2,1)$ algebra and the differential equation for the hypergeometric functions has been known [1]. Using a realization in terms of a single variable, a linear combination of them leads to the confluent hypergeometric equation. The general hypergeometric case is obtained when the Casimir operator of the algebra is considered in a two-variable realization of $SO(2,1)$ algebra. These results have been successfully applied to the study [2]—from the algebraic point of view—of those potentials for which the Schrödinger equation reduces to a hypergeometric one, namely, the Natanzon potentials (NP) [3]. For these cases the $SO(2,1)$ algebra is a Spectrum Generating Algebra (SGA).

In this paper we construct a family of Pauli-Schrödinger equations (PSEs) for which the $SO(2,1)$ algebra is the SGA using a linear combination of the generators (LSGA). It

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is found that the subset of this family for which the solution can be exhibited (at least partially) includes a large number of neutral systems and the trivial case of a free charged particle. As a result, this work describes, algebraically, neutral spin-(1/2) systems that satisfy solvable PSEs .

In [4] it was shown that a class of neutral spin-(1/2) systems that satisfy a PSE can be exactly solved if the solution of an associated pair of spin-0 Schrödinger equations whose supersymmetric partners is known; this follows from the condition imposed for solvability of the PSE. This class is extended in [5] to include quasi-solvable systems [6]. The family we are interested in is characterized by the superpotential W of the associated spin-0 supersymmetric system in the sense that for each W the scalar potential V_s and the magnetic field $\mathbf{B}$ that define a Pauli-Schrödinger Hamiltonian, H, are constructed. This hints to the role played by supersymmetric quantum mechanics (SUSYQM) [7] in the determination of the solvable neutral spin-(1/2) cases.

The LSGA technique puts into correspondence the operator $H-E$, where E is the energy, with a linear combination of the generators. For a specific choice of the parameters the resulting second-order differential operator turns out—after a tilting, if necessary—to be compact thus leading to a discrete spectrum which will, as the same time, be required to be bounded below. This is the most interesting case from the physical point of view since it can be related to the discrete spectrum of H. Previous studies of the PSE using SUSYQM have looked for a companion PSE which plays the role of a supersymmetric partner [8]. In this paper a somehow different approach is adopted: the single PSE considered must be supersymmetric by itself, no companion PSE appears [4, 5].

The organization of this paper is the following: sect. **2** is a summary of LSGA for spin-0 systems and its relation to the confluent NP. Section **3** shows the construction of the $SO(2,1)$ generators, Casimir operator and the study of the master equation, whereas in sect. **4** the role of SUSYQM is discussed . In sect. **5** we exhibit a particular example and finally sect. **6** contains some comments on the relation of this technique to other cases of the PSE.

2. – Summary of LSGA and confluent NP

In this section we will review the algebraic method to treat the confluent Natanzon potentials [2].

The LSGA technique is based on the assumption that the Schrödinger equation can be written in terms of the generators J_α ($\alpha = 0, 1, 2$) of the $SO(2,1)$ algebra: $[J_0, J_1] = iJ_2$, $[J_1, J_2] = -iJ_0$, $[J_0, J_2] = -iJ_1$,

$$[H-E]\,\Psi = G(y)\,[2\,(1+\beta)\,J_0 + 2\,(1-\beta)\,J_1 + 4\zeta J_2 - \delta]\,\Psi = 0\,, \tag{2.1}$$

where $G(y)$ is a function fixed by consistency, β,ζ and δ are constants; usually one takes $\zeta = 0$. Equation (2.1) is called the master equation. In the study of (2.1), the following steps are: i) use a realization of the J_μ in terms of a pair of operators $\widehat{a}$ and $\widehat{b}$ that satisfy $[\widehat{a}, \widehat{b}] = 1$ and ii) use a specific realization of $\widehat{a}$ and $\widehat{b}$ in terms of functions of$y$ and $\frac{\mathrm{d}}{\mathrm{d}y}$. The realization is well known [1]:

$$4J_0 = -\widehat{a}^2 + (4Q + 3/4)\,\widehat{b}^{-2} + \widehat{b}^2\,, \tag{2.2}$$

$$4J_1 = -a^2 + (4Q + 3/4)\,\widehat{b}^{-2} - \widehat{b}^2\,, \tag{2.3}$$

$$4iJ_2 = 2\widehat{b}\widehat{a} + 1\,, \tag{2.4}$$

where Q is the Casimir of $SO(2,1)$ algebra, $Q = J_0^2 - J_1^2 - J_2^2$. The condition$[\widehat{a}, \widehat{b}] = 1$ together with the requirement that $\widehat{a}^2$ does not have $\frac{\mathrm{d}}{\mathrm{d}y}$ leads to

