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Analysis of the spectrum generating algebra method for obtaining energy spectra

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We analyze and clarify how the SGA (spectrum generating algebra) method has been applied to different potentials. We emphasize that each energy level E_ν obtained originally by Morse belongs to a *different* $\mathfrak{so}(2,1)$ multiplet. The corresponding wave functions Ψ_ν are eigenfuntions of the compact generators J_0^ν with the same eigenvalue k_0, but with different eigenvalues q_ν of the Casimir operators Q. We derive a general expression for all effective potentials which have $\Psi_{\lambda_\nu,\nu+m}(r) \propto (J_+^\nu)^m \Psi_{\lambda_\nu,\nu}(r)$ as eigenfunctions, without using supersymmetry formalism. The different actions of SGA is further illustrated by two diagrams. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1895106]

I. INTRODUCTION

Pauli[1] in 1926 was the first person who calculated the energy spectrum of a Hamiltonian *algebraically*. He did it for the hydrogen atom. Since then his procedure was followed by many people. It is essentially based on relating the *total* Hamiltonian H to the Casimir operator of the *symmetry or degeneracy algebra*, whose generators commute with H. [The symmetry algebra of the N-dimensional hydrogen atom is usually identified differently for different energies E. By replacing the Hamiltonian H by its eigenvalues, one obtains $\mathfrak{so}(N+1)$, $\mathfrak{e}(N)$, and $\mathfrak{so}(N,1)$, for $E<0$, $E=0$, and $E>0$, respectively.[2–4] However, by keeping H as operator, one obtains infinite-dimensional Kac–Moody loop algebras, of standard type for even N and of twisted type for odd N (Refs. 5 and 6).]

The hydrogen atom and the isotropic oscillator have infinite number of states for each eigenvalue of angular momentum ℓ. Their symmetry algebras for the bound states are compact and therefore the raising and lowering operators of these algebras can only generate a *finite* number of states with different values of ℓ, for each degenerate energy eigenvalue E_n. To relate the infinite number of states for a fixed ℓ it is necessary to use noncompact algebras. The smallest appropriate noncompact algebra is $\mathfrak{so}(2,1)$, because it has a infinite-dimensional representations which are bounded from below, and are denoted by $D^+(\lambda)$. The $\mathfrak{so}(2,1)$ is generated by three generators K_i which commute as follows:

$$
\begin{aligned}
[K_0,K_1] &= iK_2,\\
[K_2,K_0] &= iK_1, \qquad (1)\\
[K_1,K_2] &= -iK_0.
\end{aligned}
$$

The K_0 is the compact generator, while K_1 and K_2 are the noncompact ones. Once a simultaneous eigenstate of K_0 and Q is found, then one can generate an infinite number of states by applying the raising and lowering operators $K_\pm$ repeatedly. Because of this property $\mathfrak{so}(2,1)$ has been called a

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[a)] On sabbatical leave from the Physics Department, Ben Gurion University of the Negev, 84105 Beer Sheva, Israel. Electronic mail: daboul@bgu.ac.il *spectrum-generating algebra* (SGA).

The idea of using the SGA was very popular in the 1960s and 1970s and numerous papers[7] investigated different approaches and potentials. In fact, Wybourne devoted a whole chapter of his popular book (Ref. 8, Chap. 18) to describe how one obtains the energy levels of various systems from the $\mathfrak{so}(2,1)$ SGA.

In particular, $\mathfrak{so}(2,1)$ has been applied to obtain the S-wave bound states for a important subclass of Natanzon potentials,[9] which has only a finite number of bound states, such as the Morse potential. The question arises, *where are the other infinitely many energy levels that can be generated by* $K_\pm$?

The purpose of the present paper is to show that in the many applications of $\mathfrak{so}(2,1)$ involving Natanzon-type potentials,[10] the raising and lowering operators $K_\pm$ were actually never used to generate the finite number of bound states for $\ell=0$. Instead, it turned out that for every energy level, a different $\mathfrak{so}(2,1)$ representation was used.

For clarity, we shall concentrate on the Morse potential,

$$V_M(r) := V_0[e^{-2(r-r_0)/a} - 2e^{-(r-r_0)/a}], \tag{2}$$

which is the simplest example of a Natanzon potential which has a finite number of bound states for each ℓ. The potential (2) was introduced by Morse in 1929 to obtain the vibrational levels of diatomic molecules.[11] He obtained only the S-wave solutions, by explicitly solving the Schrödinger equation. The Morse potential has since become popular, especially among chemists, because it allows disintegration of diatomic molecules, in contrast to the shifted harmonic-oscillator potential $(k/2)(r-r_0)^2$.

In 1970 Cordero and Hojman[12] (hereafter will be quoted as CH) reproduced algebraically the finite S-wave energy spectrum for the *Morse potential,*[11] by using $\mathfrak{so}(2,1)$. Therefore we shall analyze this paper in particular, and show that CH did not use a single $\mathfrak{so}(2,1)$ representation to get the energy levels, but actually a different $\mathfrak{so}(2,1)$ representation for each energy level. We explain how CH succeeded nevertheless in obtaining the correct energy spectrum. This should be worthwhile, since the CH paper succeeded for the first time to produce a finite number of bound states from a SGA.

