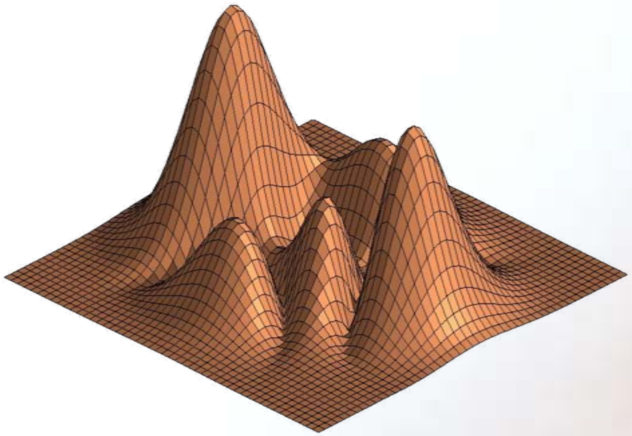
Stochastic Geometry and Bayesian SLAM



Ba-Ngu Vo

School of EECE
University of Western Australia
Perth, Australia

http://www.ee.uwa.edu.au/~bnvo/

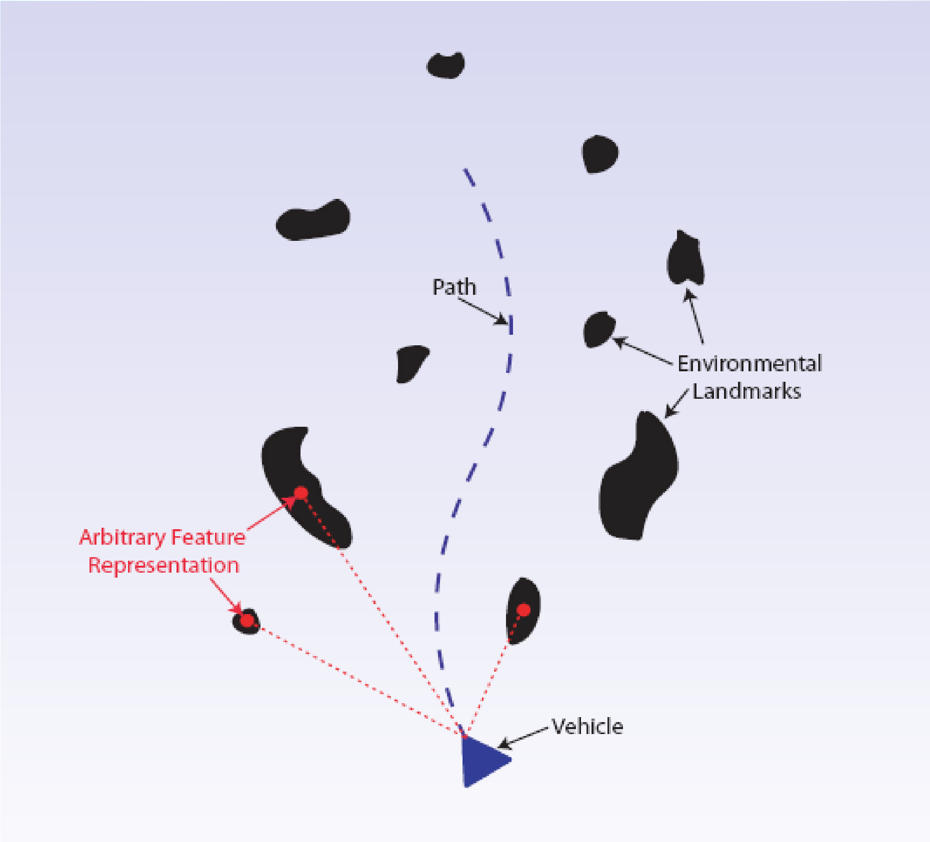
St Paul, US, May 2012

Outline

- Introduction
- Map representation
- Stochastic Geometry
- Bayesian SLAM
- Conclusions

Introduction

SLAM (Simultaneous Localisation and Mapping)



- Objective: Jointly estimate robot pose & map

Introduction

- Statistical basis: [Smith & Cheeseman]

  Estimates of landmarks are correlated with each other because of the common error in estimated vehicle location [Smith, Self & Cheeseman]

  ⇩

  *SLAM requires a **joint state** composed of **pose and every landmark position**, to be updated following each landmark observation.*

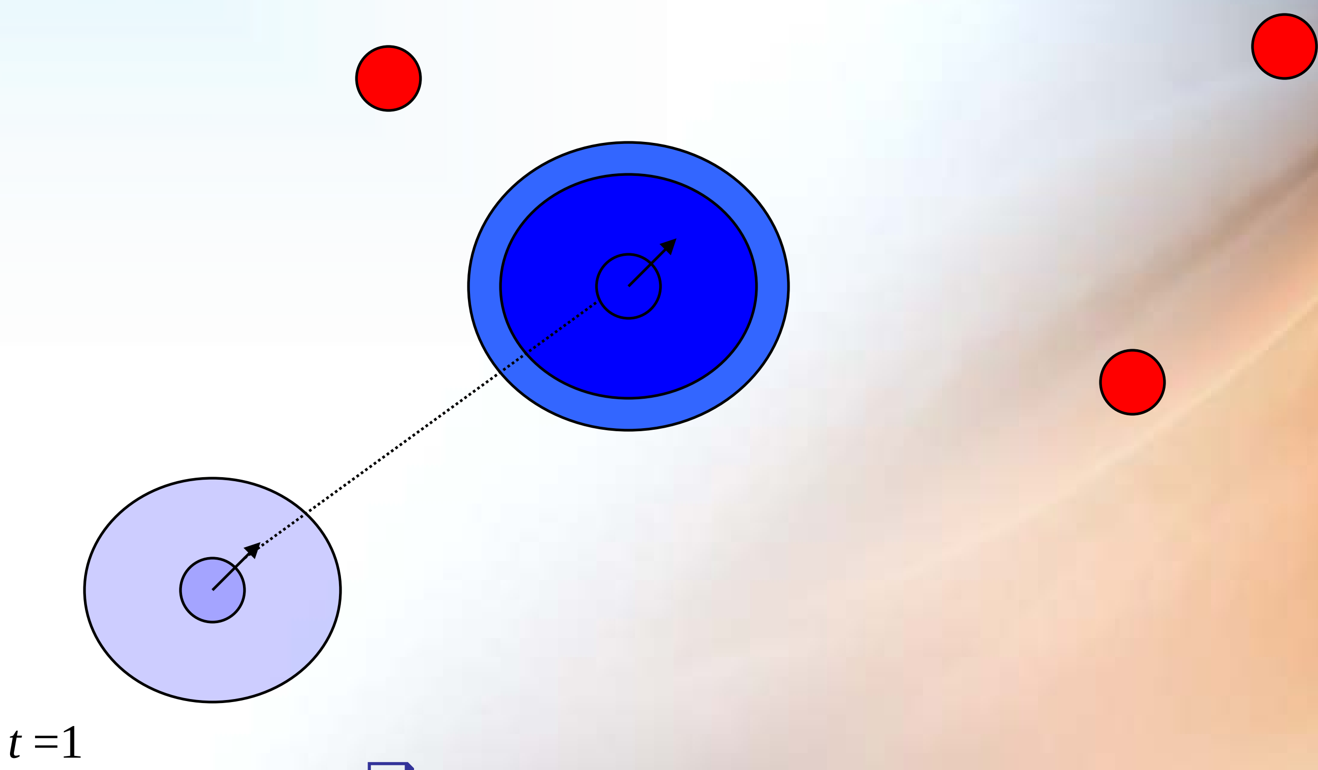
- Key problem: Geometric uncertainty [Durrent-Whyte]
- Essential theory on **convergence** [Csorba]
- Algorithms [Bailey & Durrant-Whyte], [Montermelo et al]