$$\widehat{a} = L(y)\frac{\mathrm{d}}{\mathrm{d}y} - \frac{1}{2}\frac{\mathrm{d}L(y)}{\mathrm{d}y}\,, \tag{2.5}$$

$$\widehat{b} = \int \frac{\mathrm{d}y}{L(y)}\,, \tag{2.6}$$

where $L(y)$ is an arbitrary function. It has been customary in the literature to set $L = 2\sqrt{h(y)}/h^{'}(y)$. Now the final step to obtain the confluent Natanzon potentials is to replace (2.5)and (2.6)in (2.2)-(2.4) and the result in (2.1). This leads to two equations: one involving the parameter E and the other independent of E. A formal derivation with respect to E is performed assuming that only Q, β and δ depend linearly on E; as a result, $h(y)$ satisfies

$$\frac{\mathrm{d}h(y)}{\mathrm{d}y} = \frac{2h(y)}{\sqrt{R}}\,, \tag{2.7}$$

where R is given by

$$R = s_2 h^2 + s_1 h + c_0 \tag{2.8}$$

with s_0, s_1 and c_0 constants. The resulting potential is the confluent Natanzon potential given by

$$V_{\mathrm{C}} = \frac{g_2 h^2 + g_1 h + \eta}{R} + \frac{s_1 h - s_2 h^2}{R^2} - \frac{5}{4}\frac{\Delta h^2}{R^3}\,, \tag{2.9}$$

where $\Delta = s_1^2 - 4s_2 c_0$ and g_2 , g_1 and η are constants. We follow the notation in [9].

3. – $SO(2,1)$ for spin-(1/2) systems

3˙1. *Generators.* – The realization of the generators as in (2.2) through (2.4) is not adequate to build up a PSE as in (2.5) and (2.6) because the Pauli matrices do not enter in a natural way. For this reason the algorithm of sect. **2** will be repeated but using operators $\widehat{a}$ and $\widehat{b}$ of the form (3.1) and (3.2), see below, which explicitly include the Pauli matrices. We assume then

$$\widehat{a} = A_\mu \sigma_\mu \frac{\mathrm{d}}{\mathrm{d}y} + B_\mu \sigma_\mu, \qquad \mu = 0, ..., 3\,, \tag{3.1}$$

$$\widehat{b} = C_\mu \sigma_\mu\,, \tag{3.2}$$

where σ_0 is the identity, σ_i the Pauli matrices and A_μ, B_μ and C_μ are twelve arbitrary functions of y. The relation $[\widehat{a}, \widehat{b}] = 1$ leads to $j_\mu \sigma_\mu \frac{\mathrm{d}}{\mathrm{d}y} + K_\mu = 1$, where the functions j_μ and K_μ are given by

$$K_0 \equiv A_0 C_0^{'} + \mathbf{A} \cdot C^{'} = 1, \qquad j_0 = 0\,, \tag{3.3}$$

$$\mathbf{K} \equiv i\mathbf{A} \times \mathbf{C}^{'} + A_0 \mathbf{C}^{'} + C_0^{'} \mathbf{A} + 2i\mathbf{B} \times C = 0\,, \tag{3.4}$$

$$\mathbf{j} \equiv 2i\mathbf{A} \times \mathbf{C} = 0\,. \tag{3.5}$$

These equations imply

$$\mathbf{A} = L\mathbf{C}\,, \tag{3.6}$$

$$A_0 = \frac{\mathbf{C}^2 C_0^{'}}{D}\,, \tag{3.7}$$

$$L = -\frac{\mathbf{C} \cdot \mathbf{C}^{'}}{D}\,, \tag{3.8}$$

$$\mathbf{B} = M\mathbf{C} + \frac{1}{2} L\mathbf{C}^{'} + \frac{i}{2} \frac{A_0}{\mathbf{C}^2} \mathbf{C} \times \mathbf{C}^{'}\,, \tag{3.9}$$

where M is an arbitrary function and D is given by

$$D = C_0'^2 \mathbf{C}^2 - (\mathbf{C} \cdot \mathbf{C}')^2 \,. \tag{3.10}$$

The above results leave six arbitrary functions, namely: C_μ, B_0 and M. From (2.4) the generator J_2 is given by