In Sec. II we review the SGA method for the three-dimensional (3D) oscillator and in Sec. III we review the paper of CH on the Morse potential, and point out the basic difference in the two cases. We shall see that K_0 commutes with H_ℓ in the case of the oscillator, but it does not commute with H_S in the Morse case; in the latter case the Casimir operator Q commutes with H_S. In Sec. IV we derive the effective potentials $V_M(k_m, r_m; r)$ [see (43) below], which have the functions $\Psi_{\lambda_\nu,\nu+m}(r) \propto (J_+^\nu)^m \Psi_{\lambda_\nu,\nu}(r)$ as eigenstates, where $\Psi_{\lambda_\nu,\nu}(r)$ are the wave functions of the Morse potential (2) and J_+^ν are raising operators. Finally, in Sec. V we give a summary.

II. ENERGY SPECTRUM OF THE 3D HARMONIC OSCILLATOR

In this section we review the derivation of the energy spectra for partial-wave Hamiltonians H_ℓ of the "generalized" (by adding the ε) harmonic oscillator

$$\begin{aligned} H_\ell &= -\frac{\hbar^2}{2M}\frac{d^2}{dr^2} + \frac{(\ell(\ell+1)+\varepsilon)\hbar^2}{2Mr^2} + \frac{k}{2}r^2 = \frac{\hbar^2}{2Ma^2}\left(-a^2\frac{d^2}{dr^2} + \frac{\alpha_\ell a^2}{r^2} + \frac{a^4Mk}{\hbar^2}\frac{r^2}{a^2}\right) \\ &= \mathcal{E}\left(-\frac{d^2}{dy^2} + \frac{\alpha_\ell}{y^2} + \operatorname{sgn}(k)\left(\frac{a^2M\omega}{\hbar}\right)^2 y^2\right) \end{aligned} \tag{3}$$

for all $-\infty < k < \infty$, where we use the notation

$$\omega \equiv \sqrt{|k|/M}, \quad y \equiv r/a,$$

$$\alpha_\ell \equiv \ell\,(\ell+1)+\varepsilon, \tag{4}$$

$$\mathcal{E} \equiv \frac{\hbar^2}{2Ma^2}.$$

Note that α_ℓ are dimensionless constants. We introduced the scaling factor a which has the dimension of length, and whose value will be determined below in (9). Thus, y becomes a dimensionless variable, and $\mathcal{E}$ has the dimension of energy.

Note that we defined H_ℓ for the attractive oscillator $k>0$ and also for repulsive oscillator $k<0$. Usually the oscillator is only studied for attractive case, but k was defined in Ref. 13 for all real values of k, in order to study the contraction of the algebra $\mathfrak{su}(2)$ to the Euclidean algebra $\mathfrak{e}(2)$ or to the Heisenberg algebra $\mathfrak{h}(3)$, as $k\to 0$.

A. Realization of $\mathfrak{so}(2,1)$ generators for the oscillator

It is easy to check that the following three generators satisfy the commutation relations (1) of $\mathfrak{so}(2,1)$:

$$K_0(\alpha) := -\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{\alpha}{y^2}+\frac{y^2}{16},$$

$$K_1(\alpha) := -\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{\alpha}{y^2}-\frac{y^2}{16}=K_0-\frac{y^2}{8}, \tag{5}$$

$$K_2 := \frac{-i}{2}\left(y\frac{\mathrm{d}}{\mathrm{d}y}+\frac{1}{2}\right),$$

and thus yield different realizations of $\mathfrak{so}(2,1)$ for every value of the constant α. [The generators in (5) follow from those in Ref. 8, Eq. (18.7) by multiplying K_0 and K_1 by a minus sign, which leaves the commutation relations unchanged, and then by replacing α by $-\alpha$.] The above generators are so constructed, that if they are applied to an eigenfunction of K_0, then their Casimir operator is related to α, as follows:

$$Q := K_0^2-K_1^2-K_2^2=\left(\frac{\alpha}{4}-\frac{3}{16}\right)I =: q(\alpha)I, \tag{6}$$

where I is identity operator. If we factorize q as follows:

$$q(\alpha) := \lambda(\lambda-1)=\frac{\alpha}{4}-\frac{3}{16}, \tag{7}$$

and solve the quadratic equation in (7), we obtain for λ the values

$$\lambda_\pm(\alpha)=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{1+4q}=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{\alpha+1/4}, \quad \text{so that } \alpha \geqslant -1/4. \tag{8}$$

It is interesting to note that the realization (5) for $\alpha=0$ also yields the SGA of the one-dimensional harmonic oscillator. In this case, (8) yields the well-known values $\lambda_-=1/4$ and $\lambda_+=3/4$, which define the two infinite-dimensional representations, the Fock states $|2m\rangle$ and $|2m+1\rangle$, $m=0,1,2,\ldots$, respectively.[14]