Introduction

$t = 0$

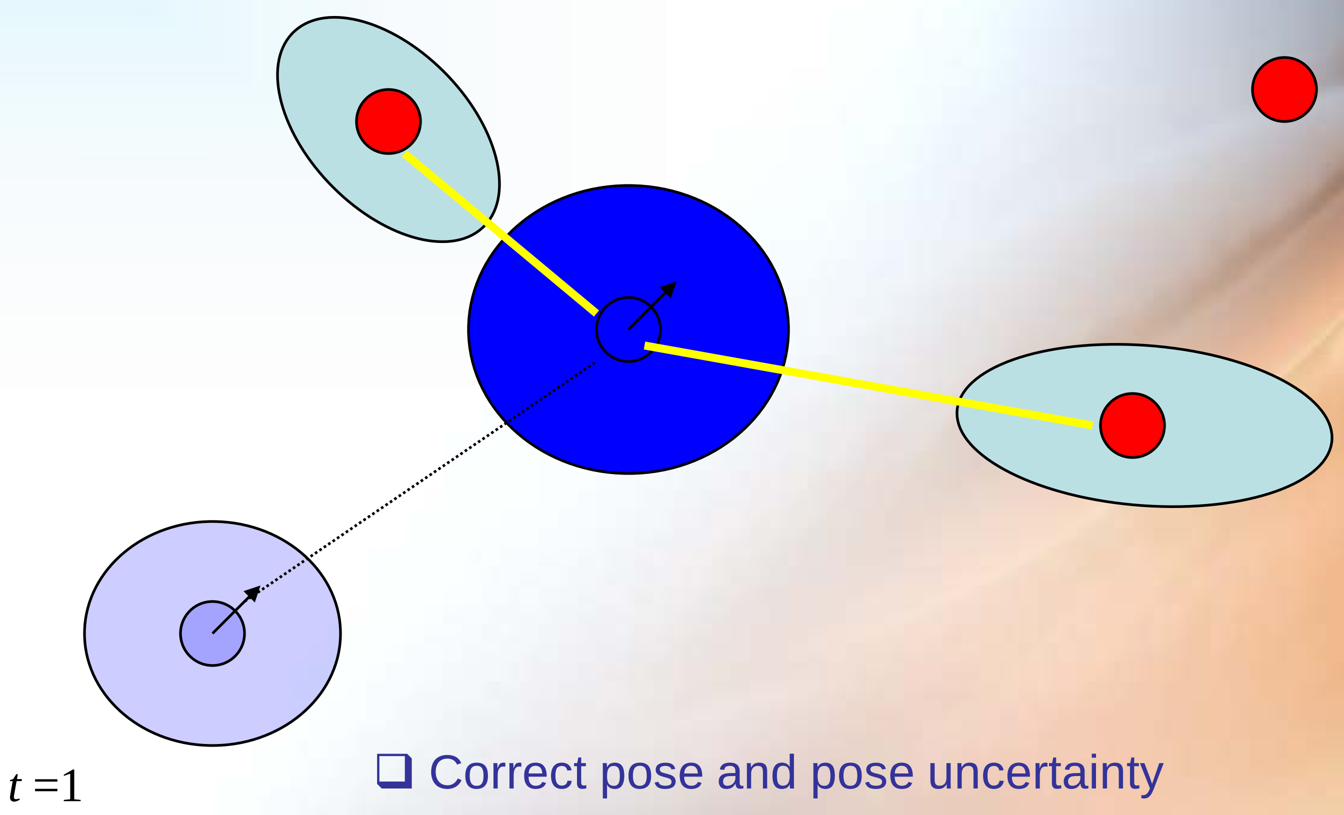
- ❑ Initial State and Uncertainty
- ❑ Using Range Measurements

Introduction



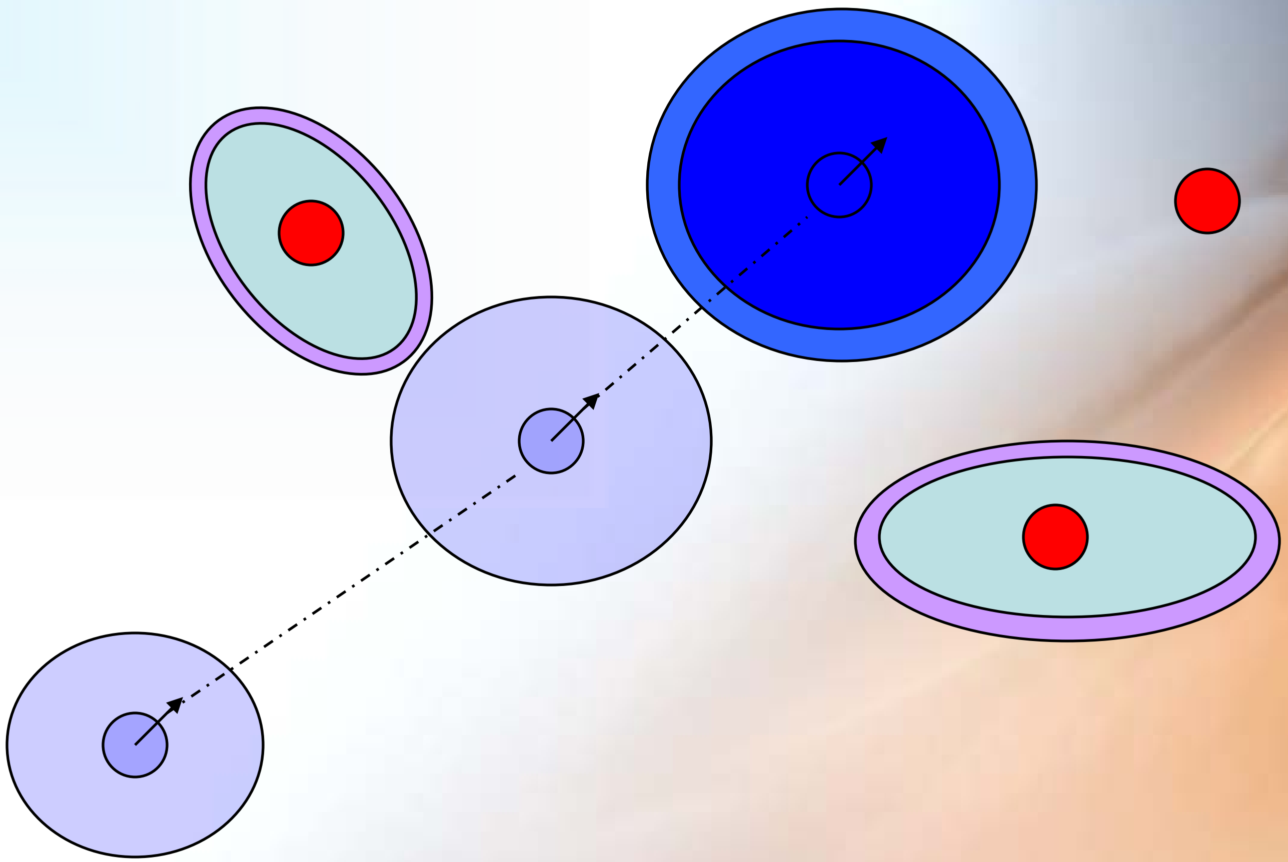
- Predict Pose and Uncertainty to time 1