$$J_2 = F_\mu \sigma_\mu \frac{\mathrm{d}}{\mathrm{d}y} + G_\mu \sigma_\mu \,, \tag{3.11}$$

where the coefficients F_μ and G_μ are

$$F_0 = -\frac{i}{2}(C_0 A_0 + \mathbf{C} \cdot \mathbf{A}) \,, \tag{3.12}$$

$$\mathbf{F} = -\frac{i}{2}(C_0 \mathbf{A} + A_0 \mathbf{C}) \,, \tag{3.13}$$

$$G_0 = -\frac{i}{4}(2C_0 B_0 + 2\mathbf{C} \cdot \mathbf{B} + 1) \,, \tag{3.14}$$

$$\mathbf{G} = -\frac{i}{2}(C_0 \mathbf{B} + B_0 \mathbf{C} + i\mathbf{C} \times \mathbf{B}) \,. \tag{3.15}$$

The result for $\widehat{a}^2$, needed for the construction of J_0 and J_1, eqs. (2.2) and (2.3), is

$$\widehat{a}^2 = T_\mu \sigma_\mu \frac{\mathrm{d}^2}{\mathrm{d}y^2} + S_\mu \sigma_\mu \frac{\mathrm{d}}{\mathrm{d}y} + P_\mu \sigma_\mu \,, \tag{3.16}$$

where

$$T_0 = A_0^2 + \mathbf{A}^2 \,, \tag{3.17}$$

$$\mathbf{T} = 2A_0 \mathbf{A} \,, \tag{3.18}$$

$$S_0 = A_0 A_0' + \mathbf{A} \cdot A' + 2A_0 B_0 + 2\mathbf{A} \cdot B \,, \tag{3.19}$$

$$\mathbf{S} = A_0 \mathbf{A}' + i\mathbf{A} \times \mathbf{A}' + 2A_0 \mathbf{B} + 2B_0 \mathbf{A} + A_0' \mathbf{A} \,, \tag{3.20}$$

$$P_0 = A_0 B_0' + \mathbf{A}\cdot B' + B_0^2 + \mathbf{B}^2 \,, \tag{3.21}$$

$$\mathbf{P} = A_0 \mathbf{B}' + B_0' \mathbf{A} + 2B_0 \mathbf{B} + i\mathbf{A}\times B' \,. \tag{3.22}$$

For $\widehat{b}^2$ and $\widehat{b}^{-2}$ we easily get

$$\widehat{b}^2 = C_0^2 + \mathbf{C}^2 + 2C_0 \mathbf{C}\cdot\boldsymbol{\sigma} \,, \tag{3.23}$$

$$\widehat{b}^{-2} = \frac{C_0^2 + \mathbf{C}^2 - 2C_0\mathbf{C}\cdot\boldsymbol{\sigma}}{(C_0^2 - \mathbf{C}^2)^2} \,. \tag{3.24}$$

3˙2. *The Casimir operator*. – To complete the computation of J_0 and J_1 the explicit form of the Casimir operator Q is needed; since $Q = J_0^2 - J_1^2 - J_2^2$ is an identity, it will be assumed that Qhas the following form:

$$Q = q_\mu \sigma_\mu \,, \tag{3.25}$$

where q_0 and $\mathbf{q}$ are functions of y to be determined from $[Q, J_\alpha] = 0, \ \alpha = 0, 1, 2$. From$[Q, J_2] = 0$ it is found

$$\begin{aligned} &\frac{i}{2}(C_0 A_0 + L\mathbf{C}^2) q_0' + \frac{i}{2}(C_0 L + A_0)\mathbf{C}\cdot\mathbf{q}' + (C_0 L + A_0)\mathbf{q}\times\mathbf{C}\cdot\boldsymbol{\sigma}\frac{\mathrm{d}}{\mathrm{d}y} = \\ &= \left[\frac{1}{2}(C_0 L + A_0)\mathbf{C}\times\mathbf{q}' - \frac{i}{2}(C_0 A_0 + L\mathbf{C}^2)\mathbf{q}' - \frac{i}{2}q_0'(C_0 L + A_0)\mathbf{C}\right]\cdot\boldsymbol{\sigma} - \\ &\qquad - [\mathbf{q}\times(C_0\mathbf{B} + B_0\mathbf{C} + i\mathbf{C}\times B)]\cdot\boldsymbol{\sigma} \,. \end{aligned} \tag{3.26}$$