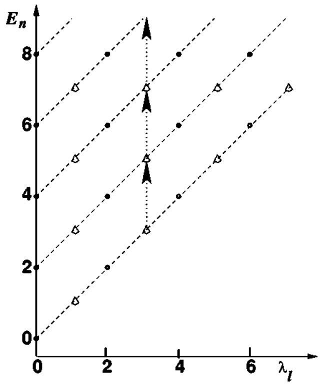

FIG. 1. The eigenstates of the harmonic oscillator for a given ℓ belong to a single irreducible representation of $\mathfrak{so}(2,1)$ which is characterized by λ_ℓ. The excited states for a given ℓ can be obtained by applying the raising operator $K_+(\alpha_\ell)$, as illustrated by the vertical arrows.

B. Relating H_ℓ to the generators of $\mathfrak{so}(2,1)$

The realization (5) is suitable for generating the oscillator states. In fact, H_ℓ becomes proportional to K_0 for $k>0$ or to K_1 for $k<0$, if we scale the coefficient of y^2 in (3) to $1/16$, i.e.,

$$\sigma \equiv \frac{a^4M^2\omega^2}{\hbar^2} \Rightarrow \frac{1}{16}$$

which is equivalent to choosing a, as follows:

$$a = \left(\frac{\hbar^2}{16M|k|}\right)^{1/4} = \left(\frac{\hbar}{4M\omega}\right)^{1/2}. \tag{9}$$

With this choice of a, we obtain

$$\mathcal{E} = \hbar^2/(2Ma^2) = 2\hbar\omega. \tag{10}$$

Hence (3) becomes

$$H_\ell = \mathcal{E}\left(-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{\alpha}{y^2} + \mathrm{sgn}(k)\frac{y^2}{16}\right) = \begin{cases} 2\hbar\omega K_0 & \text{for } k>0, \\ 2\hbar\omega K_1 & \text{for } k<0. \end{cases} \tag{11}$$

Hence, the eigenfunctions of K_0 will be the eigenfunctions of H_ℓ for $k>0$. Since the eigenvalues of K_0 are given by[8,15] $\nu+\lambda$, $\nu=0,1,2,\ldots$, it follows that the energy spectrum for the ℓ-partial wave is given, for $\alpha_\ell=\ell(\ell+1)+\varepsilon$, by Ref. 16, Sec. 36 and Ref. 8,

$$\begin{aligned} E_{\nu,\ell} &= 2\hbar\omega(\nu+\lambda_+(\alpha_\ell)), \quad \nu=0,1,2,\ldots, \\ &= \hbar\omega(2\nu+1+\sqrt{(\ell+1/2)^2+\varepsilon}) \end{aligned} \tag{12}$$

which tends, in the limit $\varepsilon \to 0$ to

$$E_{\nu,\ell} = \hbar\omega(n+3/2), \quad \text{where } n:=2\nu+\ell. \tag{13}$$

It can be seen that the shift in λ_ℓ in the eigenvalues of $K_0(\alpha_\ell)$, due to ε, is also multiplied by the factor $2\hbar\omega$. Note that for a fixed ℓ the energy levels $E_{\nu,\ell}$ increase by $2\hbar\omega$ rather than $\hbar\omega$.

In the case of the harmonic oscillator all the energy levels for a given ℓ belong to a single representation of $\mathfrak{so}(2,1)$, so that all eigenfunctions $\Psi_{\ell,\nu}$ can be obtained by applying powers of the raising operators $K_+(\alpha_\ell)$ on the ground state $\Psi_{\ell,0}$, as illustrated in Fig. 1.

III. ALGEBRAIC DERIVATION OF THE MORSE SPECTRUM

In CH the authors reproduced the S-wave energy spectrum algebraically by using $\mathfrak{so}(2,1)$. We now show that, in contrast to the oscillator, one cannot obtain the S-wave spectrum of the Morse potential by using one representation of $\mathfrak{so}(2,1)$. In fact, it turns out, that for each energy level E_ν one needs a different realization J_i^ν [see (33) below] of the $\mathfrak{so}(2,1)$ generators. This fact is essentially implied by the formulas they used, but it was never stated clearly neither in the above paper, nor in subsequent papers on the Natanzon potentials.[9] Before explaining their procedure, we introduce new notation and also note some changes of notation from that in CH.

A. New notation

We try to make the present paper self-contained. However, if the reader likes to consult the original paper of CH, he should note the following changes of notations and definitions which we have made, so that it becomes easier to check the dimensions in the formulas and to simplify some of them.

In CH the Morse potential was written as $V_{\mathrm{old}}(r) := D[e^{-2a_{\mathrm{old}}(r-r_0)} - 2be^{-a_{\mathrm{old}}(r-r_0)}]$. In this paper we define $a := 1/a_{\mathrm{old}}$, so that a has the dimension of *length*. We also replace the unnecessary parameter b by 1, by defining $V_0 = Db^2$ and adjusting the value of r_0. With the new definition the potential $V(r)$ has its minimum at $r = r_0$.