Introduction



- Correct pose and pose uncertainty
- Predict new feature positions and their uncertainties

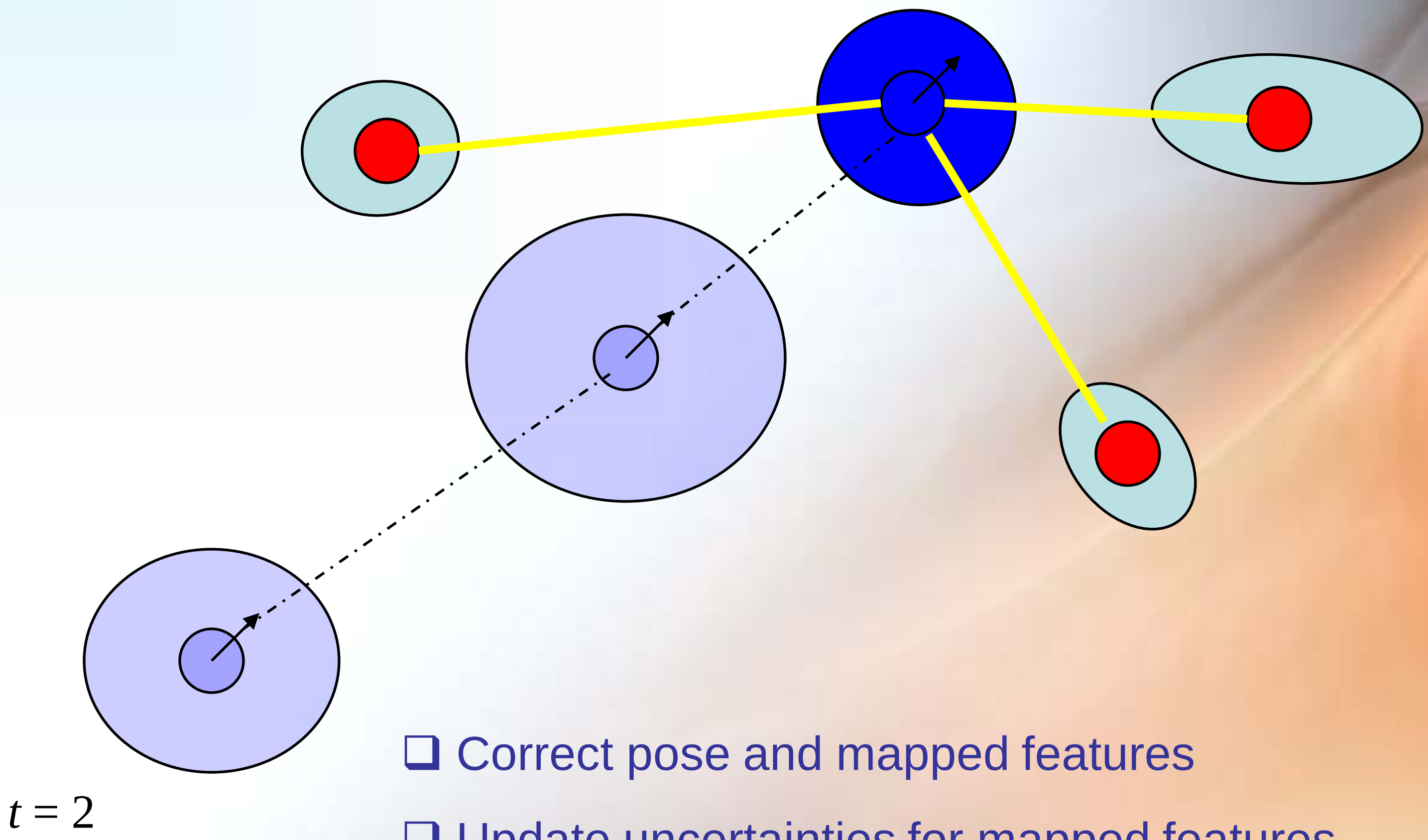
Introduction



$t = 2$

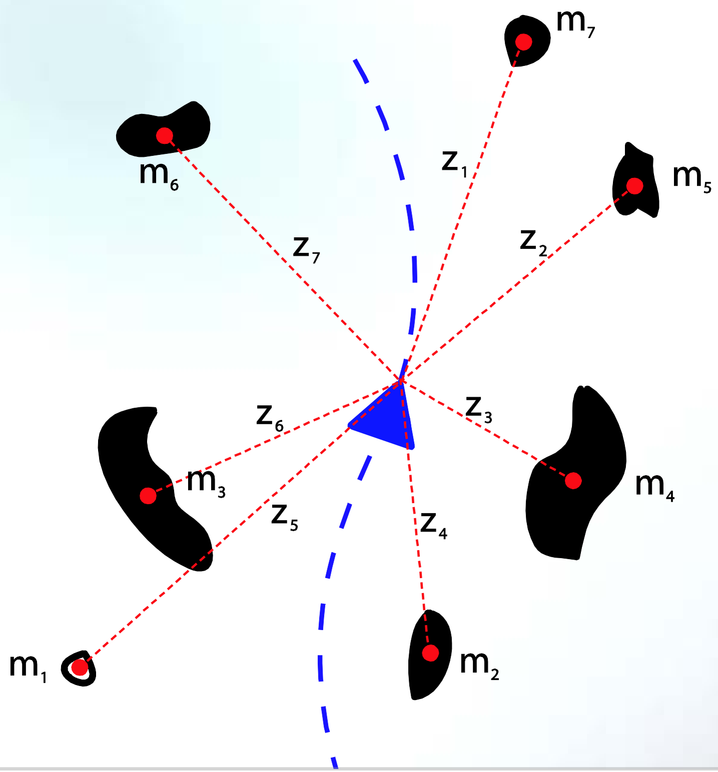
- Predict pose and uncertainty of pose at time 2
- Predict feature measurements and their uncertainties