The vanishing of the coefficient of $\frac{\mathrm{d}}{\mathrm{d}y}$ implies

$$(C_0 L + A_0)\mathbf{C}\times q = 0 \tag{3.27}$$

with the two alternatives

$$\mathbf{q} = k\mathbf{C} \,, \tag{3.28 a}$$

$$C_0 L + A_0 = \mathbf{C}^2 C_0' - C_0 \mathbf{C}\cdot C' = 0 \,. \tag{3.28 b}$$

Alternative *b*) fixes C_0 in terms of $\mathbf{C}$ and $\mathbf{C}'$ and leaves $\mathbf{q}$ arbitrary. The rest of this paper uses *a*) for simplicity. Going back to (3.25) leads to the set of equations ($R \equiv C_0 A_0 + \mathbf{A}\cdot C$; $S \equiv LC_0 + A_0$)

(3.29 *a*) $$q_0^{'}R + \mathbf{q}^{'}\cdot\mathbf{C}S = 0\,,$$

(3.29 *b*) $$\mathbf{q}^{'}R + q_0^{'}S\mathbf{C} + iS\mathbf{C}\times q^{'} + 2iC_0\mathbf{B}\times q + 2\mathbf{q}\times(C\times B) = 0\,.$$

Multiplying (3.29 *b*) scalarly by $\mathbf{C}$, one gets

(3.30 *a*) $$q_0^{'}R + \mathbf{q}^{'}\cdot\mathbf{C}S = 0\,,$$

(3.30 *b*) $$\mathbf{q}^{'}\cdot\mathbf{C}R + q_0^{'}S\mathbf{C}^2 = 0$$

The determinant of this system (in $q_0^{'}$ and $\mathbf{q}^{'}\cdot\mathbf{C}$) is $\Lambda = (C_0^2 - \mathbf{C}^2)(A_0^2 - \mathbf{A}^2)$ which will be required to be non-zero so that the master equation is well defined, see next subsection, eq. (3.33). As a result

(3.31) $$q_0^{'} = 0 \quad , \quad \mathbf{q}^{'}\cdot\mathbf{C} = 0 = k^{'}\mathbf{C}^2 + k\mathbf{C}\cdot\mathbf{C}^{'}$$

so that q_0 = constant and $k = f/\sqrt{\mathbf{C}^2}$, where f is a constant. After this result the vector equation (3.29 *b*) is satisfied identically. To complete the calculation the commutators of the Casimir operator with the two remaining generators have to be checked; they separate into two terms:

$$[\widehat{b}^2, Q] \quad and \quad \left[-\widehat{a}^2 + (4Q + \frac{3}{4})\widehat{b}^{-2}\right]\,,$$

since $\widehat{b}^2$, $\widehat{b}^{-2}$ and Q depend on $\mathbf{C}\cdot\boldsymbol{\sigma}$, we have $[\widehat{b}^2, Q] = [\widehat{b}^{-2}, Q] = 0$; therefore, the only term to look at is

$$\begin{aligned}\left[\widehat{a}^2, Q\right] = (T_0 + \mathbf{T}\cdot\sigma)\mathbf{q}^{''}\cdot\boldsymbol{\sigma} + (S_0 + \mathbf{S}\cdot\sigma)\mathbf{q}^{'}\cdot\boldsymbol{\sigma} + 2i\mathbf{P}\times\mathbf{q}\cdot\boldsymbol{\sigma} + \\ + \left\{2(T_0\mathbf{q}^{'}\cdot\boldsymbol{\sigma} + i\mathbf{T}\times\mathbf{q}^{'}\cdot\boldsymbol{\sigma}) + 2i\mathbf{S}\times\mathbf{q}\cdot\boldsymbol{\sigma}\right\}\frac{\mathrm{d}}{\mathrm{d}y}\,,\end{aligned}$$

which is shown to vanish after a straightforward calculation.

Since there are no restrictions on the functions M,B_0 and C_μ, there are still six arbitrary functions. For future reference it is worthwhile to point out that if a separation parameter —in the next section the separation parameter is the energy eigenvalue— is introduced, then the Casimir operator is written in the following form:

(3.32) $$Q = (q_0 + Eq_1) + (f_0 + Ef_1)\frac{\mathbf{C}\cdot\boldsymbol{\sigma}}{\sqrt{\mathbf{C}^2}}\,,$$

where q_0, q_1, f_0 and f_1 are constants. Notice that the spectrum of Q is $(q_0 + Eq_1) \pm (f_0 + Ef_1)$.