We find it quite useful to introduce the following dimensionless constant:

$$k_0 \equiv \frac{1}{\hbar}\sqrt{2Ma^2V_0} = \sqrt{V_0/\mathcal{E}}, \quad \text{so that } V_0 = \mathcal{E}k_0^2, \tag{14}$$

where $\mathcal{E}$ is defined by the expression (4). We shall see that k_0 is equal to the eigenvalue of K_0, when it acts on the S-wave solutions.

The following dimensionless exponential function is also very useful:

$$h(r) := k_0 e^{-(r-r_0)/a}. \tag{15}$$

Note that $h(r_0) := k_0$ and that our $h(r)$ in (15) is equal to *twice* the $h_{\mathrm{old}}(r)$ in CH; this redefinition simplifies many formulas, by making factors of $1/2$ and $1/4$ unnecessary.

We can now write the Morse potential in terms of $h(r)$ simply, as follows:

$$V_M(r) = V_0\frac{h(r)}{k_0}\left(\frac{h(r)}{k_0} - 2\right) = \mathcal{E}(h(r)^2 - 2k_0 h(r)). \tag{16}$$

B. Relating the partial Hamiltonian H_S to J_0

The following three operators were defined in CH, which depend on a parameter β, which the authors called $-E/\mathcal{E}$,

$$J_0(\beta) := \frac{1}{2h(r)}[-a^2\Delta_r + \beta + h^2(r)],$$

$$J_1(\beta) := J_0 - h, \tag{17}$$

$$J_2 := i\left(a\frac{\mathrm{d}}{\mathrm{d}r} + \frac{a}{r} - \frac{1}{2}\right),$$

where $h(r)$ is the exponential function (15) and Δ_r is the radial part of the Laplacian Δ, i.e.,

$$\Delta_r := \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} = \left(\frac{d}{dr} + \frac{1}{r}\right)^2. \tag{18}$$

These J_i commute exactly as the K_i in (1) and hence they also yield a realization of the noncompact $\mathfrak{so}(2,1)$ algebra.

The realization J_i in (17) is defined such that it acts on the radial part $\Psi_{\nu,\ell}(r) = R_{\nu,\ell}/r$ of the wave function $\Psi_{\nu,\ell}(\mathbf{r}) = R_{\nu,\ell}/rY_{lm}(\theta,\varphi)$. It can be transformed into the realization K_i of $\mathfrak{so}(2,1)$ which acts on $R(r)$, as follows:

$$K_i := rJ_i\frac{1}{r}. \tag{19}$$

The Casimir operator Q of the J_i is related to the parameter β by

$$Q := J_0^2 - J_1^2 - J_2^2 = \left(\beta - \tfrac{1}{4}\right)I =: q(\beta)I, \tag{20}$$

so that we may replace β in (17) by $q+1/4$. Note that the above expression for $q(\beta)$ for the J_i in (17) is different from that for $q(\alpha)$ in (6). The corresponding

$$\lambda_{\pm}(\beta) = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4q} = \tfrac{1}{2} \pm \sqrt{\beta}, \tag{21}$$

require that $\beta \geqslant 0$ in order that $\lambda_{\pm}$ to be real.

The "*S-wave Hamiltonian*" for the Morse potential

$$H_S := -\frac{\hbar^2}{2M}\Delta_r + V_M(r) \tag{22}$$

is related to J_0, as follows:

$$H_S = \mathcal{E}[2h(r)(J_0(\beta) - k_0) - \beta]. \tag{23}$$

Note that J_0 does not commute with H_S, because J_0 does not commute with $h(r)$.

C. Condition on the wave functions $\Psi_\nu(r)$

Let $\Psi_\nu(r)$ denote the eigenfunction of H_S associated to the eigenvalue E_ν, $\nu = 0,1,2,\ldots,\nu_{\max}$, i.e.,

$$[H_S - E_\nu]\Psi_\nu(r) = [\mathcal{E}(2h(J_0 - k_0) - \beta) - E_\nu]\Psi_\nu(r) = 0. \tag{24}$$

This equation can be satisfied, iff $\Psi_\nu(r)$ are eigenfunctions of J_0, with the *same* (!) eigenvalue k_0 for all the allowed ν, i.e.,

$$J_0\Psi_\nu(r) = k_0\Psi_\nu(r), \quad \text{for } \nu = 0,1,2,\ldots,\nu_{\max}, \tag{25}$$

and if for every E_ν we choose β_ν, such that

$$-\frac{E_\nu}{\mathcal{E}} = \beta_\nu = q_\nu + \frac{1}{4} = \lambda_\nu(\lambda_\nu - 1) + \frac{1}{4} = (\lambda_\nu - 1/2)^2. \tag{26}$$

But since k_0 is an eigenvalue of the compact generator, it must be related to λ_ν as follows:[8,15]

$$k_0 = m_\nu + \lambda_\nu, \tag{27}$$

where m_ν is some integer.