Introduction



- Correct pose and mapped features
- Update uncertainties for mapped features
- Estimate uncertainty of new features

Introduction



- **Traditional "Bayes" SLAM:**
  - Vector representation of map
  - Performs data association
  - Applies "Bayes"-SLAM filter

$$Z_k = [z_1, z_2, z_3, z_4, z_5, z_6, z_7]$$

$$M_k = [m_1, m_2, m_3, m_4, m_5, m_6, m_7]$$

"Bayes"-SLAM filter

$$\cdots \longrightarrow p_{k-1}(M_{k-1}, x_{k-1}|Z_{1:k-1}) \xrightarrow{\textit{prediction}} p_{k|k-1}(M_k, x_k|Z_{1:k-1}) \xrightarrow{\textit{data-update}} p_k(M_k, x_k|Z_{1:k}) \longrightarrow \cdots$$

$$\int\int f_{k|k-1}(M_k, x_k \mid M, x)\, p_{k-1}(M, x \mid Z_{1:k-1})\, dM\, dx$$

$$\frac{g_k(Z_k \mid M_k, x_k)\, p_{k|k-1}(M_k, x_k \mid Z_{1:k-1})}{\int\int g_k(Z_k \mid M, x)\, p_{k|k-1}(M, x \mid Z_{1:k-1})\, dM dx}$$

Map Representation

Q: *What is the purpose of estimation?*

A: *To get good estimate!*

- What is the type object that we're trying to estimate?
- What is a "good" estimate?

- **Error metric:**

  Quantifies how close an estimate is to the true value

  Fundamental in **estimation**

  Well-understood for localization: Euclidean distance, MSE, …

  What about mapping?

Map Representation

Q: *Why do we need mapping error, localisation error alone is sufficient, **since good** localization implies **good mapping** anyway?*

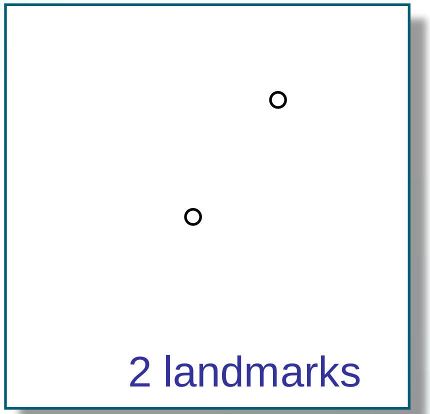
A: *How do we know it's a **good mapping** if we don't know how to quantify **mapping error**?*

Map Representation

- Traditional feature-based SLAM: stack landmarks into a large **vector**!

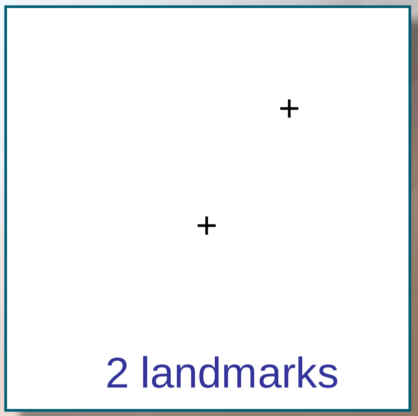
$$M = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

True Map



$$\hat{M} = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$$

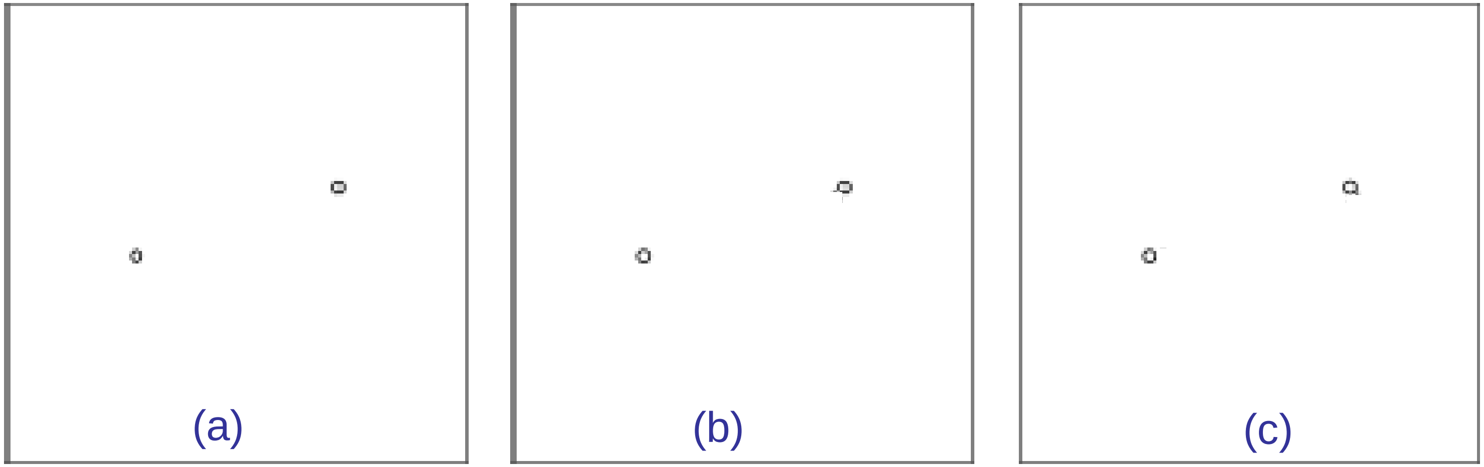
Estimated Map



Estimate is correct but estimation error $|| M - \hat{M} || = 2$?

Remedy: use $\min_{perm(M)} || M - \hat{M} || = 0$

Map Representation



o : True landmarks
+ : Estimated landmarks

What are the estimation errors?

Which map estimate is better?

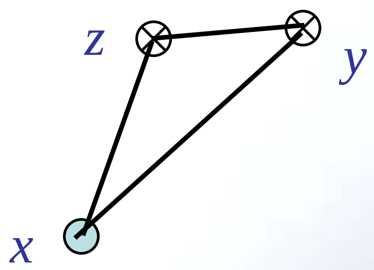
Map Representation

- Need the mapping error metric to
  - be a metric
  - have meaningful interpretation
  - capture errors in number of landmarks and their positions

Map Representation

Q: *Why do we need a metric?*

- **Metric:** $d(\bullet\ ,\ \bullet)$
  - (identity) $d(x, y) = 0$ iff $x = y$;
  - (symmetry) $d(x, y) = d(y, x)$ for all x, y
  - (triangle inequality) $d(x, y) \leq d(x, z) + d(z; y)$ for all x, y, z.