3˙3. *The master equation.* – This equation relates the second-order differential operator of the PSE to the generators of the $SO(2,1)$ algebra, the relation is

$$G(y)(-\widehat{a}^2 + (4Q + \frac{3}{4})\widehat{b}^{-2} + \beta\widehat{b}^2 + 4\zeta J_2 - \delta) = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + V - E\,, \tag{3.33}$$

where $G(y)$ is in this case a 2×2 operator matrix determined so that the coefficient of the second derivative is the same in both sides of (3.33). Using the result given in eq. (3.16), one concludes that $G(y)$ is then given by

$$G(y) = \frac{T_0 - \mathbf{T}\cdot\boldsymbol{\sigma}}{T_0^2 - \mathbf{T}^2}\,. \tag{3.34}$$

Now it is assumed that the Casimir operator, β, ζ and δ are linear functions of the energy E:

$$\begin{cases} Q \;=\; (q_0 + Eq_1) + (f_0 + Ef_1)\dfrac{\mathbf{C}\cdot\boldsymbol{\sigma}}{\sqrt{\mathbf{C}^2}}\,, \\ Q \;\equiv\; Q_0 + EQ_1\,, \\ \beta \;=\; \beta_0 + \beta_1 E\,, \\ \delta \;=\; \delta_0 + \delta_1 E\,, \\ \zeta \;=\; \zeta_0 + \zeta_1 E\,. \end{cases} \tag{3.35}$$

It is convenient at this point to rewrite the expression of the Casimir operator in terms of two vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ defined as

$$\mathbf{g}_1 = f_0\frac{\mathbf{C}}{\sqrt{\mathbf{C}^2}}\,, \qquad \mathbf{g}_2 = f_1\frac{\mathbf{C}}{\sqrt{\mathbf{C}^2}}\,, \tag{3.36}$$

therefore

$$Q_0 = q_0 + \mathbf{g}_1\cdot\boldsymbol{\sigma}\,, \quad Q_1 = q_1 + \mathbf{g}_2\cdot\boldsymbol{\sigma}\,. \tag{3.37}$$

The energy E is considered as a separation parameter so that (3.33) splits into two equations, each one the coefficient of E^θ, $\theta = 0, 1$, obtaining for $\theta = 1$

$$T_0 + \mathbf{T}\cdot\boldsymbol{\sigma} = \delta_1 - \beta_1\widehat{b}^2 - 4Q_1\widehat{b}^{-2} - 4\zeta_1(F_\mu\sigma_\mu\frac{\mathrm{d}}{\mathrm{d}y} + G_\mu\sigma_\mu)\,. \tag{3.38}$$

Substitution of eqs. (3.16) through (3.23) into (3.38) leads to the following four equations:

$$T_0 = \delta_1 - \beta_1(C_0^2 + \mathbf{C}^2) - \frac{4q_1(C_0^2 + \mathbf{C}^2) - 8C_0\mathbf{g}_2\cdot\mathbf{C}}{(C_0^2 - \mathbf{C}^2)^2} - 4\zeta_1(F_0\frac{\mathrm{d}}{\mathrm{d}y} + G_0)\,, \tag{3.39}$$

$$\mathbf{T} = -2\beta_1 C_0 \mathbf{C} + \frac{8q_1 C_0 \mathbf{C} - 4(C_0^2 + \mathbf{C}^2)\mathbf{g}_2}{(C_0^2 - \mathbf{C}^2)^2} - 4\zeta_1 (\mathbf{F}\frac{\mathrm{d}}{\mathrm{d}y} + \mathbf{G})\,. \tag{3.40}$$

Since there are no terms proportional to $\frac{\mathrm{d}}{\mathrm{d}y}$ in T_0 or $\mathbf{T}$, see (3.17) and (3.18), ζ_1 must vanish. This implies that the vector equation reduces to only one since both sides are proportional to $\mathbf{C}$. The set of four equations amounts only to two conditions thus leaving four arbitrary functions.