To fix the integers m_ν, we proceed as follows: First, we order for definiteness the E_ν, such that $E_0 < E_1 < \cdots < E_{\nu_{\max}}$. By noting (26) we conclude that the ground state Ψ_0 must belong to the highest possible value of λ_ν consistent with the condition (27). Hence, we must choose $m_0 = 0$ for λ_0. Following similar arguments, we finally obtain

$$k_0 = \nu + \lambda_\nu, \quad \text{or} \quad \lambda_\nu = k_0 - \nu. \tag{28}$$

To obtain (28) we implicitly assume that every permissible solution is a physical eigenfunction. Substituting this expression for λ_ν into (26), we obtain the same energy spectrum for the *S*-wave bound states as in Ref. 12,

$$E_\nu(\ell = 0) = -\mathcal{E}\left[k_0 - \frac{1}{2} - \nu\right]^2 = -\frac{\hbar^2}{2Ma^2}\left[\frac{\sqrt{2Ma^2V_0}}{\hbar} - \frac{1}{2} - \nu\right]^2, \tag{29}$$

where

$$\nu = 0,1,2\ldots,\nu_{\max} = \lfloor k_0 - 1/2 \rfloor_- . \tag{30}$$

This spectrum was first obtained by Morse[11] by solving the Schrödinger equation. The value of $\nu_{\max}=\lfloor k_0-1/2\rfloor_-$, where we use the notation $\lfloor x \rfloor_-$ to denote the largest integer which is *smaller* (not equal to) than x, because $s := k_0 - \frac{1}{2} - \nu > 0$ in order for the solution to be normalizable[11] (see also the comment in Sec. III D). Hence, for $k_0 \leqslant 1/2$ there are no bound states.

It is important to note that the energy levels E_ν in (29) do not depend on r_0.

The main observation in this section is that all eigenfunctions of H_S must be eigenfunctions of different J_0^ν, but with the same eigenvalue k_0. In contrast, all eigenfunctions of H_ℓ of the oscillator are eigenfunctions of Casimir operator Q for a fixed $q_\ell = \lambda_\ell(\lambda_\ell - 1)$ and different eigenvalues of $K_0(\alpha_\ell)$, namely $k_\nu = \nu + \lambda_\ell, \nu = 0,1,2,\ldots$.

D. The traditional derivation of the bound state solutions

For completeness and for comparison, we review the traditional derivation of the bound state solutions.[11,16] Making the change of variables

$$\xi = 2h(r) = 2k_0 e^{-(r-r_0)/a},$$

in Schrödinger's equation (24) and using $\Psi(r) = R(\xi)/r$, we obtain

$$R''(\xi) + \frac{1}{\xi}R'(\xi) + \left(-\frac{1}{4} + \frac{k_0}{\xi} + \frac{E}{\mathcal{E}\xi^2}\right)R(\xi) = 0. \tag{31}$$

Again, substituting $R(\xi) = e^{-\xi/2}\xi^s F(\xi)$ into (31), where $s = \sqrt{-E/\mathcal{E}}$, yields a differential equation for F,

$$F''(\xi) + (2s + 1 - \xi)F'(\xi) + (k_0 - 1/2 - s)F(\xi) = 0,$$

whose solutions are the confluent hypergeometric functions ${}_1F_1(s+1/2-k_0, 2s+1; \xi)$. These functions become polynomials and yield normalizable wave functions for $s = k_0 - 1/2 - \nu \geqslant 0$ where ν is non-negative integer. This condition yields the energy levels

$$-E_\nu = \mathcal{E}s_\nu^2 = \mathcal{E}(k_0 - 1/2 - \nu)^2 > 0, \quad \nu = 0,1,2,\ldots,\nu_{\max},$$

and the corresponding wave functions

$$\Psi_\nu(r) = \frac{1}{r}R(2h(r)) \propto \frac{1}{r}e^{-h}h^{(k_0-1/2-\nu)}{}_1F_1(-\nu,\ 2k_0 - 2\nu;\ 2h). \tag{32}$$

It is interesting to note that if $k_0 = n + 1/2$ we obtain $E_n = 0$, but this solution does not correspond to a bound state, since the solution is not normalizable for $s = 0$.[11] This result can be understood intuitively, because we are dealing with *S*-wave solutions, so that there is no potential barrier which can prevent the particle from escaping to infinity. In contrast, an $E = 0$ solution would probably be normalizable for $\ell > 1$, since in the latter case the effective potential $U(r) = \ell(\ell+1)/(2Mr^2) + V_M(r)$ approaches $r \to \infty$ from above, and thus provides a potential barrier of infinite range. This was illustrated by Daboul and Nieto,[17] who studied $E=0$ solutions for a class of potentials.