Why triangle inequality?

Suppose estimate z is "close" to the true state x.

If estimate y is "close" to z, then y is also "close" to x

A: *Necessary for comparisons/bounds/convergence*

Map Representation

Q: *Why do we even care about error in the number of landmarks?*

A:

Catastrophic consequences in applications such as search & rescue, obstacle avoidance**,** UAV mission…

Map Representation

- **Vector representation** doesn't admit map error metric!
- **Finite set representation** admits map error metric, e.g.

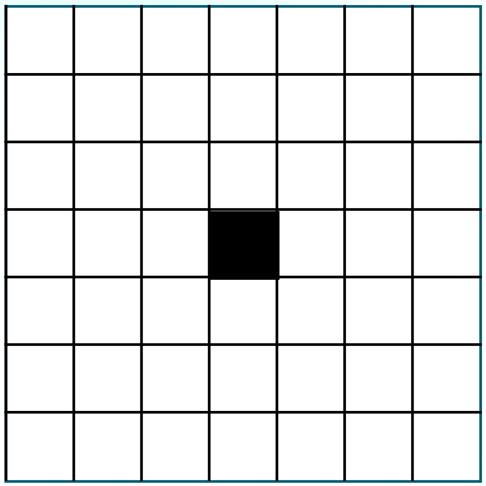
  Hausdorff, Wasserstein, OSPA

- The **map** is fundamentally a **set** (of landmarks)

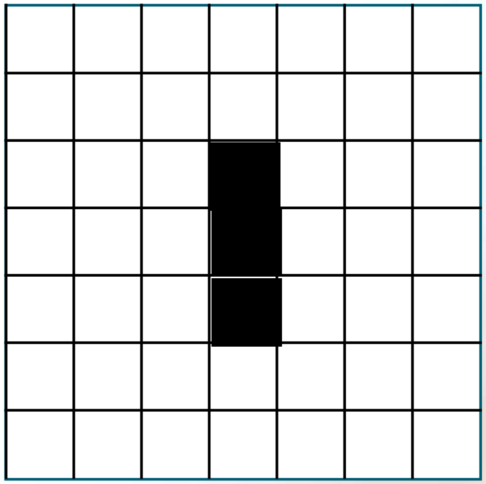
  The realization that the map is a set is found in [Durrant-Whyte]

Map Representation

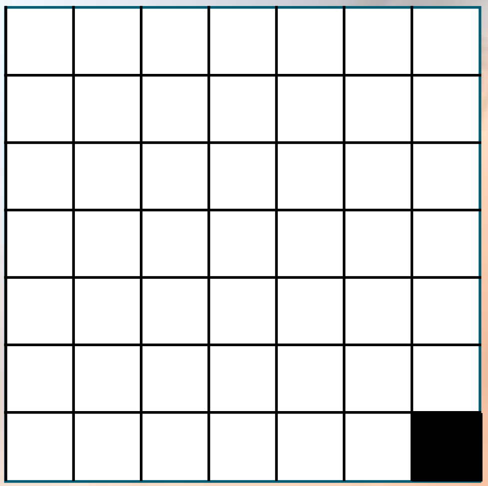
- What about grid-based maps?



True Map



Estimate 1



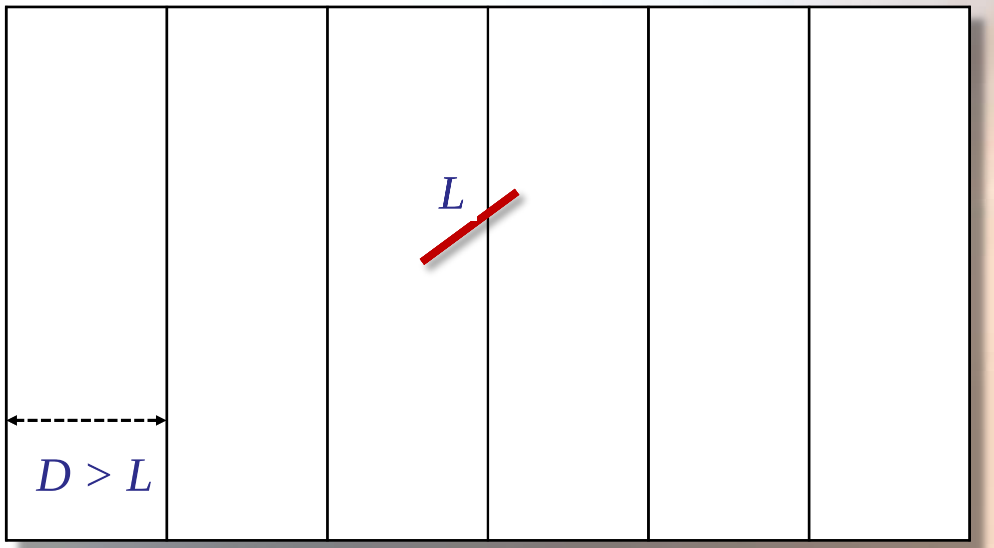
Estimate 2

- When treated as vectors, estimates 1 and 2 have the same error, even though intuitively estimate 1 is better than 2

Stochastic Geometry

- **Essence**: Connections between **Geometry** and **Probability**
- **Origin** 18th century**: geometric probability**

(Buffon's needle 1777) *What is the chance that a needle dropped randomly on a floor marked with equally spaced parallel lines crosses 1 of the lines?*

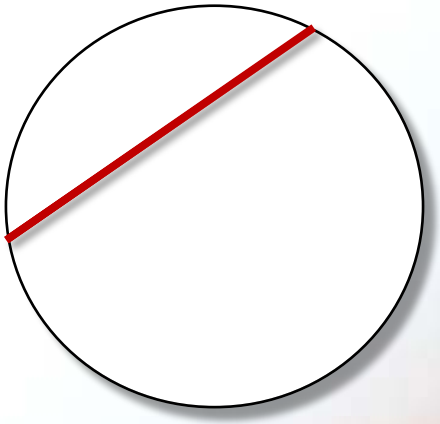


Ans = $\frac{2L}{\pi D}$

Stochastic Geometry

- Other well-known Geometric Probability problems

What is the mean length of a random chord of a unit circle?



(cf. Bertrand's paradox)

What is the chance that 3 random points in the plane form an acute triangle?

What is the mean area of the polygonal regions formed when randomly-oriented lines are spread over the plane?