At this point we are in a position to write down the potential V that comes from the energy independent part ($\theta = 0$) of eq. (3.33) and is given by

$$V = V_{\rm sc} + \boldsymbol{\sigma} \cdot \boldsymbol{\Sigma} \tag{3.41}$$

where the scalar and spin parts of V, $V_{\rm sc}$ and $\boldsymbol{\Sigma}$ are

$$V_{\rm sc} = \frac{T_0\alpha_0 - \mathbf{T}\cdot\boldsymbol{\alpha}}{T_0^2 - \mathbf{T}^2}\,, \tag{3.42}$$

$$\boldsymbol{\Sigma} = -\frac{T_0\boldsymbol{\alpha}\mathbf{T}\alpha_0 + i\mathbf{T}\times\boldsymbol{\alpha}}{T_0^2 - \mathbf{T}^2} \tag{3.43}$$

with α_0 and $\boldsymbol{\alpha}$ defined as follows:

$$\begin{aligned}\alpha_0 = {} & (4\xi_0 F_0 - S_0)\frac{\mathrm{d}}{\mathrm{d}y} + \beta_0(C_0^2 + \mathbf{C}^2) - P_0 + 4\xi_0 G_0 + \\ & + \frac{(4q_0 + \frac{3}{4})(C_0^2 + \mathbf{C}^2)}{(C_0^2 - \mathbf{C}^2)^2} - \frac{8C_0\mathbf{g}_1\cdot\mathbf{C}}{(C_0^2 - \mathbf{C}^2)^2} - \delta_0\,,\end{aligned} \tag{3.44}$$

$$\begin{aligned}\boldsymbol{\alpha} = {} & (4\xi_0\mathbf{F} - \mathbf{S})\frac{\mathrm{d}}{\mathrm{d}y} + 2\beta_0 C_0\mathbf{C} - \mathbf{P} + 4\xi_0\mathbf{G} - \\ & - \frac{2C_0(4q_0 + \frac{3}{4})}{(C_0^2 - \mathbf{C}^2)^2}\mathbf{C} + \frac{4(C_0^2 + \mathbf{C}^2)\mathbf{g}_1}{(C_0^2 - \mathbf{C}^2)^2}\,.\end{aligned} \tag{3.45}$$

We remind the reader that in the formalism there are still four arbitrary functions. If these functions are given as input, eqs. (3.42) and (3.43) define completely the potential which may contain first derivatives. On the other hand, if $V_{\rm sc}$ and $\boldsymbol{\Sigma}$ are given as an input, then the problem is much more complicated since one should invert (3.42) and (3.43) to obtain the four arbitrary functions. However, the terms with first derivatives must be equal on both sides and the same for those without first derivatives. This amount to eight equations for the four unknown functions. Only those cases in which this overdetermined system is satisfied correspond to the solution of the problem.

4. – Supersymmetry enters the game

Up to this point supersymmetry (SUSYQM) has not played any role. To bring SUSYQM into play define the following operator:

$$\Gamma = i\sigma_2 \frac{\mathrm{d}}{\mathrm{d}y} + W\sigma_1 \,, \tag{4.1}$$

where W is the superpotential. The Hamiltonian is constructed as

$$H_{\mathrm{SUSY}} = \Gamma^2 = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + W^2 + W^{'}\sigma_3 \,, \tag{4.2}$$

which is the well-known form of a SUSYQM problem for a pair of Schrödinger Hamiltonians $H_\pm$ which are supersymmetric partners $\left(H_\pm = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + W^2 \pm W^{'}\right)$. It will be assumed in the following that the solution to this problem is known (eigenvalues and eigenfunctions), at least partially.

Now introducing the Hamiltonian (k, d constants) [5]

$$H_{\mathrm{SUSY,PS}} = (\Gamma + k)^2 + d\sigma_3 \tag{4.3}$$

and writing it explicitly a Pauli-Schrödinger equation is obtained. In fact

$$H_{\mathrm{SUSY,PS}} = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + 2i\sigma_2 k \frac{\mathrm{d}}{\mathrm{d}y} + v_{\mathrm{s}}(y) + B(y)\left[\sin(2\tau(y))\sigma_1 + \cos(2\tau(y))\sigma_3\right] \,, \tag{4.4}$$

where the functions $v_{\mathrm{s}}(y)$, $B(y)$ and $\tau(y)$ are determined by the superpotential W as

$$v_{\mathrm{s}}(y) = W^2 + k^2 \,, \tag{4.5}$$

$$B(y)\sin(2\tau(y)) = 2Wk \,, \tag{4.6}$$

$$B(y)\cos(2\tau(y)) = W^{'} + d \,. \tag{4.7}$$

The eigenfunctions of $H_{\mathrm{SUSY,PS}}$ are

$$\Phi_n(y) = \left[\begin{array}{c} \chi^+_{n-1}(y)\cos\alpha \\ \chi^-_n(y \sin\alpha \end{array} \right] \,, \tag{4.8}$$

where $\left[\begin{array}{c} \chi^+_{n-1}(y) \\ \chi^-_n(y) \end{array} \right]$ are eigenfunctions of Γ^2, the constant α is given by

$$\tan\alpha^{\pm} = \frac{2kE_n}{d \pm (d^2 + 4k^2 E_n)^{1/2}} \tag{4.9}$$

and the spectrum is found to be

$$E_n^{\pm} = E_n + k^2 \pm (d^2 + 4k^2 E_n)^{1/2}\,, \tag{4.10}$$

where $\{E_n\}$ is the spectrum of Γ^2 .