IV. EFFECTIVE MORSE POTENTIALS GENERATED BY $J_{\pm}^{\nu} \equiv J_{\pm}(|E_\nu|/\mathcal{E})$

Equations (17) define a realization $J_i(\beta)$ of the $\mathfrak{so}(2,1)$ algebra for every value of the parameter $\beta \geqslant 0$. However, since we are interested in eigenfunctions of J_0 which are normalizable, we restrict the values of β to the discrete set $\beta_\nu = -E_\nu/\mathcal{E}$, where the E_ν are the discrete eigenvalues of H_S. We denote the corresponding generators by

$$J_i^{\nu} := J_i(\beta_\nu) = J_i(-E_\nu/\mathcal{E}), \quad \nu = 0,1,2,\ldots\,. \tag{33}$$

In (25) we found that

$$J_0^{\nu}\Psi_{\lambda_\nu,\nu}(r) = k_0 \Psi_{\lambda_\nu,\nu}(r). \tag{34}$$

Using the raising and lowering operators

$$J_{\pm} := J_1 \pm iJ_2, \tag{35}$$

which obey

$$[J_0,J_{\pm}] = \pm J_{\pm}, \quad [J_+,J_-] = -2J_0, \tag{36}$$

the following functions can be defined, by acting with $(J_{\pm}^{\nu})^m$ onto $\Psi_{\lambda_\nu,\nu}$:

$$\Psi_{\lambda_\nu,\nu\pm m}(r) := (J_{\pm}^{\nu})^m \Psi_{\lambda_\nu,\nu}(r). \tag{37}$$

If these states exist and are normalizable, then they must be eigenfunctions of J_0 with eigenvalues $k_0 \pm m$, since

$$J_0^{\nu}\Psi_{\lambda_\nu,\nu\pm m}(r) = J_0^{\nu}(J_{\pm}^{\nu})^m\Psi_{\lambda_\nu,\nu}(r) = (k_0 \pm m)(J_{\pm}^{\nu})^m\Psi_{\lambda_\nu,\nu}(r) = (k_0 \pm m)\Psi_{\lambda_\nu,\nu\pm m}(r), \tag{38}$$

where we used the following general relations:

$$J_0^{\nu}(J_{\pm}^{\nu})^m = (J_{\pm}^{\nu})^m J_0^{\nu} + [J_0^{\nu},(J_{\pm}^{\nu})^m] = (J_{\pm}^{\nu})^m J_0^{\nu} \pm m (J_{\pm}^{\nu})^m.$$

Multiplying (38) by the factor $\mathcal{E}2h(r)$ and substituting the expression (23) for the J_0^{ν} operator, yields the following differential equations:

$$0 = \mathcal{E}2h(r)[J_0^{\nu} - (k_0 \pm m)]\Psi_{\lambda_\nu,\nu\pm m}(r) = \mathcal{E}\left[-a^2\Delta_r - \frac{E_\nu}{\mathcal{E}} + h^2(r) - (k_0 \pm m)2h(r)\right]\Psi_{\lambda_\nu,\nu\pm m}(r).$$

The radial functions $\Psi_{\lambda_\nu,\nu+m}(r)$ can therefore be interpreted as eigenstates of the S-wave Schrödinger equation for the following potentials:

$$V_{\mathrm{eff}}(m,r) = \frac{\hbar^2}{2Ma^2}[h^2(r) - (k_0 + m)2h(r)] = V_0\left[e^{-2(r-r_0)/a} - 2\frac{k_m}{k_0}e^{-(r-r_0)/a}\right], \tag{39}$$

where

$$k_m \equiv k_0 + m. \tag{40}$$

Each of these effective potentials has its minimum at r_m, where

$$\frac{r_m}{a} = \frac{r_0}{a} - \ln\left(\frac{k_0 + m}{k_0}\right) = \frac{r_0}{a} - \ln\left(\frac{k_m}{k_0}\right), \tag{41}$$

so that

$$e^{-(r-r_0)/a} = \frac{k_m}{k_0} e^{-(r-r_m)/a}. \tag{42}$$

By substituting this expression into (39), we obtain

$$V_M(k_m, r_m; r) = V_m[e^{-2(r-r_m)/a} - 2e^{-(r-r_m)/a}] = \mathcal{E}k_m^2[e^{-2(r-r_m)/a} - 2e^{-(r-r_m)/a}]. \tag{43}$$

The effective potentials (43) look exactly as the original Morse potential (2), except that the parameters (k_0, r_0) get changed into (k_m, r_m). As m increases the effective potentials (43) will have shorter range and their minima

$$V_{\mathrm{eff}}(r_m) = -V_0\left(\frac{k_0 + m}{k_0}\right)^2 = -\mathcal{E}k_m^2 \tag{44}$$

become deeper and deeper, decreasing almost quadratically with m. The associated energy eigenvalues are

$$\begin{aligned} E_\nu^{(m)} &= -\mathcal{E}\left(k_m - \frac{1}{2} - \nu\right)^2, \quad \nu = 0, 1, \ldots, \lfloor k_m - 1/2 \rfloor_-, \\ &= -\mathcal{E}\left((k_0 + m) - \frac{1}{2} - \nu\right)^2 = E_{\nu+n}^{(m+n)} \quad \text{for } n > -\nu. \end{aligned} \tag{45}$$