Monograph: [H. Solomon, *Geometric Probability,* Philadelphia, PA: SIAM,1978]

Stochastic Geometry

Geometric Probability

Integral Geometry

- Development of expected values associated with geometric objects derived from random points
- Theory of measures that are invariant under symmetry groups

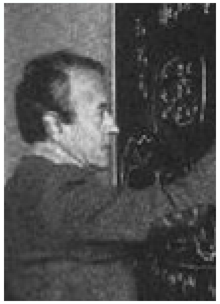
Stochastic Geometry

- Focus on the random geometrical objects, e.g. models for random lines, random tessellations, random sets.
- Study of random processes whose outcomes are geometrical objects or spatial patterns

The terms **Stochastic Geometry** and **Geometric Probability** are some times used interchangeably

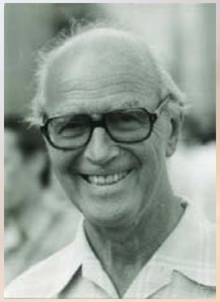
Stochastic Geometry

- Modern stochastic geometry deals with random subsets of arbitrary forms, even randomly generated fractals
- Foundation (1960s-1970s): mostly due to independent work by **Matheron** and **Kendall**, both of whom gave credits to earlier work by **Choquet**



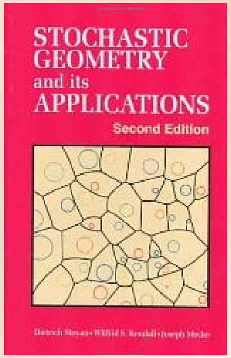
G. Matheron (1930-2000)

D. Kendall (1918-2007)



G. Choquet (1915-2006)

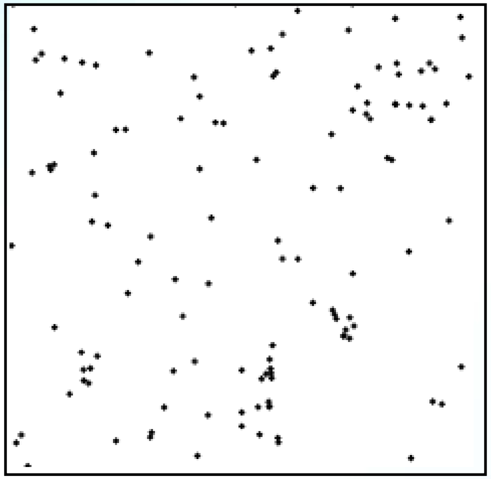
- Applications: physics, biology, sampling theory, stereology, spatial statistics, agriculture, forestry, geology, epidemiology, material science, image analysis, telecommunications, data fusion, target tracking …



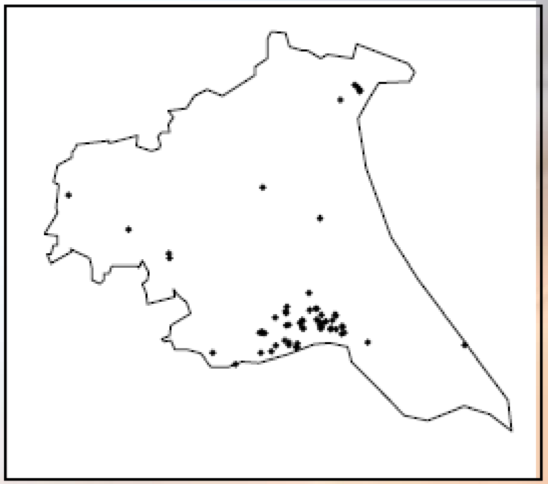
Stochastic Geometry

- **Random Finite Set**: Special case of Matheron's random closed set

  Examples of point pattern data (realisations of RFS)



Pine saplings in a Finish forest [Kelomaki & Penttinen]

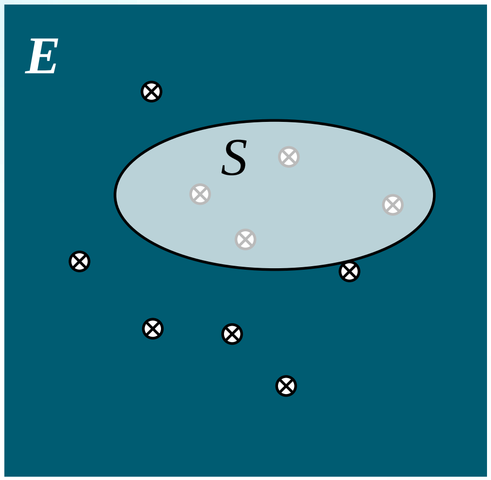


Childhood leukaemia & lymphoma in North Humberland [Cuzich & Edwards]

- The **number of points** is **random**,
- The **points** have **no ordering** and are **random**
- An RFS is a finite-set-valued random variable
- AKA: (simple finite) **point process** or **random point pattern**

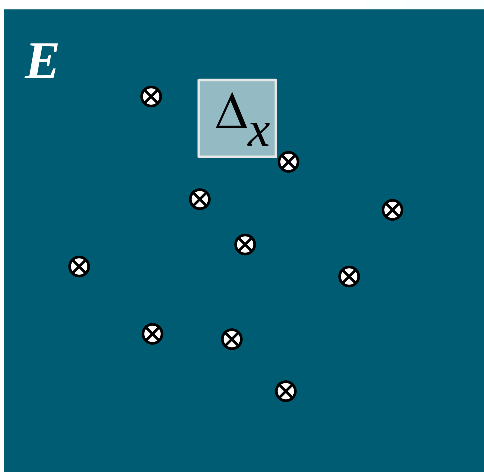
Stochastic Geometry

- **What is the expectation of a random finite set?**



$$V_\Sigma(S) = \mathbf{E}[|\Sigma \cap S|] = \text{expected No. points of } \Sigma \text{ in } S$$