After a rotation in spin space [5] to define $\Psi(y) = \exp[2iky\sigma_2]\Phi(y)$ the well-known PSE for a neutral spin-(1/2) system is obtained:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + v_{\mathrm{s}}(y) + B(y)\left[\cos(2\beta(y))\sigma_3 + \sin(2\beta(y))\sigma_1\right] - E\right)\Phi(y) = 0\,, \tag{4.11}$$

where $\beta(y) = 2(ky + \tau(y))$, see [4], which will be considered as the PSE in a fixed reference frame while (4.4) is the same equation in a rotated frame. Equation (4.11) will be called a supersymmetric spin-(1/2) system.

The next step consists in writing the Hamiltonian of (4.11) in terms of the $SO(2,1)$ generators constructed in sect. **3**. The situation in dealing with (4.11) is that V_{sc} and $\mathbf{\Sigma}$ are given in (4.5) through (4.7). This information must be used as input in (3.42) and (3.43) and from these equations the four arbitrary functions must be determined. Since we have not been able to invert this system of equations we present in the next section an example in order to show that the scheme is not empty.

5. – An example

Let us assume $\mathbf{C} = (C_1, 0, C_3)$, where C_1 and C_3 are arbitrary constants, for simplicity we choose $|\mathbf{C}| = 1$. This implies

$$\mathbf{A} = 0\,, \quad A_0 = \frac{1}{C_0'}\,, \quad L = 0\,. \tag{5.1}$$

For F_μ, G_μ , S_μ , P_μ and T_μ, $(\mu = 0, ..., 3)$ we obtain from (3.12)-(3.22)

$$F_0 = -\frac{1}{2}\frac{iC_0}{C_0'}\ , \quad \mathbf{F} = -\frac{1}{2}\frac{i\mathbf{C}}{C_0'}\,, \tag{5.2}$$

$$G_0 = -\frac{i}{4}(2C_0 B_0 + 2M + 1)\,, \quad \mathbf{G} = -\frac{i}{2}(C_0 M + B_0)\mathbf{C}\,, \tag{5.3}$$

$$S_0 = -\frac{C_0''}{C_0'^3} + 2\frac{B_0}{C_0'}\,, \quad \mathbf{S} = 2\frac{M}{C_0'}\mathbf{C}\,, \tag{5.4}$$

$$P_0 = \frac{B_0^{'}}{C_0^{'}} + B_0^2 - \xi_0^2 \,, \qquad \mathbf{P} = -2i\xi_0 B_0 \mathbf{C} \,, \tag{5.5}$$

$$T_0 = \frac{1}{C_0^{'2}} \,, \quad \mathbf{T} = \mathbf{0} \,. \tag{5.6}$$

Now we turn to the master equation; with the above results it is found from (3.39) and (3.40),

$$\frac{1}{C_0^{'2}} = \delta_1 - \beta_1(C_0^2 + 1) - \frac{4q_1(C_0^2+1) - 8C_0 f_1}{(C_0^2-1)^2} \,, \tag{5.7}$$

$$\mathbf{0} = (-2\beta_1 C_0 + \frac{8q_1 C_0 - 4(C_0^2+1)f_1}{(C_0^2 - \mathbf{C}^2)^2})\mathbf{C} \,. \tag{5.8}$$

A simple way to satisfy this last couple of equation is with the choice

$$\beta_1 = q_1 = f_1 = 0 \,, \tag{5.9}$$

so that

$$C_0 = \frac{y}{\sqrt{\delta_1}} \,. \tag{5.10}$$

If no first-order derivatives should appear in the scalar and spin parts of the potential V, eqs. (3.41) and (3.42), S_μ and F_μ must satisfy the following relations:

$$S_0 = 4\xi_0 F_0 \,, \qquad \mathbf{S} = 4\xi_0 \mathbf{F} \tag{5.11}$$

that fix B_0 as a function of C_0 and M in terms of ξ_0, namely

$$B_0 = \frac{1}{2}\left[\frac{-2iC_0}{C_0^{'}} + \frac{C_0^{''}}{C_0^{'2}}\right] C_0^{'} \,, \qquad M = -i\xi_0 \,. \tag{5.12}$$