A. Connection to SUSY–QM

Using the relation (45) we obtain immediately

$$E_\nu \equiv E_\nu^{(0)} = -\mathcal{E}\left(k_0 - \frac{1}{2} - \nu\right)^2 = -\mathcal{E}\left(k_{-\nu} - \frac{1}{2}\right)^2 = E_0^{(-\nu)} \quad \text{for } \nu = 1, 2, \ldots, \nu_{\max}. \tag{46}$$

Hence the energies E_ν, $\nu = 1, 2, \ldots, \nu_{\max}$ of the excited states of the original Morse potential (2) are equal to *ground state* energies $E_0^{(-\nu)}$ of the effective potentials $V_{\mathrm{eff}}(-\nu, r)$, $\nu = 1, 2, \ldots, \nu_{\max}$. This is one of the interesting results of the quantum-mechanical supersymmetry (SUSY–QM) formalism.[18] We derived it here without using the latter formalism and without the need of finding out the relevant supersymmetric potential $W(x, a_i)$.

To understand the above result more thoroughly we give a second proof that the $\Psi_{\lambda_\nu, \nu+m}(r) \equiv (J_+^\nu)^m \Psi_{\lambda_\nu, \nu}(r)$ are eigenfunctions of H_S with the potential $V_{\mathrm{eff}}(m, r)$.

For this, we first note the important relation (42), it tells us that the $h(r)$ defined in (39) is invariant under the transformation (k_0, r_0) to (k_m, r_m), i.e.,

$$h(r) \equiv h(k_0, r_0, r) = k_0 e^{-(r-r_0)/a} = k_m e^{-(r-r_m)/a} = h(k_m, r_m, r). \tag{47}$$

Hence, also the generators

$$J_i^\nu \equiv J_i(k_0, r_0, -E_\nu/\mathcal{E}; r) = J_i(k_m, r_m, -E_\nu/\mathcal{E}; r), \tag{48}$$

as defined in (17), do not depend on m, but differ for different $E_\nu/\mathcal{E}$.

Now we compare the following two equations:

$$J_0^\nu(k_0, r_0; r)\Psi_{k_0-\nu, \nu+m}(k_0, r_0; r) = (k_0 + m)\Psi_{k_0-\nu, \nu+m}(k_0, r_0; r), \tag{49}$$

$$J_0^\nu(k_m, r_m; r)\Psi_{k_m-(\nu+m), \nu+m}(k_m, r_m; r) = k_m \Psi_{k_m-(\nu+m), \nu+m}(k_m, r_m; r), \tag{50}$$

where the first follows from (38) and the second follows from the condition (34) on the Morse eigenfunctions for the potential $V_{\mathrm{eff}}(m, r)$. The two Ψ are solutions of the same differential operator J_0^ν with the same eigenvalue. Since the eigenvalues of J_0^ν are not degenerate, the two

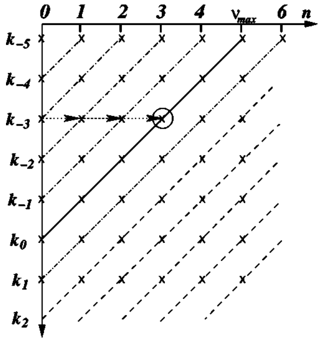

FIG. 2. The eigenstates for effective Morse potentials $V_{\text{eff}}(m,r)$, $m \geqslant -5$ are displayed in the (k_m,n) plane, where k_m is related to the ground-state energies by $k_m = \sqrt{|E_0^m|/\mathcal{E}} + 1/2$ and $n \equiv \nu \pm m$. The eigenstates for a single effective Morse potential $V_{\text{eff}}(m,r)$ (39) are given along a diagonal line. The number of bound states is given by $\lfloor k_m \rfloor_- + 1 = \lfloor k_0 \rfloor_- + m + 1 = \nu_{\text{max}} + m + 1$. In particular, the original potential $V_M(r)$ in (2), which corresponds to $m=0$, has $\nu_{\text{max}}+1$ bound states, which lie along the solid diagonal line. Thus, for $m = -\lfloor k_0 \rfloor_-$ there is only one bound state. In this diagram we choose $k_0 = 5.7$ so that $V_M(r)$ has six bound states. In contrast to Fig. 1 all the states with a fixed energy E_ν belong to the same irreducible representation of $\mathfrak{so}(2,1)$, and lie along a horizonal line. The ground states for different $V_{\text{eff}}(m,r)$ lie along the vertical line $n=0$. By moving along the horizonal lines we go from eigenstates of one effective potential to another. This is illustrated by the horizintal arrows. By applying $J_+^{\nu=-3}$ three times to the ground state of $V_{\text{eff}}(-3,r)$, we obtain the third excited state of the original Morse potential $V_M(r)$.

functions must be the same, except for a constant factor. Hence, $\Psi_{\lambda_\nu,\nu+m}(r)$ is the $(\nu+m)$th excited state of the Morse potential $V_{\text{eff}}(m,r)$.