intensity measure or 1st moment measure



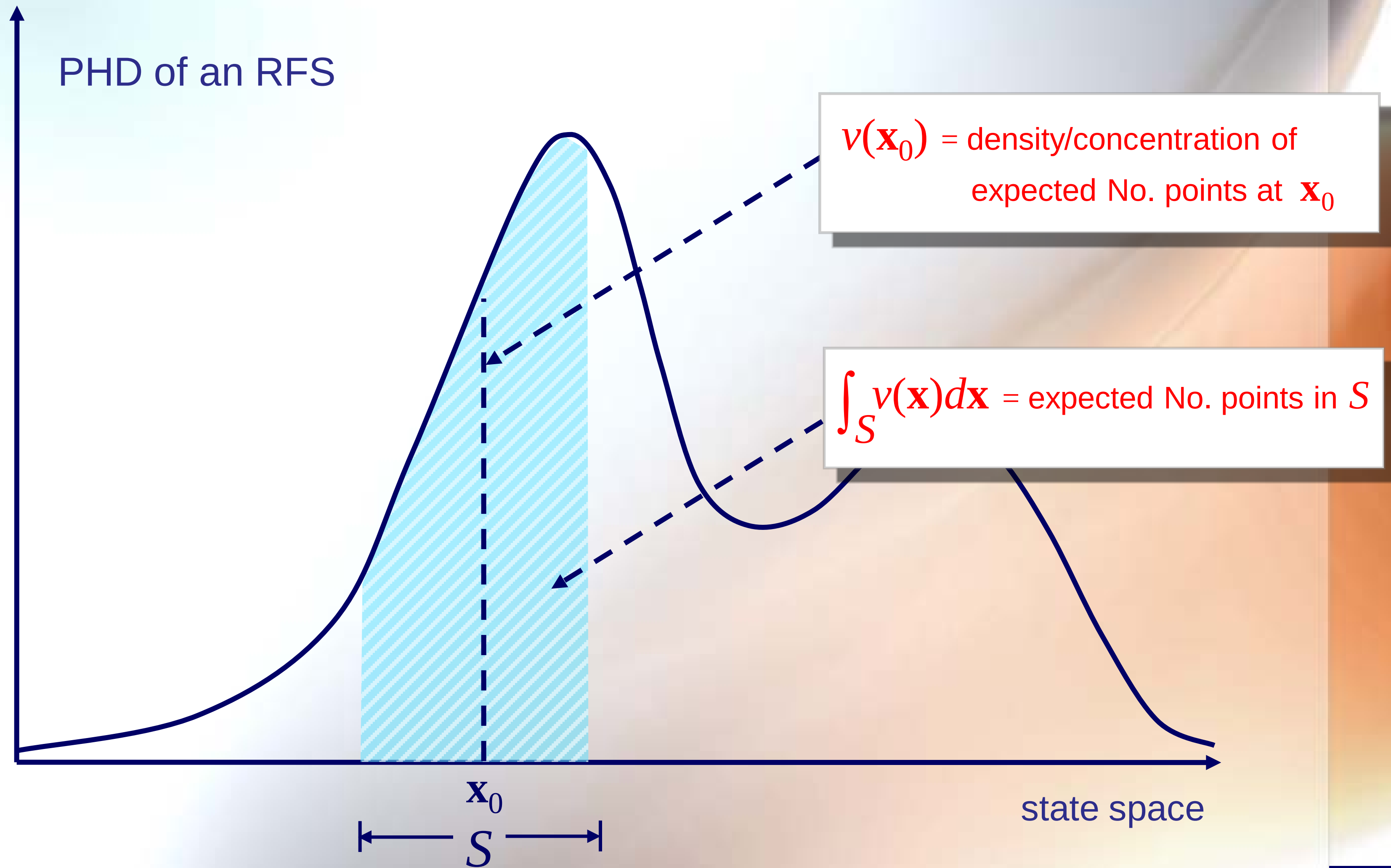
$$v_\Sigma(x) = \lim_{vol(\Delta_x)\to 0} \frac{V_\Sigma(\Delta_x)}{vol(\Delta_x)} = \frac{V_\Sigma(dx)}{vol(dx)}$$

intensity function or PHD (Probability Hypothesis Density)

$$\boxed{\mathbf{E}[|\Sigma \cap S|] = \int_S v_\Sigma(x)dx}$$

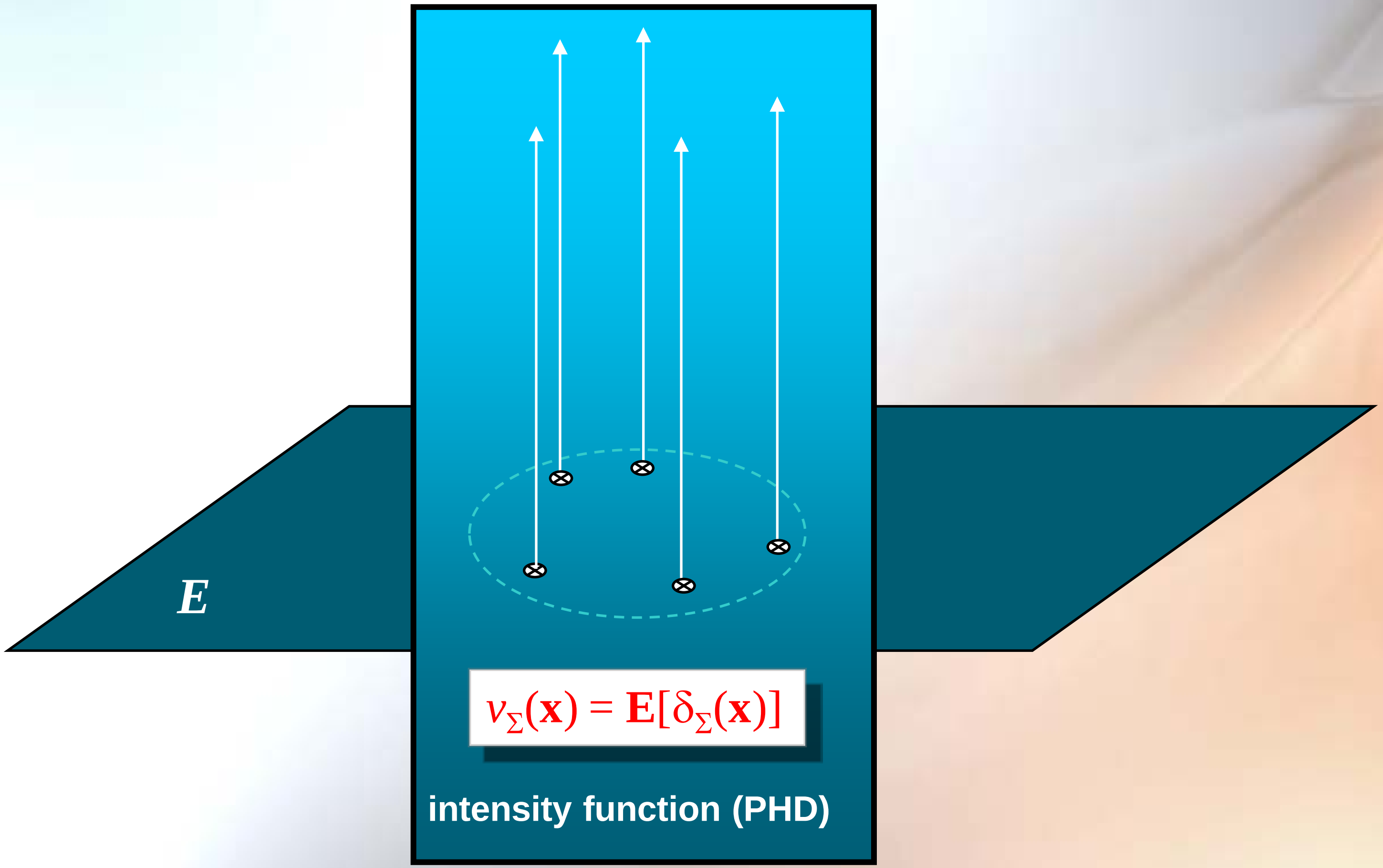
Stochastic Geometry

- **Physical** interpretation of the PHD



Stochastic Geometry

- Engineering interpretation of the PHD as the “expected set”