We can make further assumptions if we look at the expression for α_0 and $\boldsymbol{\alpha}$, eqs. (3.43) and (3.44), since both contain terms of the following form:

$$\frac{4q_0 + \frac{3}{4}}{\left(\frac{y^2}{\delta_1} - 1\right)^2} \quad and \quad \frac{f_0}{\left(\frac{y^2}{\delta_1} - 1\right)^2}$$

which give rise to complicated potentials; we assume then, for simplicity

(5.13) $$q_0 = -\frac{3}{16}\,, \quad f_0 = 0\,;$$

with these choices one finds for $V_{\rm sc}$, see (3.39),

(5.14) $$V_{\rm sc} = \frac{(\beta_0 - \xi_0^2)}{\delta_1^2}\, y^2 + \frac{\beta_0 - \xi_0^2 - \delta_0}{\delta_1},$$

and the spin part $\mathbf{\Sigma}$, (3.40), is then

(5.15) $$\mathbf{\Sigma} = -2\frac{(\beta_0 + \xi_0^2)}{\delta_1^{3/2}} y\mathbf{C}\,,$$

The superpotential W and k is obtained from (4.5) after identifying $v_{\rm s}(y)$ with $V_{\rm sc}$, given in (5.14):

(5.16 *a*) $$W = \frac{\sqrt{(\beta_0 - \xi_0^2)}}{\delta_1}\, y\,,$$

(5.16 *b*) $$k^2 = \frac{\beta_0 - \xi_0^2 - \delta_0}{\delta_1}\,.$$

With these results the magnetic field $B(y)$ and $\tau(y)$ are easily evaluated from (4.6) and (4.7). From (3.1) and (3.2) the operators $\widehat{a}$ and $\widehat{b}$ are found to be

(5.17 *a*) $$\widehat{a} = \sqrt{\delta_1}\frac{\mathrm{d}}{\mathrm{d}y} - i\xi_0\widehat{b}\,,$$

(5.17 *b*) $$\widehat{b} = \frac{y}{\sqrt{\delta_1}} + C_1\sigma_1 + C_3\sigma_3\,.$$

Then from eqs. (2.2)-(2.4) the generators are easily obtained.

Consider now the following particular case: $d = -W'$ and $C_3 = 0$. The magnetic field is in this case $\mathbf{B} = 2kW\widehat{i}$ in the rotatory frame, see (4.4). In the fixed frame, eq. (4.11), the magnetic field $\mathbf{B}_{\rm fix} = 2kW\left[\sin(2(ky+\tau))\,\widehat{i} + \cos(2(ky+\tau))\,\widehat{k}\,\right]$ rotates around the y-axis with step $\frac{2\pi}{k}$. The scalar potential is harmonic, see example ii) of [4]. As a curiosity, the Casimir operator for this example is the same as for the usual harmonic oscillator.

6. – Final comments

In sect. **3** and **4** we have shown how the LSGA technique is used to algebraize a precisely defined family of neutral spin-(1/2) PSEs. In this way an explicit solution to all cases included is exhibited in terms of the superpotential W of a pair of Schrödinger partner Hamiltonians. With W as input also the generators of the $SO(2,1)$ algebra—which is the SGA for the specific problem at hand—are also shown to be defined; the algebrization of the case in point is taken in this sense.

As a final comment, if only the $SO(2,1)$ algebra is considered —namely, only the results of sect. **3**— then the set of differential equations is much larger than the one we have studied and the possibility is open to algebrize the PSE for a charged system under the influence of both a scalar potential and a magnetic field. Thus the case included in de Crombrugghe and Rittenberg (see [8]) with a unique supersymmetric charge

$$\Lambda = \phi + (\mathbf{p} - e\mathbf{Z}) \cdot \boldsymbol{\sigma} \tag{6.1}$$

and the Hamiltonian given by $2H = \Lambda^2$ is seen to be a particular case of (3.41). For those choices of the scalar ϕ and vector potential $\mathbf{Z}$ which satisfy (3.42) and (3.43) the system (6.1) is algebrized by our technique. Particular cases of (6.1) have been studied in the literature [10].

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