In Fig. 2 we display the states $\Psi_{\lambda_\nu,n}(r)$, defined in (37), in the (λ_ν,n) plane, where $\lambda_\nu = k_0 - \nu = \sqrt{|E_\nu|/\mathcal{E}} - 1/2$ and $n \equiv \nu \pm m$. All the eigenstates for a fixed effective potential $V_M(k_m,r_m;r)$ lie on a single diagonal line. But, the states along a horizontal line have the energy eigenvalue, but belong to different potentials $V_{\text{eff}}(m,r)$. The states $\Psi_{\lambda_\nu,\nu}(r)$ for the original Morse potential (2) can be obtained from the ground states of $V_{\text{eff}}(-\nu,r)$ by applying $(J_+^\nu)^\nu$, as illustrated in the figure by horizontal arrows.

V. SUMMARY AND CONCLUSIONS

The *S*-wave energy levels of the Morse potential have been known since 1929. Thus, the algebraic derivation of these levels has not brought anything new, as far as applications are concerned. What is interesting in the algebraic treatment is how the mathematical formalism works. In the present paper we did clarify when it works and how it works.

We showed in particular that $\mathfrak{so}(2,1)$ is applied in a completely different manner to the oscillator and the Morse potentials: for the harmonic oscillator a single raising operator $K_+(\ell)$ yields all the eigenstates for the given ℓ. In contrast, for the Morse potential the raising operators $J_+^\nu(\ell=0)$ map the wave functions of the original Morse potential onto wave functions of other Morse potentials. This contrast is illustarted by Figs. 1 and 2. Thus, the message of the present paper is that one should be more critical and check carefully how the SGA are applied.

For example, with our present insight we wanted to check how Wybourne obtained the spectrum of the Morse potential in Ref. 8, Sec. 18.8. It turned out that Wybourne did not even give the $\mathfrak{so}(2,1)$ generators for every potential, as was done, for example, in Refs. 9 and 12. Instead, he used a slightly different version of the realization (5) of $\mathfrak{so}(2,1)$ for the harmonic oscillator, and showed that the algebra describes the energy spectrum of the following differential equation [Ref. 8, Eq. (18.19)]:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{a}{y^2}+by^2+c\right)\Psi(y)=0. \tag{51}$$

Then he simply transformed the Schrödinger equation of different potentials to the above differential equation. But this is essentially what Morse did, already in 1929, by transforming the *S*-wave Schrödinger equation to a differential equation, which was solved by Schrödinger. It is also what Landau and Lifschitz (Ref. 16, Sec. 23) did for many potentials, by transforming their Schrödinger equations to the confluent hypergeometric equation.

There is no doubt that group theory helps us understand many results in physics, such as the degeneracies of the eigenstates of the hydrogen atom. It has many useful applications in elementary particles such as flavor and color SU(3). However, it seems to us that many of the papers on the SGA can even mislead nonexperts, as we demonstrated in the present paper. They might believe, for example, that one could obtain finite number of states by just using the raising operators of a single realization of the $\mathfrak{so}(2,1)$ algebra, as is the case for the oscillator.

Apparently, some experts have noticed that. In a well written and easy to read article,[18] the authors give a review of the supersymmetry (SUSY–QM) formalism and of shape invariant potentials and mention that the Morse potential is of the invariant type. This means that one can obtain the excited states of the Morse potential from its ground state, not by applying powers of a raising operator, as one naively expects, but by applying the supersymmetric raising and lowering operators $A^\dagger(a_i)$ and $A(a_i)$ operators, which have the same structure, but which depend on different parameters a_i. We write this statement, using their notation, as follows:

$$\psi_{n+1}^{(-)}(x,a_0)\sim A^\dagger(a_1)A^\dagger(a_2)\cdots A^\dagger(a_n)\psi_0^{(-)}(x,a_n).$$

We showed in Sec. IV A that for the Morse case the following equivalent statement holds:

$$\Psi_{k_0-\nu,\nu}(r,k_0)\sim(J_+^\nu)^\nu\Psi_{k_{-\nu},0}(r,k_{-\nu}),$$

where all the raising operators are equal, as we showed in (48). Thus, we gave an explanation of why the excited states $\Psi_{k_0-\nu,\nu}(r,k_0)$ of the original Morse potential (2) are related to the ground states $\Psi_{k_{-\nu},0}(r,k_{-\nu})$ of related effective potentials V_{eff}, without using the SUSY formalism. We illustrate this action in Fig. 2 by applying $J_+^{\nu=3}$ three times on the ground state, $\Psi_{k_{-3},0}(r,k_{-3})$, of $V_{\mathrm{eff}}(-3,r)$ and obtain the third excited state of $V_{\mathrm{eff}}(0,r)$. A more recent and detailed review of the SUSY formalism can be found in Ref. 19.

It is interesting to note that by using a direct approach, it was possible to obtain many results,[20] among them a construction of new quasiexactly solvable deformation of the Morse potential, which the authors have not been able to obtain by Lie-algebraic methods.

The Morse potential is one of the simplest examples of the general class of Natanzon's potentials[10] which have been studied by using SGA algebraic methods.[9] Therefore a deeper understanding of the SGA method in the Morse's case should help us understand the more general cases as well.

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