Bayesian SLAM

- Map = **finite set** of landmarks
- Bayesian SLAM requires modelling uncertainty in maps by RFS

Bayes-SLAM prediction

$$p_{k|k-1}(x_k, M_k \mid Z_{1:k-1}, u_{1:k-1}, x_0) = \int\int f_{k|k-1}(x_k, M_k \mid x_{k-1}, M_{k-1}, u_{k-1})\, p_{k-1}(x_{k-1}, M_{k-1} \mid Z_{1:k-1}, u_{1:k-2}, x_0)\, \delta M_{k-1} dx_{k-1}$$

Controls

Transition density

Set integral

Measurements

Map

Robot pose

RFS-SLAM
[Mullane et. al. 08]

Bayes-SLAM update

$$p_k(x_k, M_k \mid Z_{1:k}, u_{1:k-1}, x_0) = \frac{g_k(Z_k \mid x_k, M_k)\, p_{k|k-1}(x_k, M_k \mid Z_{1:k-1}, u_{1:k-1}, x_0)}{\int\int g_k(Z_k \mid x_k, M_k)\, p_{k|k-1}(x_k, M_k \mid Z_{1:k-1}, u_{1:k-1}, x_0)\, \delta M_k dx_k}$$

Measurement likelihood

Set integral

Bayesian SLAM

Bayes Risk: Expected posterior cost/penalty of incorrect estimate

$$R(\tilde{M}) \triangleq \mathbf{E}\left[C(\tilde{M}(Z_{1:k}),\tilde{M})\right] = \int C(\tilde{M}(Z_{1:k}),M)\,p\left(Z_{1:k}\mid M\right)p\left(M\right)\delta M\,\delta Z_{1:k}$$

Bayes risk

Penalty of using $\tilde{M}(Z_{1:k})$ as an estimate of M

set integrals

Optimal Bayes estimator:

$$\hat{M}^{Bayes} : Z_{1:k} \mapsto \hat{M}^{Bayes}(Z_{1:k}) = \arg\min_{\tilde{M}} R(\tilde{M}\,|Z_{1:k})$$

Joint multi-target estimator [Mahler07]: given a D

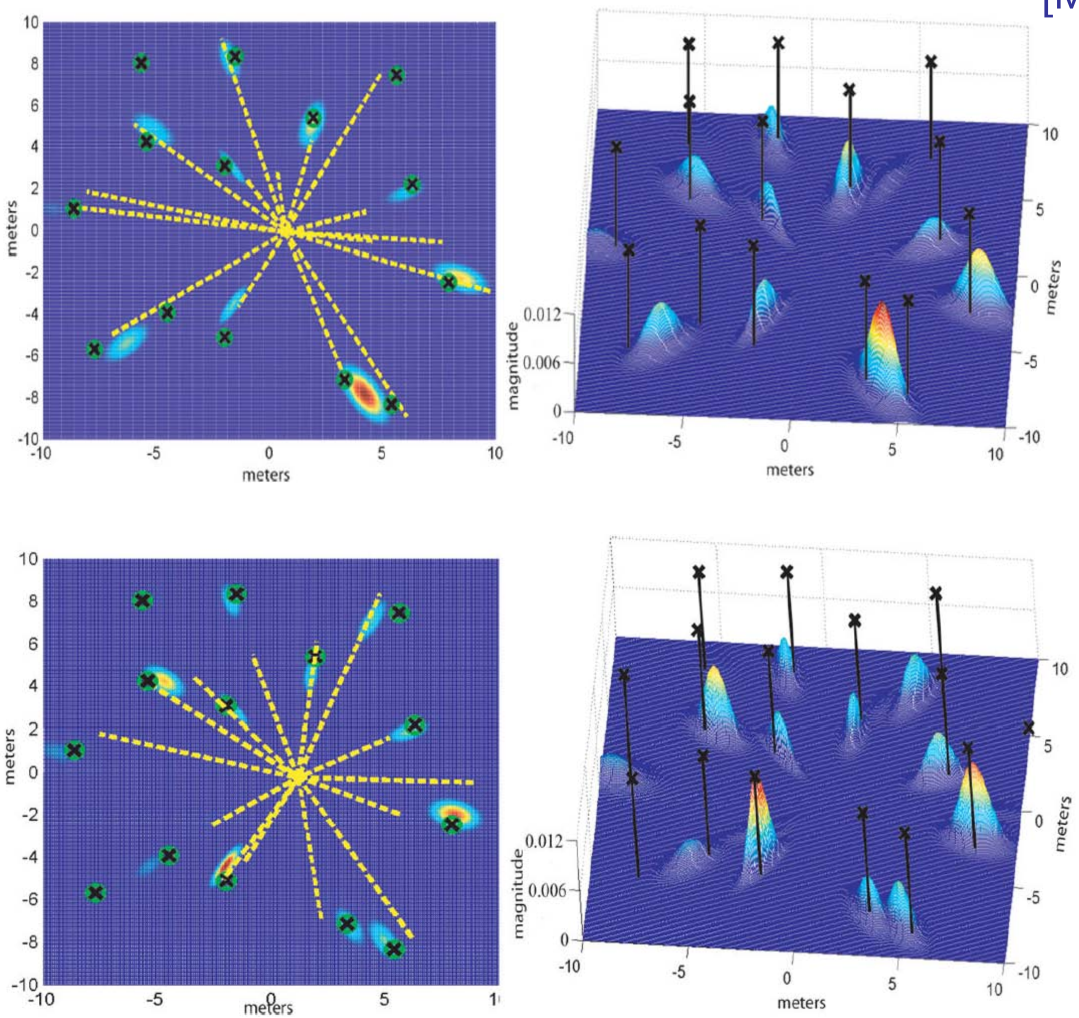
$$\hat{M}_D^{JoM} = \arg\sup_{M}\, p(M\mid Z_{1:k})\frac{D^{|M|}}{|M|!}$$

Bayes Optimal & converges as k tends to infinity

Bayesian SLAM

Using the PHD as the expected map

RFS-SLAM
[Mullane et. al. 08]



Bayesian SLAM

SLAM Formulation & Solutions [Mullane et. al. 2008, 2010, 2011]

SLAM SMC-PHD [Kalyan et. al. 2010]

Mapping [Lundquist et. al. 2011]

Collaborative SLAM [Moratuwage et. al. 2010, 2012]

SLAM with cluster processes [Clark et. al. 2012]

Conclusion

- Mapping error is of fundamental importance
- The (feature) map is a finite set
- Bayesian SLAM requires random finite set
  - Borne out of practical & fundamental **necessity**
  - Fully integrates uncertainty in data association & landmarks under one umbrella.
- The rest is up to you ...

Thank You!